On local times of Martin-Löf random Brownian motion

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In this project, we study the local times of Brownian motion from the point of view of algorithmic randomness. We introduce the notion of effective local time and show that *any* path which is Martin-Löf random with respect to the Wiener measure has continuous effective local times at every computable point.

A Brownian motion on the unit interval is a real-valued function $X : (t, \omega) \mapsto X(t, \omega)$ defined on $[0, 1] \times \Omega$, where Ω is the underlying space of some probability space, such that $X(0, \omega) = 0$, the function $t \mapsto X(t, \omega)$ is continuous for any ω, and for any finite sequence $0 < t_1 < \ldots < t_n \leq 1$, the functions $\omega \mapsto X(t_1, \omega), X(t_2, \omega) - X(t_1, \omega), \cdots, X(t_n, \omega) - X(t_{n-1}, \omega)$ are random variables statistically independent and normally distributed with means 0 and variances $t_1, t_2 - t_1, \cdots, t_n - t_{n-1}$, respectively. In this paper, we shall consider the canonical Brownian motion X defined by $X(t, \omega) = \omega(t)$ where $\Omega = C[0, 1]$ is the set of continuous real functions defined on $[0, 1]$ and vanishing at the origin, with its uniform norm topology and endowed with the *Wiener measure* $\mathbb{P}$. The random variable $\omega \mapsto X(t, \omega)$ will be denoted $X(t)$.

The *occupation measure of the Brownian motion X up to time t* is the random Borel measure defined by

$$\mu(t, \omega, A) = \lambda\{s \in [0, t] : X(s, \omega) \in A\}, \ A \text{ Borel in } \mathbb{R}, \ \omega \in \Omega.$$

Here λ is the Lebesgue measure.

Lévy proved that for almost all $\omega \in \Omega$, the occupation measure $\mu(t, \omega, .)$ is absolutely continuous, that is, there exists a function

$$\mathscr{L}(t, \omega, .) : \mathbb{R} \to \mathbb{R}, \ x \mapsto \mathscr{L}(t, \omega, x)$$

such that

$$\mu(t, \omega, A) = \int_A \mathscr{L}(t, \omega, x) dx, \ (A \text{ Borel in } \mathbb{R}).$$

The number $\mathscr{L}(t, \omega, x)$ is called the "local time of ω at x up to time t". Lévy called it the "mesure du voisinage" and it represents the "time that the Brownian path ω spends at the point x during the time interval $[0, t]$". It is clear, by the Lebesgue density theorem that, for almost every $x \in \mathbb{R}$ (with respect to the Lebesgue measure),

$$\mathscr{L}(t, \omega, x) = \lim_{\epsilon \to 0^+} \frac{1}{2\epsilon} \lambda\{s \leq t : |X(s, \omega) - x| \leq \epsilon\}. \tag{1}$$

Trotter [10] proved later that the occupation measure $\mu(t, \omega, .)$ has continuous density for almost all ω, that is, $\mathscr{L}(., \omega, .) : [0, 1] \times \mathbb{R}, (t, x) \to \mathscr{L}(t, \omega, x)$ is continuous. (See also [8, Theorem 6.19].) This has the implication that, for every $x \in \mathbb{R}$, almost surely, for all $t \in [0, 1]$,

$$\mathscr{L}(t, \omega, x) = \lim_{\epsilon \to 0^+} \frac{1}{2\epsilon} \lambda\{s \leq t : |X(s, \omega) - x| \leq \epsilon\}. \tag{2}$$

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We shall consider the local time at 0 and it will be denoted $\mathscr{L}(t,\omega)$ instead of $\mathscr{L}(t,\omega,0)$. That is,

$$\mathscr{L}(t,\omega) = \lim_{\epsilon\to 0^+} \frac{1}{2\epsilon}\lambda\{s \leq t : |X(s,\omega)| \leq \epsilon\}. \tag{3}$$

The notion of local time has been extended to many stochastic processes. It is an important key concept in the study of fine properties of stochastic processes, for example fractal dimensions of level sets, regularity of sample paths.

We can, of course, associate with any real-valued function on the unit interval, an occupation density, and ask whether it is absolutely continuous with respect to Lebesgue measure. In this case the Radon-Nykodym derivative will be its local time. As indicated by Geman and Horowitz [6], it is an interesting open problem to investigate the existence of local times of particular classical functions such as the Weierstrass nowhere differentiable function and to determine which functions representable as Fourier series are such that the occupation measure is absolutely continuous and to compute the local times in terms of the Fourier coefficients.

One aim of this project, is to show that, we can explicitly define a continuous function on the unit interval, that does have local time, in the sense that the occupation density of this function is absolutely continuous with respect to Lebesgue measure. It will turn out we can find such a function which is the complex oscillation $\Phi(\Omega)$ associated with a Chaitin real Ω with Φ as constructed in Fouché [3]. As was pointed out by George Davie, it follows s from [1] that this function can be directly computed from such an Ω. A more detailed exposition of the first stage of this project can be found in [5] which is a precursor of a paper with the aim of viewing the construction local time of $\Phi(\Omega)$ from the angle of layerwise computability as in [2] and [7]. We shall argue that this approach does solve some open problems in real analysis as formulated in [6].

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