# Optimal control and state constraints, with a model for a crawling robot

Giovanni Colombo

Università di Padova

May 10, 2019

Many thanks to those who worked for this event!

# Outline of the Talk

- Differential equations and state constraints
  - Generalities
  - Constraints that are active in the dynamics
- A control problem with a motivating model (in collaboration with Paolo Gidoni)
- Methods and results

# Differential equations and state constraints

It is well known that the Cauchy problem

$$\begin{cases} \dot{x} & = f(x) \\ x(0) & = x_0 \end{cases}$$

has one and only one (local in time) solution, provided $f$ is smooth in a neighborhood of the initial point $x_0$.

# Differential equations and state constraints

It is well known that the Cauchy problem

$$\begin{cases} \dot{x} & = f(x) \\ x(0) & = x_0 \end{cases}$$

has one and only one (local in time) solution, provided $f$ is smooth in a neighborhood of the initial point $x_0$.
What happens if one imposes a constraint that the state $x(t)$ must satisfy for all $t$?

# Differential equations and state constraints

It is well known that the Cauchy problem

$$\begin{cases} \dot{x} & = f(x) \\ x(0) & = x_0 \end{cases}$$

has one and only one (local in time) solution, provided $f$ is smooth in a neighborhood of the initial point $x_0$.
What happens if one imposes a constraint that the state $x(t)$ must satisfy for all $t$?
In other words, let $K$ be given and consider the problem

$$\begin{cases} \dot{x} & = f(x) \\ x(0) & = x_0 \\ x(t) & \in K \quad \forall t \end{cases}$$

It is clear that in order to let solutions exist there must be a compatibility condition between $f$ and $K$:

It is clear that in order to let solutions exist there must be a compatibility condition between $f$ and $K$: when the state hits the boundary of $K$, that is denoted by $\partial K$, then the prescribed velocity must not "point outside $K$".

It is clear that in order to let solutions exist there must be a compatibility condition between $f$ and $K$: when the state hits the boundary of $K$, that is denoted by $\partial K$, then the prescribed velocity must not "point outside $K$". Formally: if $K = \{x : g(x) \leq 0\}$ and $\nabla g(x) \neq 0$ whenever $g(x) = 0$, then the compatibility condition reads as

$$f(x) \cdot \nabla g(x) \leq 0 \quad \forall x \text{ such that } g(x) = 0.$$

It is clear that in order to let solutions exist there must be a compatibility condition between $f$ and $K$: when the state hits the boundary of $K$, that is denoted by $\partial K$, then the prescribed velocity must not "point outside $K$". Formally: if $K = \{x : g(x) \le 0\}$ and $\nabla g(x) \neq 0$ whenever $g(x) = 0$, then the compatibility condition reads as

$$f(x) \cdot \nabla g(x) \le 0 \quad \forall x \text{ such that } g(x) = 0.$$

There is a lot of literature that studies optimal control problems with state constraints

It is clear that in order to let solutions exist there must be a compatibility condition between $f$ and $K$: when the state hits the boundary of $K$, that is denoted by $\partial K$, then the prescribed velocity must not "point outside $K$". Formally: if $K = \{x : g(x) \le 0\}$ and $\nabla g(x) \ne 0$ whenever $g(x) = 0$, then the compatibility condition reads as

$$f(x) \cdot \nabla g(x) \le 0 \quad \forall x \text{ such that } g(x) = 0.$$

There is a lot of literature that studies optimal control problems with state constraints, that is
maximize some objective depending on $(x, u)$, subject to

$$\begin{cases} \dot{x} & = f(x,u),\ x(0) = x_0,\ u \in U \\ x(t) & \in K \quad \forall t \end{cases}$$

It is clear that in order to let solutions exist there must be a compatibility condition between $f$ and $K$: when the state hits the boundary of $K$, that is denoted by $\partial K$, then the prescribed velocity must not "point outside $K$". Formally: if $K = \{x : g(x) \leq 0\}$ and $\nabla g(x) \neq 0$ whenever $g(x) = 0$, then the compatibility condition reads as

$$f(x) \cdot \nabla g(x) \leq 0 \quad \forall x \text{ such that } g(x) = 0.$$

There is a lot of literature that studies optimal control problems with state constraints, that is
maximize some objective depending on $(x, u)$, subject to

$$\begin{cases} \dot{x} & = f(x,u),\ x(0) = x_0,\ u \in U \\ x(t) & \in K \quad \forall t \end{cases}$$

Necessary conditions of PMP type, even of higher order (Rampazzo, Vinter, Frankowska, ...)

# Constraints that are active in the dynamics: Moreau's sweeping process

In this kind of problems, the state constraint is juxtaposed to the dynamics.

# Constraints that are active in the dynamics: Moreau's sweeping process

In this kind of problems, the state constraint is juxtaposed to the dynamics.
There is another way to deal with state constraints, that essentially goes back to J.J. Moreau, in the Seventies.

# Constraints that are active in the dynamics: Moreau's sweeping process

In this kind of problems, the state constraint is juxtaposed to the dynamics.

There is another way to deal with state constraints, that essentially goes back to J.J. Moreau, in the Seventies.

Some notations: given a (reasonable) set $K$, denote by $N_K(x)$ the external normal cone to $K$ at $x$.

# Constraints that are active in the dynamics: Moreau's sweeping process

In this kind of problems, the state constraint is juxtaposed to the dynamics.
There is another way to deal with state constraints, that essentially goes back to J.J. Moreau, in the Seventies.

Some notations: given a (reasonable) set $K$, denote by $N_K(x)$ the external normal cone to $K$ at $x$. Let $K = \{x : g(x) \leq 0\}$:

if $g(x) < 0$, $N_K(x) = \{0\}$
if $g(x) = 0$, $N_K(x) = \{\lambda \nabla g(x) : \lambda \geq 0\}$
if $g(x) > 0$, $N_K(x) = \emptyset$.

Consider the dynamics

$$\begin{cases} \dot{x} & \in -N_K(x) + f(x) \\ x(0) & = x_0 \in K \end{cases} \tag{SP}$$

Consider the dynamics

$$\begin{cases} \dot{x} & \in -N_K(x) + f(x) \\ x(0) & = x_0 \in K \end{cases} \qquad \text{(SP)}$$

This is no longer a differential equation, but a differential inclusion.

Consider the dynamics

$$\begin{cases} \dot{x} & \in -N_K(x) + f(x) \\ x(0) & = x_0 \in K \end{cases} \tag{SP}$$

This is no longer a differential equation, but a differential inclusion. The meaning is the following: as long as the state $x(t)$ remains in the interior of $K$, the dynamics is that of $\dot{x} = f(x)$. Instead, when $x(t)$ touches $\partial K$, there is a reaction of the constraint. This reaction points towards the interior of $K$ and keeps the trajectory inside $K$.

Consider the dynamics

$$\begin{cases} \dot{x} & \in -N_K(x) + f(x) \\ x(0) & = x_0 \in K \end{cases} \tag{SP}$$

This is no longer a differential equation, but a differential inclusion. The meaning is the following: as long as the state $x(t)$ remains in the interior of $K$, the dynamics is that of $\dot{x} = f(x)$. Instead, when $x(t)$ touches $\partial K$, there is a reaction of the constraint. This reaction points towards the interior of $K$ and keeps the trajectory inside $K$.

It can be proved: (SP) is well posed (forward in time) provided $K$ is reasonable. The solution remains in $K$ all the time long by the very definition.

Consider the dynamics

$$\begin{cases} \dot{x} & \in -N_K(x) + f(x) \\ x(0) & = x_0 \in K \end{cases} \qquad \text{(SP)}$$

This is no longer a differential equation, but a differential inclusion. The meaning is the following: as long as the state $x(t)$ remains in the interior of $K$, the dynamics is that of $\dot{x} = f(x)$. Instead, when $x(t)$ touches $\partial K$, there is a reaction of the constraint. This reaction points towards the interior of $K$ and keeps the trajectory inside $K$.

It can be proved: (SP) is well posed (forward in time) provided $K$ is reasonable. The solution remains in $K$ all the time long by the very definition.

Observe however that the dynamics is discontinuous w.r.t. the state. In some sense it is a hybrid dynamics

Consider the dynamics

$$\begin{cases} \dot{x} & \in -N_K(x) + f(x) \\ x(0) & = x_0 \in K \end{cases} \tag{SP}$$

This is no longer a differential equation, but a differential inclusion. The meaning is the following: as long as the state $x(t)$ remains in the interior of $K$, the dynamics is that of $\dot{x} = f(x)$. Instead, when $x(t)$ touches $\partial K$, there is a reaction of the constraint. This reaction points towards the interior of $K$ and keeps the trajectory inside $K$.

It can be proved: (SP) is well posed (forward in time) provided $K$ is reasonable. The solution remains in $K$ all the time long by the very definition.

Observe however that the dynamics is discontinuous w.r.t. the state. In some sense it is a hybrid dynamics, but the state may leave the interior of the constraint, or change face of the boundary if $\partial K$ is not smooth (e.g., a square), on any type of time sets, not only on intervals.

There is a growing interest on the optimal control of this type of dynamics. There are models ranging from population dynamics in a crowded environment to electric circuits with diodes, and to mechanics. However, this topic needs new methods, as it does not fall into any classical setting, due to the discontinuity w.r.t. $x$ of the right hand side of the equation.

There is a growing interest on the optimal control of this type of dynamics. There are models ranging from population dynamics in a crowded environment to electric circuits with diodes, and to mechanics. However, this topic needs new methods, as it does not fall into any classical setting, due to the discontinuity w.r.t. $x$ of the right hand side of the equation.

Let us consider first the simplest optimal control problem, together with a toy example.

There is a growing interest on the optimal control of this type of dynamics. There are models ranging from population dynamics in a crowded environment to electric circuits with diodes, and to mechanics. However, this topic needs new methods, as it does not fall into any classical setting, due to the discontinuity w.r.t. $x$ of the right hand side of the equation.

Let us consider first the simplest optimal control problem, together with a toy example.

maximize $\varphi(x(T))$, over state control pairs $(x, u)$ such that

$$\begin{cases} \dot{x} & \in -N_K(x) + f(x, u), \quad \text{a.e. on } [0, T], \ u \in U \\ x(0) & = x_0 \in K. \end{cases} \tag{CSP}$$

There is a growing interest on the optimal control of this type of dynamics. There are models ranging from population dynamics in a crowded environment to electric circuits with diodes, and to mechanics. However, this topic needs new methods, as it does not fall into any classical setting, due to the discontinuity w.r.t. $x$ of the right hand side of the equation.

Let us consider first the simplest optimal control problem, together with a toy example.
maximize $\varphi(x(T))$, over state control pairs $(x, u)$ such that

$$\begin{cases} \dot{x} & \in -N_K(x) + f(x, u), \quad \text{a.e. on } [0, T], \ u \in U \\ x(0) & = x_0 \in K. \end{cases} \tag{CSP}$$

**Example.** Let $K = \{(x_1, x_2) : x_2 \leq \min\{0, -x_1\}\}$. Minimize $x_2(1)$ over $(x, u)$ such that $u \in [-1, 1]$ and

$$\begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} & \in -N_K\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} u \\ 0 \end{pmatrix} \\ x_1(0) & = -\frac{1}{2}, \ x_2(0) = 0. \end{cases}$$

disegna

# A motivating model

A (minimal) model for a crawler, in collaboration with Paolo Gidoni.

# A motivating model

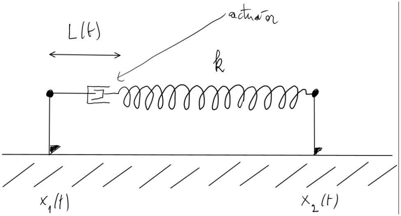
A (minimal) model for a crawler, in collaboration with Paolo Gidoni.

from a paper by Noselli and De Simone

# A motivating model

A (minimal) model for a crawler, in collaboration with Paolo Gidoni.

from a paper by Noselli and De Simone




$(x_1, x_2)$ describes the position of each contact point; $z = \frac{x_2 - x_1}{2}$ describes the length of the device and $y = \frac{x_1 + x_2}{2}$ its barycenter, i.e., the displacement we are interested in. The actuator modifies the length of the spring by changing $L(t)$, that is our control. It is a one dimensional model.

The energy of the spring is

$$\mathscr{E}(t,x) = \frac{k}{2}\big(x_2 - x_2 - L(t)\big)^2$$

The friction is

$$c_1|\dot{x}_1| + c_2|\dot{x}_2| =: \Psi(\dot{x}), \quad \text{where } x = (x_1, x_2) \text{ and } (c_1 \neq c_2).$$

The (quasi-static) force balance law is

$$0 \in \partial\Psi(\dot{x}) + D_x\mathscr{E}(t,x), \qquad \text{(BL)}$$

where $\partial$ stands for the "generalized derivative" of $y \mapsto |y|$, that can be defined – as a set – also at $y = 0$.

The energy of the spring is

$$\mathscr{E}(t,x) = \frac{k}{2}\big(x_2 - x_2 - L(t)\big)^2$$

The friction is

$$c_1|\dot{x}_1| + c_2|\dot{x}_2| =: \Psi(\dot{x}), \quad \text{where } x = (x_1, x_2) \text{ and } (c_1 \neq c_2).$$

The (quasi-static) force balance law is

$$0 \in \partial\Psi(\dot{x}) + D_x\mathscr{E}(t,x), \qquad \text{(BL)}$$

where $\partial$ stands for the "generalized derivative" of $y \mapsto |y|$, that can be defined – as a set – also at $y = 0$.
We want $L$ and $x_2 - x_1$ to be $T$-periodic, in order to be able to repeat the movement.

Via some magics of Convex Analysis, one can rewrite (BL) as

$$\begin{cases} \dot{z} & \in -N_K(z) + u, \qquad K = [-\mu_n, \mu_+], \quad u(= \dot{L}) \in [-1, 1] \\ \dot{y} & = |u - \dot{z}| \\ z(0) & = z(T) \\ y(0) & = 0 \\ \int_0^T u(t)\, dt & = 0. \end{cases}$$

We wish to maximize $y(T)$.

Via some magics of Convex Analysis, one can rewrite (BL) as

$$\begin{cases} \dot{z} & \in -N_K(z) + u, \qquad K = [-\mu_n, \mu_+], \quad u(= \dot{L}) \in [-1, 1] \\ \dot{y} & = |u - \dot{z}| \\ z(0) & = z(T) \\ y(0) & = 0 \\ \int_0^T u(t)\, dt & = 0. \end{cases}$$

We wish to maximize $y(T)$.

The optimal control problem that we wish to study is non-classic for two reasons:

Via some magics of Convex Analysis, one can rewrite (BL) as

$$\begin{cases} \dot{z} & \in -N_K(z) + u, \qquad K = [-\mu_n, \mu_+], \quad u(= \dot{L}) \in [-1, 1] \\ \dot{y} & = |u - \dot{z}| \\ z(0) & = z(T) \\ y(0) & = 0 \\ \int_0^T u(t)\, dt & = 0. \end{cases}$$

We wish to maximize $y(T)$.

The optimal control problem that we wish to study is non-classic for two reasons:

1) the dynamics is discontinuous w.r.t. the state

Via some magics of Convex Analysis, one can rewrite (BL) as

$$\begin{cases} \dot{z} & \in -N_K(z) + u, \qquad K = [-\mu_n, \mu_+], \quad u(= \dot{L}) \in [-1, 1] \\ \dot{y} & = |u - \dot{z}| \\ z(0) & = z(T) \\ y(0) & = 0 \\ \int_0^T u(t)\, dt & = 0. \end{cases}$$

We wish to maximize $y(T)$.

The optimal control problem that we wish to study is non-classic for two reasons:
1) the dynamics is discontinuous w.r.t. the state
2) the control must have zero mean.

Via some magics of Convex Analysis, one can rewrite (BL) as

$$\begin{cases} \dot{z} & \in -N_K(z) + u, \qquad K = [-\mu_n, \mu_+], \quad u(= \dot{L}) \in [-1, 1] \\ \dot{y} & = |u - \dot{z}| \\ z(0) & = z(T) \\ y(0) & = 0 \\ \int_0^T u(t)\, dt & = 0. \end{cases}$$

We wish to maximize $y(T)$.

The optimal control problem that we wish to study is non-classic for two reasons:
1) the dynamics is discontinuous w.r.t. the state
2) the control must have zero mean.

Observe that $\dot{y}$ represents the size of the reaction of the constraint.

One can apply common sense in order to design optimal strategies: since we want to maximize the reaction of the constraint, we want to maximize the time that $z$ spends on the boundary of the constraint: optimal strategies are expected to be bang-bang (with a balance between the time when it is 1 and $-1$, in order to satisfy the zero mean requirement).

One can apply common sense in order to design optimal strategies: since we want to maximize the reaction of the constraint, we want to maximize the time that $z$ spends on the boundary of the constraint: optimal strategies are expected to be bang-bang (with a balance between the time when it is 1 and $-1$, in order to satisfy the zero mean requirement).

However our job is also trying to develop a theory that covers this case.

One can apply common sense in order to design optimal strategies: since we want to maximize the reaction of the constraint, we want to maximize the time that $z$ spends on the boundary of the constraint: optimal strategies are expected to be bang-bang (with a balance between the time when it is 1 and $-1$, in order to satisfy the zero mean requirement).

However our job is also trying to develop a theory that covers this case.

We will see that the theory is still behind the model: there is still work to do, that I believe is worth to.

# Methods

Why do classical methods fail?

Why do classical methods fail?

The classical method relies on 1) constructing tangent vectors to the reachable sets, that are obtained via variations (=small perturbations) of the optimal control, 2) transporting them along the flow, and 3) using the optimality.

# Methods

Why do classical methods fail?

The classical method relies on 1) constructing tangent vectors to the reachable sets, that are obtained via variations (=small perturbations) of the optimal control, 2) transporting them along the flow, and 3) using the optimality.

What is missing - up to now - is the transportation of tangent vectors along the flow.

# Methods

Why do classical methods fail?

The classical method relies on 1) constructing tangent vectors to the reachable sets, that are obtained via variations (=small perturbations) of the optimal control, 2) transporting them along the flow, and 3) using the optimality.

What is missing - up to now - is the transportation of tangent vectors along the flow. We will see that *there is* and adjoint equation, but its interpretation as the transporter of tangent vectors is missing.

# Methods

Why do classical methods fail?

The classical method relies on 1) constructing tangent vectors to the reachable sets, that are obtained via variations (=small perturbations) of the optimal control, 2) transporting them along the flow, and 3) using the optimality.

What is missing - up to now - is the transportation of tangent vectors along the flow. We will see that *there is* and adjoint equation, but its interpretation as the transporter of tangent vectors is missing.

To solve the problem, there are up to now two main approaches.

## a) Discretization

Approximating an optimal pair (trajectory/control) with piecewise linear/constant maps

## a) Discretization

Approximating an optimal pair (trajectory/control) with piecewise linear/constant maps; consider (finite dimensional) discrete optimization problems and write necessary conditions for them

## a) Discretization

Approximating an optimal pair (trajectory/control) with piecewise linear/constant maps; consider (finite dimensional) discrete optimization problems and write necessary conditions for them; passing to the limit.

## a) Discretization

Approximating an optimal pair (trajectory/control) with piecewise linear/constant maps; consider (finite dimensional) discrete optimization problems and write necessary conditions for them; passing to the limit.

## b) Penalization

Consider the family of smooth problems

$$\begin{cases} \dot{x} & = -\frac{1}{\varepsilon}\nabla d_K^2(x) + f(x,u), \quad \varepsilon > 0 \\ x(0) & = x_0 \end{cases}$$

(the solution may leave the constraint, but it can be proved that it converges to the solution of (CSP) as $\varepsilon \to 0$)

## a) Discretization

Approximating an optimal pair (trajectory/control) with piecewise linear/constant maps; consider (finite dimensional) discrete optimization problems and write necessary conditions for them; passing to the limit.

## b) Penalization

Consider the family of smooth problems

$$\begin{cases} \dot{x} & = -\frac{1}{\varepsilon}\nabla d_K^2(x) + f(x,u), \quad \varepsilon > 0 \\ x(0) & = x_0 \end{cases}$$

(the solution may leave the constraint, but it can be proved that it converges to the solution of (CSP) as $\varepsilon \to 0$) ; write necessary conditions (adjoint equation + PMP)

## a) Discretization

Approximating an optimal pair (trajectory/control) with piecewise linear/constant maps; consider (finite dimensional) discrete optimization problems and write necessary conditions for them; passing to the limit.

## b) Penalization

Consider the family of smooth problems

$$\begin{cases} \dot{x} & = -\frac{1}{\varepsilon}\nabla d_K^2(x) + f(x,u), \quad \varepsilon > 0 \\ x(0) & = x_0 \end{cases}$$

(the solution may leave the constraint, but it can be proved that it converges to the solution of (CSP) as $\varepsilon \to 0$) ; write necessary conditions (adjoint equation + PMP); pass to the limit along them.

# Results

**Theorem.** Let $(\bar{x}, \bar{u})$ be an optimal pair for (CSP). Then there exist a $BV$ function $p$ and a measure $\mu$ (that is supported on $\{t : \bar{x}(t) \in \partial K\}$) such that

$$\begin{aligned}
& dp = -p^\top D_x f(\bar{x}, \bar{u}) + d\mu \\
& p(T) = \nabla\varphi(\bar{x}(T)) \\
& p(t) \cdot f(\bar{x}(t), \bar{u}(t)) = \max_{u \in U} p(t) \cdot f(\bar{x}(t), u)
\end{aligned}$$

# Results

**Theorem.** Let $(\bar{x}, \bar{u})$ be an optimal pair for (CSP). Then there exist a $BV$ function $p$ and a measure $\mu$ (that is supported on $\{t : \bar{x}(t) \in \partial K\}$) such that

$$\begin{aligned} dp &= -p^{\top} D_x f(\bar{x}, \bar{u}) + d\mu \\ p(T) &= \nabla \varphi(\bar{x}(T)) \\ p(t) \cdot f(\bar{x}(t), \bar{u}(t)) &= \max_{u \in U} p(t) \cdot f(\bar{x}(t), u) \end{aligned}$$

(via penalization; assumption: $K$ smooth).

For the problem connected with the crawler (in collaboration with Paolo Gidoni):

Let $(\bar{z}, \bar{u})$ be an optimal pair. Then there exist $\lambda \geq 0$, $\omega \in \mathbb{R}$, $\eta^+, \eta^- \in \partial|\bar{u}(t)|$, and a $BV$ function $p$ such that

$$
\begin{aligned}
&p(0) = p(T) \\
&\omega + p(t) - \lambda(\eta^+ - \eta^-) \in N_U(\bar{u}(t)) \\
&\lambda + \|p\|_\infty \neq 0.
\end{aligned}
$$

For the problem connected with the crawler (in collaboration with Paolo Gidoni):

Let $(\bar{z}, \bar{u})$ be an optimal pair. Then there exist $\lambda \geq 0$, $\omega \in \mathbb{R}$, $\eta^+, \eta^- \in \partial|\bar{u}(t)|$, and a $BV$ function $p$ such that

$$\begin{aligned} &p(0) = p(T) \\ &\omega + p(t) - \lambda(\eta^+ - \eta^-) \in N_U(\bar{u}(t)) \\ &\lambda + \|p\|_\infty \neq 0. \end{aligned}$$

(via discretization)

For the problem connected with the crawler (in collaboration with Paolo Gidoni):

Let $(\bar{z}, \bar{u})$ be an optimal pair. Then there exist $\lambda \geq 0$, $\omega \in \mathbb{R}$, $\eta^+, \eta^- \in \partial|\bar{u}(t)|$, and a $BV$ function $p$ such that

$$
\begin{aligned}
& p(0) = p(T) \\
& \omega + p(t) - \lambda(\eta^+ - \eta^-) \in N_U(\bar{u}(t)) \\
& \lambda + \|p\|_\infty \neq 0.
\end{aligned}
$$

(via discretization)

$\omega$ is concerned with the zero mean control;

For the problem connected with the crawler (in collaboration with Paolo Gidoni):

Let $(\bar{z}, \bar{u})$ be an optimal pair. Then there exist $\lambda \geq 0$, $\omega \in \mathbb{R}$, $\eta^+, \eta^- \in \partial|\bar{u}(t)|$, and a $BV$ function $p$ such that

$$\begin{aligned}
&p(0) = p(T) \\
&\omega + p(t) - \lambda(\eta^+ - \eta^-) \in N_U(\bar{u}(t)) \\
&\lambda + \|p\|_\infty \neq 0.
\end{aligned}$$

(via discretization)

$\omega$ is concerned with the zero mean control; there is still a lot to be understood: degeneration of information.

THANK YOU FOR YOUR ATTENTION.