Narrowing the gap between point-free topology and exact real computation

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In the continuity, computability and constructivity community, we have three well-established ways of presenting topology without points. These correspond to different strengths of the foundations: classical point–set topology to the axiom of choice, locale theory to topos logic, formal topology to Martin-Löf type theory and abstract Stone duality to arithmetic universes. The free arithmetic universe is composed of computable functions, which points towards computation.

On the other side, interval or "reliable" computation has a remarkable ability to give provably correct solutions to arbitrary precision to problems in integral calculus and solid modelling. However, whilst the programming is very clever, the subject lacks conceptual principles that can guide us past the original arithmetic operations in one real dimension.

Rather than defining some new kind of numbers, intervals should be seen as *using crude arithmetic to derive strong logical properties of functions.* Underlying the fact that it gives outer bounds to real-valued functions is that evaluating a *predicate* intervalwise provides an approximation to the *universal quantifier*. This is made precise by dividing up the interval sufficiently finely.

Using *back-to-front* intervals similarly approximates the *existential* quantifier, although this is poorly understood. Similarly, the potentially chaotic behaviour of the Newton–Raphson algorithm may be tamed by replacing its *pointwise* differentials with intervalwise ones.

These techniques were the basis of Andrej Bauer's *Marshall* calculator for evaluating Dedekind cuts. The operational principle is that each application of these ideas splits an interval into smaller ones on which some predicate is clearly true, clearly false or still indeterminate. The next stage is to devise new interval-like methods to extend these results to $\mathbb{R}^n$.

Dedekind cuts are easily equivalent to the formal points of point-free topology. These methods capture a problem in a *purely* mathematical way, whereas Cauchy sequences *pre-judge* the way a computation is to be done. Beware that, because of this essential difference, *actual computation with cuts or formal points does not make traditional numerical analysis redundant, but demands that we re-formulate the whole of it within our subject.*

That is the chasm, seen from the practical side, so what can we offer from the theoretical side? In place of intervals, we need *parts* of a space that can be sub-divided.

- In Formal Topology, these capture basic *open* subspaces using an *infinitary* cover relation $\triangleleft$.
- In Abstract Stone Duality, they are intuitively *pairs* of subspaces, one *compact* and the other *open*, with a *finitary* cover relation $\prec\!\!\!\prec$.

These axiomatisations are given respectively in the Formal Topology literature and my draft paper *Local Compactness and Bases in various formulations of Topology*,

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and their similarity should promote a transfer of methods.

Completeness, *i.e.* that an abstract $a \prec\!\!\prec b$ actually captures $K_a \subset U_a$, for the four formulations of topology relies, as we expect, on the corresponding logic. The hardest case is the classical one, in general requiring not only Choice but proving the constructive one first, although the countable form can be done directly using Dependent Choice.

However, the account of $\prec\!\!\prec$ goes further than completeness, refining the axioms so that a model of ASD can be constructed "with bare hands" from an arithmetic universe. In this, an auxiliary category naturally arises that is simpler than the one composed of continuous functions and has an *affine* product and related exponential.

The two are related by a structure-preserving functor. What is does concretely is to make the sufficiently fine sub-division that yields exact answers for ordinary interval methods,

It is difficult to know where to go from here towards computation for locally compact spaces. This is because $\mathbb{R}$ is the only space for which we know how to capture the topology entirely based on "arithmetic" operations plus the ASD axioms.

However, returning to theory, when the details of this "bare hands" model of ASD have been fully worked out, we should be able to state more precisely its relationship to the logic of arithmetic universes. This question was first raised in my paper *Inside every model of Abstract Stone Duality lies an Arithmetic Universe* (namely its full subcategory of overt discrete spaces).

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