Spread representation of point-free real numbers

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Brouwer's quest for continuum as infinitely proceeding sequences led him to introduce the notion of spread. A spread is a certain subset of Baire space $\mathbb{N}^{\mathbb{N}}$ from which intended mathematical objects are obtained as its continuous image (cf. Heyting [3, 3.1.2]). In particular, the real numbers can be obtained as a continuous image of the ternary spread [3, Section 3.3]. Using this representation of real numbers, together with the intuitionistic principles such as fan theorem and the continuity principle,[1] Brouwer showed that every real-valued function on the unit interval is uniform continuous [2].

In this talk, we take another look at the spread representation of real numbers from the view point of point-free topology [5]. Point-free topology can be seen as a constructive basis of Brouwer's mathematics without assuming intuitionistic principles such as fan theorem and the continuity principle; thus, the argument in point-free topology can be carried out constructively in the sense of Bishop's constructive mathematics [1]. We see that the notion of real numbers by the spread representation is geometric, so they can be naturally understood as points of certain point-free topology. This gives rise to a new notion of point-free real numbers. With respect to this notion, the Heine–Borel theorem and the uniform continuity of real-valued functions on $[0, 1]$ follow. In addition, approximate versions of the intermediate value theorem and Brouwer's fixed-point theorem can be naturally formulated in this point-free setting, which provide point-free and choice free proofs of these theorems (cf. Kawai [4]).

References

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[1] The fan theorem is classically equivalent to the weak König's lemma. The continuity principle says that every function on real numbers is point-wise continuous.