Polynomial-time equivalent representations of compact sets in Euclidean spaces

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Brattka and Weihrauch [2, Section 4] compared several natural representations of the space $\mathscr{K}$ of compact subsets of Euclidean space, which induce computability notions that are now well studied and generalized [1, 4]. One of the representations, $\delta^{>}_{\mathscr{K}}$, is based on the fact that compact sets are bounded and closed: a $\delta^{>}_{\mathscr{K}}$-name of $K \in \mathscr{K}$ consists of a bound on K (i.e., a finite set of rational balls whose union contains K) together with an infinite list enumerating basic sets (rational balls) contained in the complement of K. Another representation, δ_{cover}, is based on the characterization of compact sets using covers: a δ_{cover}-name of $K \in \mathscr{K}$ enumerates a set of finitely many basic sets that covers K. These representations are shown to be computably equivalent.

We discuss possible ways to refine their argument to polynomial-time computability. To make the representations meaningful in the time-bounded context, we need to replace enumeration by recognition: that is, instead of an infinite list of basic sets or covers, we consider a predicate that roughly tells us whether a given set intersects K, or whether given sets cover K. This leads to the following representations of $\mathscr{K}$:

- A $\hat{\delta}_{\mathscr{K}}$-name of $K \in \mathscr{K}$ is a pair (w, D) of a bound $w \in \Sigma^*$ on K and a predicate $D \subseteq \Sigma^*$ such that for any rational ball B, we have $B \in D$ if K intersects B, and $B \notin D$ if K does not intersect $2B$, the ball with the same centre and twice the radius as B.
- A $\hat{\delta}_{\text{cover}}$-name of $K \in \mathscr{K}$ is a pair (w, C) of a bound $w \in \Sigma^*$ on K and a predicate $C \subseteq \Sigma^*$ such that for any finite list $l = (B_0, \dots, B_{k-1})$ of rational balls, we have $l \in C$ if $K \subseteq B_0 \cup \dots \cup B_{k-1}$, and $l \notin C$ if $K \nsubseteq 2B_0 \cup \dots \cup 2B_{k-1}$.

Note that we are using "soft" recognition: the expected behaviour of the predicates C and D is unspecified on some inputs. Computing a $\hat{\delta}_{\mathscr{K}}$-name of K can be thought of as drawing K on a computer screen with various precision, and was the notion used in the study of complexity of some fractal sets [5, 3]. We show that $\hat{\delta}_{\mathscr{K}}$ and $\hat{\delta}_{\text{cover}}$ are polynomial-time equivalent.

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The computable equivalence of $\delta^{>}_{\mathscr{K}}$ and δ_{cover} (as well as some other representations) can be generalized to computable metric spaces satisfying the properties called nice closed balls and effective covering [1, Theorem 4.10]. Our argument for polynomial-time equivalence seems to rely on more special properties of Euclidean spaces, and it remains open whether similar generalization is possible.

References

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