Spreen spaces and the synthetic KLST theorem

Andrej Bauer

I take a constructive look at Dieter Spreen's treatment of the Kreisel-Lacombe-Shoenfield-Tseitin (KLST) continuity theorem for effective T-spaces. He formulated his ideas and techniques in the context of numbered sets and computability theory, but with some reverse realizability translation they can be rephrased in Bishop-style constructive mathematics, and more specifically in the context of synthetic computability. From the analysis a new kind of space is identified – one in which open sets separate semidecidable sets from overt ones. These deserve to be called Spreen spaces. The synthetic KLST theorem can then be stated simply as "every map from an overt Spreen space to a regular space is pointwise continuous". It becomes very easy to prove the theorem in Bishop's constructive mathematics, but difficult to find non-trivial instances of it. In order to produce some, we need further assumptions, such as those valid in the effective topos, or Russian constructivism. So in the second part I show that the axioms of synthetic computability imply that countably based sober spaces, and more, are Spreen spaces.