On the relationship between well-founded sets and inductive generation of pointfree topology

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The development of topology in a constructive and predicative foundation led to the introduction of formal topology in [12, 13] with a severe distinction between a set of basic opens and a collection or class (not generally a set!) of formal opens. In turn the need of building predicative and constructive examples of formal topologies, including the point-free topology of Dedekind real numbers, inspired the advent of powerful inductive methods of topological generation put forward in [3]. Since then, it was clear that some kind of well-founded set constructor was enough to formalize such a topological induction in Martin-Löf's type theory as shown in detail in [14]. Moreover, it was also underlined that the main difficulty in generating inductive topologies reduces essentially to that of generating inductive suplattices, named *inductively generated basic covers*, because the structure of inductive frame can be easily instantiated as a special case of inductive suplattice like shown in [3] and extensively explained in [1, 2].

Recently, in [10], the Curry-Howard representation of intuitionistic connectives and quantifiers as types has been extended by giving a proof-relevant presentation of inductively generated basic covers within a two-level extension of the Minimalist Foundation [8]. Moreover, combining Th.4.9 of [10] with Th.5.3 of [6] it follows that a version of Martin-Löf's type theory with well-founded sets has the same proof-theoretic strength as the one with inductively generated basic covers. This led to the following question: *can we establish directly in some version of Martin-Löf's type theory an equivalence between well-founded sets and proof-relevant inductively generated basic covers?*

Inspired by the results in [5] and [7], in this talk we show that over intensional first-order Martin-Löf's type theory, well-founded sets are enough to represent dependent well-founded sets. In turn, one can define proof-relevant inductively generated basic covers with the latter. As a corollary, well-founded sets are enough to represent proof-relevant inductive basic covers over the intensional level of the Minimalist Foundation. If we also assume function extensionality, the converse is also true, namely inductively generated basic covers represent dependent well-founded sets. As a consequence, well-founded sets, dependent well-founded sets and proof-relevant inductively generated basic covers are equivalent over

Homotopy Type Theory in [11]. Finally, combining our result with the one in [4], we will also outline further applications to quotient completions in [9].

A file with all the proofs checked in Agda by P. Sabelli is available at `http://github.com/PietroSabelli/W-DW-IBC`.

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