Categories with a base of computability

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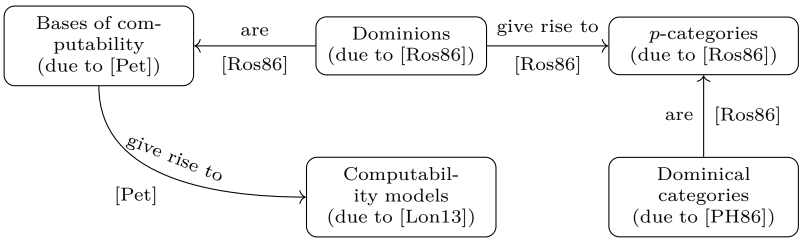
The category of computability models and simulations was introduced by Longley in [Lon13], and studied further in his book with Norman [LN15]. Within it, different notions of computability, like Turing computability, as well as more abstract notions from mathematical logic, such as untyped λ-calculus, were studied in a unified way. The novelty of Longley's approach lies in his distinction between data types and computable functions, whereas in earlier work the computable functions themselves constituted a data type of their own. It is worth noting that Cockett and Hofstra arrived at a similar notion independently from by Longley in [CH08]. The definition of a computability model bears a striking resemblance to the definition of a category, so it was natural to ask whether a computability model gives rise to a category or the other way around, and if not, what would be needed to obtain each direction. It was already known by Longley how to obtain a category from a given computability model through its associated category of assemblies. Petrakis [Pet22; Pet] showed that by endowing a category with a so called *base of computability* and with a pullback-preserving presheaf, one obtains a computability model in a canonical way. The notion of a base of computability in a category is almost identical to the notion of a dominion, introduced by Rosolini in [Ros86]. Dominions were originally used to arrive at p-categories that facilitate an intrinsic recursion theory, similarly to the work of Di Paola and Heller [PH86]. The similarities and relations between these notions are sketched as follows:



Our goal is to elaborate the theory of categories with a base of computability. We first define the category $\mathsf{CatBaseComp}$ of categories with a base of computability and the so called *computability transfers*. We then show that $\mathsf{CatBaseComp}$ has a cartesian structure, and even more, all finite limits. We show that there is a functor from the category $\mathsf{CatBasePshv}$ of categories with a base of computability and a pullback-preserving presheaf to the category $\mathsf{CompMod}$ of computability models and simulations that assigns to each such category its canonical computability model. Even more, all simulations obtained through this functor are natural, in the sense defined in [Pet]. Furthermore, we show that the functor assigning to each computability model its category of assemblies is right adjoint to the previous functor, once one restricts to appropriate subcategories.

Finally, we define the notions of prebase of computability and computability premodel and we associate to these constructs the "smallest" base and computability model containing them. As we show, these associations do not commute with the canonical computability model assignment.

References

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