WEIHRAUCH REDUCTIONS BETWEEN STEP FUNCTIONS

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When it comes to the study of Weihrauch reductions between concrete problems, the difference between using Weihrauch reducibility (which requires the reduction witnesses to be computable) and continuous Weihrauch reducibility (where the witnesses merely have to be continuous) usually does not matter: The positive results will use computable reductions, and the separation arguments will either directly involve topological arguments, or will relativize (and ultimately, a topological separation argument is the same as a relativizing one).

The Weihrauch degrees and the continuous Weihrauch degrees also share many structural properties, a phenomenon somewhat explained (in very different ways) in [Pau17] and [BP18]. On the other hand, the Weihrauch degrees obviously have a much finer structure: The lower cone of id_{2^ω} in the Weihrauch degrees is isomorphic to the reversed Medvedev degrees with a top-element added; in the continuous Weihrauch degrees, there is just the degree of id_{2^ω} and the degree of the no-where defined function.

As a first step towards understanding the structure of the Weihrauch degrees inside "simple" continuous Weihrauch degrees, the second author proposed to study the Weihrauch degrees of step functions $s_x : 2^\omega \to \{0,1\}$ and $s_x : [0,1] \to \{0,1\}$ with $s(y) = 0$ for $y < x$ and $s(y) = 1$ for $y \geq x$ at a Dagstuhl seminar [Wes].

We report some results on this question. The resulting structure turns out to be rather wide, e.g. $s_x \leq_{\mathrm{W}} s_y$ already implies $x \equiv_{\mathrm{T}} y$. On the other hand, we can construct complex structures of reductions and non-reductions inside c.e. degrees. The question of whether we work in Cantor space or in the unit interval turns out to be very relevant, with the latter yielding a richer structure.

Our results follow up on an initial unpublished theorem by Kihara and Westrick. The generalization of tt-degrees to computable Polish spaces by Kihara [Kih19] turns out to be relevant, as does the notion of m-reducibility between functions from [DDW19].

References

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