Categorifying Computable Reducibilities

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Categorical methods and language have been employed in many areas of Mathematics. In Mathematical Logic, they are widely used in Model Theory and Proof Theory, less so in Set Theory. In Recursion Theory or Computability, there is a long tradition of categorical methods in Realizability studies, expounded, for example, in the recent book "Realizability: An Introduction to its Categorical Side" [vO08]. One of the first uses of categorical methods in Realizability was Hyland's introduction of the Effective Topos [Hyl82], a categorical 'universe' of recursive mathematics. Hyland described how to use Kleene's notion of recursive realizability [Kle45] to build what came to be known as the effective topos. Then Pitts defined the notion of a 'tripos' and together with Hyland and Johnstone [HJP80] developed the initial properties of set-based triposes, including the tripos-to-toposes construction. Hofstra's work on "All Realizability is Relative" [Hof06] follows this trend of category-theoretic approaches to the theory of computation. Hofstra proved that most well-known realizability triposes, (e.g. the "effective" tripos, the "modified realizability" tripos and the "dialectica" tripos) are instances of a more general notion of tripos associated to a given partial combinatory algebra (PCA). He showed that all these triposes can be presented as "triposes for a given PCA", all these notions differ only in the choice of the associated PCA, and so we could say that all realizability is relative to a choice of a PCA. His formulation explains and systematizes the different kinds of triposes for realizability developed independently before.

In a related, but independent perspective, previous work of Trotta and Maietti [MT21] showed that every realizability tripos (in the general sense of a tripos constructed from a PCA) is an instance of the generalized **existential completion**. So every realizability tripos is obtained by freely adding left adjoints along the class of all morphisms of the base category. Hence, in particular, the dialectica tripos (Biering in [Bie08]), the tripos of modified realizability, etc... are all instances of the generalized existential completion. This reformulation explains and systematizes the different kinds of triposes for realizability in terms of an existential completion of doctrines.

In our work, we want to extend the use of categorical methods to further areas of Computability. We want to discuss categorical formulations of Medvedev, Muchnik and Weyrauch reducibility [Hin12, BGP21]. To categorify these notions of reducibility we provide categorical doctrines for the reducibilities and we connect these reducibility doctrines to different categorical doctrines used in previous work [TSdP22]. In particular, we manage to relate Medvedev and Weyrauch doctrines to the Dialectica doctrine, showing that all these doctrines can be conceptualized in terms of quantifier (existential and universal) completions.

Putting together a categorical understanding of reducibility in computability with categorical logic semantic descriptions in terms of Lawvere '(hyper)doctrines' shows a bridge between these two areas of mathematical logic that seems helpful to both sides. For categorical proof-theorists, it encompasses models of both functional interpretations and computability under the same kind of construction: both realizability/Dialectica triposes and reducibility degrees are versions of quantifier completions. For recursion theorists it extends the categorification of realizability (described e.g. in [vO08]) to degrees of reducibility (Medvedev, Muchnik and Weihrauch) not usually considered in categorical terms. The categorical formulation allows us to provide more abstract, easier proofs of our results.

Since this work is an application of categorical methods to computability, especially to notions of reducibility, we first recall the general computability definitions we need for our process of generalization. Then we recall the versions of (Lawvere) doctrines and the recent quantifier completion results[TSdP22] we will need for the categorification of reducibilities. After that, we start the real work, and formulate Medvedev and Muchnik reducibility in categorical terms, showing that their doctrines correspond exactly to universal completions and how they interact.

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Then we get to the central part of this work. Our most exciting results come in, when we categorify Weihrauch reducibility, and its 'strong' and extended versions. While all realizability-like doctrines, including Medvedev and Muchnik doctrines, are defined on the base category of sets and functions, to achieve a tight categorification of Weihrauch we have to consider a different base category, as for Weihrauch reducibility we have to keep track of the 'inputs'.

We consider strong and extended notions of Weihrauch reducibility as predicates of doctrines on the base category of modest sets [Ros90]. Considering this category as base for our doctrines allows us to have direct access to the 'inputs' of our predicates, and then to fully abstract Weihrauch reducibility in categorical terms. After defining the Weihrauch doctrines and showing that they generalize the usual notion of Weihrauch reducibility, we prove that both Wiehrauch and strong Weihrauch doctrines are instances of the existential completion.

These results allow us to employ the universal properties of the existential and universal completions to easily establish a formal connection with other already known doctrines, such as the Gödel doctrines. Recall that previous work [TSdP21] has shown how to categorify Gödel's Dialectica interpretation [GF+86] in terms of doctrines, called Gödel doctrines. This is done in the faithful way that internal language theorems allow us: results in the logic can be translated into categorical theorems and conversely, categorical results can be re-interpreted in a logical language, which is, sometimes, easier to understand than 'categorical diagram chasing'.

Finally, we compare Medvedev and Weihrauch doctrines to Gödel ones, via their associated universal properties. As our main result, we show that Weihrauch doctrines are reflective subdoctrines of suitable Gödel doctrines. Hence they are both universal and existential completions.

We conclude with further suggestions on how to continue this process of categorification of Computability, making sure that most important logic concepts can be seen as free categorical constructions, whenever possible.

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