Sets completely separated by functions in Bishop Set Theory

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Constructive mathematics (CM), although hard to define, is not only mathematics within intuitionisitic logic. It also encompasses certain "attitudes" towards the definition of mathematical concepts, in order to reveal the computational content of mathematical proofs of theorems concerning these concepts. Some of these attitudes, which are not uniformly and consistently followed by the practitioners of CM though, are the following: *predicativity*, the addition of *witnessing information* so that even weak choice principles, such as countable choice, are avoided in proofs, preference for *positive definitions* of concepts over negative ones i.e., definitions using negation, and preference for *function-based concepts* over set-based ones, as functions suit better to CM than sets. In this proposed talk, based on [13], we highlight the combination of the last two attitudes in the study of sets with an inequality, determined in a positive way from a given set of functions. As many results shown in [13] reveal an analogy between these sets with the completely regular topological spaces, we call them *completely separated sets*. Although the notion of an inequality induced by real-valued functions is implicitly used by Bishop [1], p. 66, in his definition of a complemented subset, an elaborate study of sets equipped with such an inequality is missing. Ruitenburg's related work [14] is set-based.

We work within *Bishop Set Theory* (BST), an informal, constructive theory of totalities and assignment routines, elaborated in [6]-[12], that serves as a "completion" of Bishop's original theory of sets in [1, 2]. BST highlights fundamental notions that were suppressed by Bishop in his account of the set theory underlying Bishop-style constructive mathematics (BISH), and serves as an intermediate step between Bishop's informal theory of sets and an *adequate* and *faithful* formalisation of BISH, in Feferman's sense [3]. Similarly to Martin-Löf Type Theory (MLTT) (see e.g., [5]), BST behaves like a high-level programming language.

- We present the category **SetIneq** of sets with an inequality and strongly extensional functions, and we define the Sigma-and the Pi-set of a family of sets with an inequality.
- We define the canonical equality $=_{(X,F)}$ and inequality $\neq_{(X,F)}$ induced on a set $(X, =_X)$ by an extensional subset F of the real-valued functions $\mathbb{F}(X)$ on X. The extensionality and the tightness of the inequality $\neq_{(X,F)}$ avoid completely the use of negation.
- We introduce the category **SetComplSep** of completely separated sets, a full subcategory of **SetIneq**, and the category **SetAffine** of affine sets, a subcategory of **SetComplSep** with affine arrows only. We define the notion of a family of completely separated sets over an index-completely separated set, we describe its corresponding Pi-set, and we provide a sufficient condition, in order to get a restricted form of a Sigma-set for it. By introducing the notion of a global family of completely separated sets over an index-completely separated set, we manage to describe its Sigma-set as a completely separated set, and to generalise the notion of a strongly extensional function to dependent functions. The second projection of the Sigma-set of such a global family is shown to be a strongly extensional dependent function.
- We define the free completely separated set $\varepsilon\boldsymbol{X}$ on a given set $(X, =_X)$, we prove its universal property, and we show that the functor `Free`: **Set** $\to$ **SetComplSep** is left adjoint to the corresponding forgetful functor `Frg`: **SetComplSep** $\to$ **Set**. The description of $\varepsilon\boldsymbol{X}$ corresponds to the type-theoretic fact that the setoid $(X, =_X)$, where $=_X$ is the equality type family on the type X in intensional MLTT, is the free setoid on the type X.

- We prove a purely set-theoretic version of the Stone-Čech theorem in classical topology, according to which, to any topological space corresponds a completely regular one such that the two spaces have isomorphic rings of continuous, real-valued functions, and the corresponding functor is a reflector. Replacing topological spaces with function spaces i.e., triplets $(X, =_X; F)$ as above, and completely regular spaces with completely separated sets, we correspond to any function space a completely separated set with the same carrier set and (separating) set of functions. The corresponding functor $\rho\colon \mathbf{FunSpace} \to \mathbf{SetAffine}$ is shown to be a reflector, hence left adjoint to the corresponding embedding functor $\mathtt{Emb}\colon \mathbf{SetAffine} \to \mathbf{FunSpace}$. Moreover, $\mathtt{Emb}$ is also left adjoint to ρ. The category **FunSpace** of function spaces and affine maps is studied in [4].

- We prove a purely set-theoretic version of the Tychonoff embedding theorem in classical topology, according to which, a T_1 topological space is completely regular if and only if it is topologically embedded into a product of $[0, 1]$. According to our version of this theorem, if $(X, =_X; F)$ is a function space, and if the induced inequality $\neq_{(X,F)}$ on X is tight, then there is an affine embedding of the completely separated set $(X, =_X, \neq_{(X,F)}; F)$ into the completely separated set $\boldsymbol{R}^F$. Conversely, if if $e\colon (X, =_X; F) \to (\mathbb{R}^F, =_{\mathbb{R}^F}; \bigotimes_{f\in F}\{\mathrm{id}_{\mathbb{R}}\})$ is an affine embedding in **FunSpace**, the induced inequality $\neq_{(X,F)}$ on X is tight. The latter result provides a criterion for the generation of a completely separated set from a given function space.

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