Induction and coinduction for computing exact overlaps of fractals

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In this talk, we consider simple self-similar fractals whose definition does not contain rotations. For a set $D \subset \mathbb{R}^n$ of k^2 points called a digit set, we denote by $\mathcal{F}^n(k, D)$ the n-dimensional self-similar fractal generated by the IFS (iterated function system) $\{f_d(x) = \frac{x+d}{k} \mid d \in D\}$. Self-similar fractals are defined through greatest/least fixed points. $\mathcal{F}^n(k, D)$ itself is the greatest fixed point of the operator $\Phi_{k,D}(X) = (X + D)/k$ for $+$ the Minkowski sum, and the expansion set $\mathcal{E}^n(k, D)$, which is the least fixed point of the "dual" operator $\Psi_{k,D}(X) = kX + D \cup \{0^n\}$, is an important tool for studying $\mathcal{F}^n(k, D)$. Thus, it is expected that induction / coinduction are used effectively in their studies. Spreen and Berger introduced general notion of digit space [1, 2] based on coinduction and developed a theory of fractal representations in the framework of IFP (intuitionistic fixed point logic) [3].

In this talk, we characterize directions from which three fractal three-dimensional objects are projected to sets with positive measures, and from the (co)inductive part of the proof, we extract some information on the projections. Our fractal objects are Sierpinski tetrahedron (i.e., three-dimensional Sierpinski gasket), H fractal and T fractal, which are $\mathcal{F}^3(2, D_S)$, $\mathcal{F}^3(3, D_H)$ and $\mathcal{F}^3(3, D_T)$ for some appropriate digit sets D_S, D_H and D_T, respectively.

These three fractals have a remarkable property that they are projected to squares along three orthogonal directions just like a cube. This property is called imaginary cubes([6]), and Sierpinski tetrahedron is the only fractal imaginary cube expressible as $\mathcal{F}^3(2, D)$ for some digit set D, and H fractal and T fractal are the only two fractal imaginary cube expressible as $\mathcal{F}^3(3, D)$. In addition, these fractals generate figures with positive Lebesgue measures when they are projected from many other directions as the below picture shows.

A projected image of a fractal $\mathcal{F}^3(k, D)$ has the form $\mathcal{F}^2(k, D')$ for D' the image of D, and it is a special case of a self-affine tile that has been intensively studied from the 1990's [4, 5]. Kenyon [4] proved that for any non-colinear four point set $D' = \{\mathrm{O}, \mathrm{P}, \mathrm{Q}, \mathrm{R}\}$, the fractal $\mathcal{F}^2(2, D')$ has positive Lebesgue measure if and only if $p\vec{\mathrm{OP}} + q\vec{\mathrm{OQ}} + r\vec{\mathrm{OR}} = 0$ for odd numbers p, q, r, and this result characterizes directions along which a Sierpinski tetrahedron is projected to a set with positive measure. In this talk, we reformulate his proof through the notion of projection of differenced radix expansion set so that it could be applied to projections of other fractals of the form $\mathcal{F}^3(k, D)$.

For $D \in \{D_S, D_H, D_T\}$, our characterization theorem on $\mathcal{F}^3(k, D)$ consists of three parts: (1) for projections along some rational directions, projected images have positive measure; (2) for projections along other rational directions, two portions of the fractal exactly overlap in the projection, and therefore projected images have measure zero; (3) for projections along irrational directions, projected images have measure zero. In this talk, we focus on the proof of (2). In the proof, we need to characterize (some part of) expansion $\mathcal{E}^n(k, \Delta(D))$ for $\Delta(D) = D - D$ (Minkowski difference). This part is proved by well-founded induction. We also show that $\mathcal{E}^n(k, \Delta(D))$ is also the greatest fixed point of $\Psi_{k,\Delta(D)}$ and therefore coinduction can also be used for the proof. From these proofs, one can extract two addresses of finite portions of $\mathcal{F}^3(k, D)$ that exactly overlap through the projection.

References

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