Department of Information Engineering

Control Days Workshop

# On the relation between the eigenvalues induced by a class of circulant graphs and the Dirichlet kernel

Speaker: Ph.D. candidate Marco Fabris
Supervisor: Prof. Angelo Cenedese

May $9^{\text{th}}$, 2019

Università degli Studi di Padova

# Outline

1. Preliminaries
   - Circulant graphs and their applications
   - $\kappa$-ring graphs
2. Main results
   - Spectral characterization
   - Fiedler value
   - Spectral radius of the Laplacian
   - Stochastic spectral radius
3. Discussion
   - Conjecture on the stochastic spectral radius characterization
   - Eigenvalue distribution examples
4. Conclusions and future directions

Circulant matrix

$$\mathbf{F} = \mathrm{circ}(\boldsymbol{\varpi}) := \begin{bmatrix} \varpi_0 & \varpi_1 & \cdots & \varpi_{n-2} & \varpi_{n-1} \\ \varpi_{n-1} & \varpi_0 & \cdots & \varpi_{n-3} & \varpi_{n-2} \\ \vdots & \ddots & \cdots & \ddots & \vdots \\ \varpi_2 & \varpi_3 & \cdots & \varpi_0 & \varpi_1 \\ \varpi_1 & \varpi_2 & \cdots & \varpi_{n-1} & \varpi_0 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

# Preliminaries: Circulant graphs & their applications

Circulant matrix

$$\mathbf{F} = \mathrm{circ}(\boldsymbol{\varpi}) := \begin{bmatrix} \varpi_0 & \varpi_1 & \cdots & \varpi_{n-2} & \varpi_{n-1} \\ \varpi_{n-1} & \varpi_0 & \cdots & \varpi_{n-3} & \varpi_{n-2} \\ \vdots & \ddots & \cdots & \ddots & \vdots \\ \varpi_2 & \varpi_3 & \cdots & \varpi_0 & \varpi_1 \\ \varpi_1 & \varpi_2 & \cdots & \varpi_{n-1} & \varpi_0 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

Circulant matrix spectrum

$$\lambda^{\mathbf{F}}(j) = \sum_{k=0}^{n-1}\left[\varpi_k \exp\left(-\frac{2k\pi\mathbf{i}}{n} j\right)\right] \quad \text{for } j = 0 \ldots n-1$$

# Preliminaries: Circulant graphs & their applications

Circulant matrix

$$\mathbf{F} = \mathrm{circ}(\boldsymbol{\varpi}) := \begin{bmatrix} \varpi_0 & \varpi_1 & \dots & \varpi_{n-2} & \varpi_{n-1} \\ \varpi_{n-1} & \varpi_0 & \dots & \varpi_{n-3} & \varpi_{n-2} \\ \vdots & \ddots & \dots & \ddots & \vdots \\ \varpi_2 & \varpi_3 & \dots & \varpi_0 & \varpi_1 \\ \varpi_1 & \varpi_2 & \dots & \varpi_{n-1} & \varpi_0 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

Circulant matrix spectrum

$$\lambda^{\mathbf{F}}(j) = \sum_{k=0}^{n-1}\left[\varpi_k \exp\left(-\frac{2k\pi\mathbf{i}}{n}j\right)\right] \quad \text{for } j = 0 \dots n-1$$

Randić matrix relation + $\mathrm{d}$-regularity

$$\mathbf{F} := \mathbf{D}^{-1}\mathbf{A} = \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2} =: \mathscr{R}$$

# Preliminaries: Circulant graphs & their applications

Circulant matrix

$$\mathbf{F} = \mathrm{circ}(\boldsymbol{\varpi}) := \begin{bmatrix} \varpi_0 & \varpi_1 & \dots & \varpi_{n-2} & \varpi_{n-1} \\ \varpi_{n-1} & \varpi_0 & \dots & \varpi_{n-3} & \varpi_{n-2} \\ \vdots & \ddots & \dots & \ddots & \vdots \\ \varpi_2 & \varpi_3 & \dots & \varpi_0 & \varpi_1 \\ \varpi_1 & \varpi_2 & \dots & \varpi_{n-1} & \varpi_0 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

Circulant matrix spectrum

$$\lambda^{\mathbf{F}}(j) = \sum_{k=0}^{n-1} \left[ \varpi_k \exp\left( -\frac{2k\pi\mathbf{i}}{n} j \right) \right] \quad \text{for } j = 0 \dots n-1$$

Randić matrix relation + $\mathrm{d}$-regularity

$$\mathbf{F} := \mathbf{D}^{-1}\mathbf{A} = \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2} =: \mathscr{R}$$

Laplacian matrix relation + $\mathrm{d}$-regularity

$$\mathbf{L} := \mathbf{D} - \mathbf{A} = \mathrm{d}\boldsymbol{\mathcal{L}} = \mathrm{d}(\mathbf{I}_n - \mathscr{R})$$

# Preliminaries: Circulant graphs & their applications

Circulant matrix

$$\mathbf{F} = \mathrm{circ}(\boldsymbol{\varpi}) := \begin{bmatrix} \varpi_0 & \varpi_1 & \dots & \varpi_{n-2} & \varpi_{n-1} \\ \varpi_{n-1} & \varpi_0 & \dots & \varpi_{n-3} & \varpi_{n-2} \\ \vdots & \ddots & \dots & \ddots & \vdots \\ \varpi_2 & \varpi_3 & \dots & \varpi_0 & \varpi_1 \\ \varpi_1 & \varpi_2 & \dots & \varpi_{n-1} & \varpi_0 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

Circulant matrix spectrum

$$\lambda^{\mathbf{F}}(j) = \sum_{k=0}^{n-1} \left[\varpi_k \exp\left(-\frac{2k\pi\mathbf{i}}{n} j\right)\right] \quad \text{for } j = 0 \dots n-1$$

Randić matrix relation + $\mathrm{d}$-regularity

$$\mathbf{F} := \mathbf{D}^{-1}\mathbf{A} = \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2} =: \mathscr{R}$$

Laplacian matrix relation + $\mathrm{d}$-regularity

$$\mathbf{L} := \mathbf{D} - \mathbf{A} = \mathrm{d}\boldsymbol{\mathcal{L}} = \mathrm{d}(\mathbf{I}_n - \mathscr{R})$$

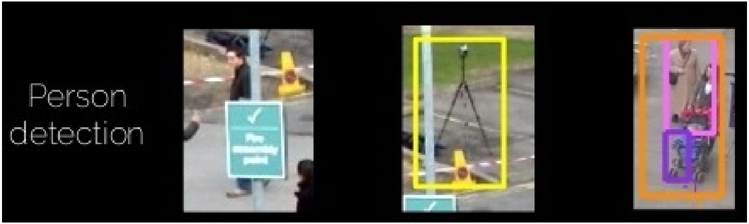
Spectral equivalence between normalize Laplacian and Randić matrices

$$\lambda^{\mathbf{F}}(j) = \lambda^{\mathscr{R}}(j) = 1 - \lambda^{\boldsymbol{\mathcal{L}}}(j) \quad \text{for } j = 0 \dots n-1$$

# Preliminaries: Circulant graphs & their applications

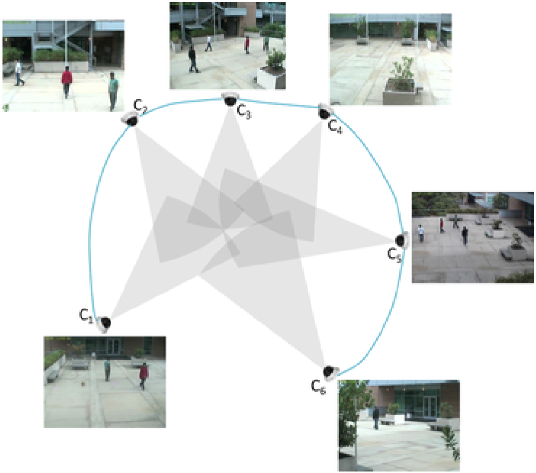
Intelligent surveillance of public spaces


Tracking-by-Detection



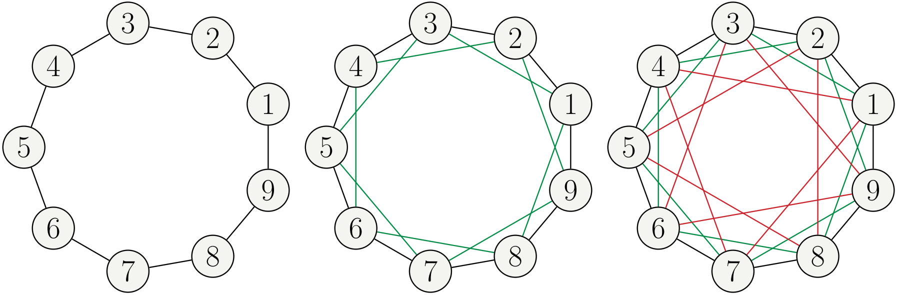
# Preliminaries: Circulant graphs & their applications

Distributed Consensus-like algorithms



# Preliminaries: $\kappa$-ring graphs

$\kappa$-ring graphs $C_n(1,\kappa)$ are a class of circulant graphs constructed by multiple circulant edge layers




$C_9(1,1)$ $C_9(1,2)$ $C_9(1,3)$

| #Vertices | #Edges | Diameter | Radius | Girth | Regularity |
|---|---|---|---|---|---|
| $\|\mathcal{V}\| = n \geq 4$ | $\|\mathcal{E}\| = n\kappa$ | $\phi = \lceil n/2^\kappa \rceil$ | $\mathrm{r} = \phi$ | $\mathrm{g} = \begin{cases} n, & \text{if } \kappa = 1 \\ 3, & \text{otherwise} \end{cases}$ | $\mathrm{d} = 2\kappa$ |

# Main results: Spectral characterization

- General aim: investigate stability, performances of graph-based protocols and the communication exchange over networks.

# Main results: Spectral characterization

- General aim: investigate stability, performances of graph-based protocols and the communication exchange over networks.

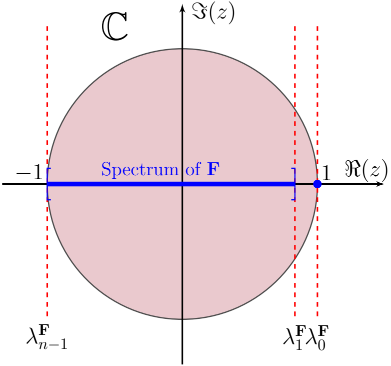
- Eigenvalues of $\mathbf{F}$ tell us more about the convergence of linear dynamic multi-agent systems.

$$\mathbf{x}(t+1) = \mathbf{F}\mathbf{x}(t) + \mathbf{u}$$

# Main results: Spectral characterization

- General aim: investigate stability, performances of graph-based protocols and the communication exchange over networks.
- Eigenvalues of $\mathbf{F}$ tell us more about the convergence of linear dynamic multi-agent systems.
- We expect the spectrum of $\mathbf{F}$ to be a real subset of the unit circle, since $\mathbf{F}$ is row-stochastic and symmetric.

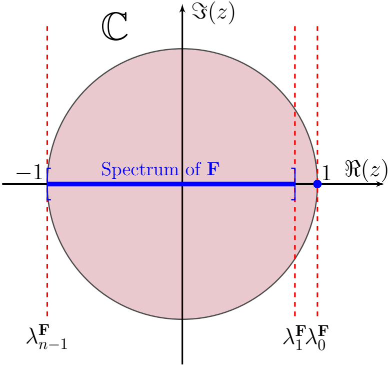
$$\mathbf{x}(t+1) = \mathbf{F}\mathbf{x}(t) + \mathbf{u}$$



# Main results: Spectral characterization

- General aim: investigate stability, performances of graph-based protocols and the communication exchange over networks.
- Eigenvalues of $\mathbf{F}$ tell us more about the convergence of linear dynamic multi-agent systems.
- We expect the spectrum of $\mathbf{F}$ to be a real subset of the unit circle, since $\mathbf{F}$ is row-stochastic and symmetric.
- The spectrum of $\mathbf{F}$ is linked to the spectrum of the Laplacian $\mathbf{L}$.

$$\mathbf{x}(t+1) = \mathbf{F}\mathbf{x}(t) + \mathbf{u}$$




$$\lambda_j^{\mathbf{F}} = 1 - \mathrm{d}^{-1}\lambda_j^{\mathbf{L}}$$

# Main results: Spectral characterization

**Definition (Dirichlet kernel)**

$\mathcal{D}_\kappa : \mathbb{R} \to \mathbb{R}$ of order $\kappa \in \mathbb{N}$ such that

$$\mathcal{D}_\kappa(x) := \begin{cases} \dfrac{\sin((\kappa + 1/2)x)}{2\sin(x/2)}, & \text{if } x \neq 2\pi l, \ \forall l \in \mathbb{Z}; \\ \kappa + 1/2, & \text{otherwise.} \end{cases}$$

# Main results: Spectral characterization

**Definition (Dirichlet kernel)**

$\mathcal{D}_\kappa : \mathbb{R} \to \mathbb{R}$ of order $\kappa \in \mathbb{N}$ such that

$$\mathcal{D}_\kappa(x) := \begin{cases} \dfrac{\sin((\kappa + 1/2)x)}{2\sin(x/2)}, & \text{if } x \neq 2\pi l,\ \forall l \in \mathbb{Z}; \\ \kappa + 1/2, & \text{otherwise.} \end{cases}$$

**Theorem (Spectral characterization of $\kappa$-ring graphs)**

$\mathbf{L}$ graph Laplacian of $\kappa$-ring graph $C_n(1,\kappa)$, $\theta := \pi/n$. Eigenvalues $\lambda^{\mathbf{L}}(j) \in \Lambda(\mathbf{L})$ can be expressed in function of the Dirichlet kernel as

$$\begin{aligned} \lambda^{\mathbf{L}}(j) &= 1 + 2\left(\kappa - \mathcal{D}_\kappa(2\theta j)\right), && \text{for } j = 0 \ldots \lfloor n/2 \rfloor; \\ \lambda^{\mathbf{L}}(n-j) &= \lambda^{\mathbf{L}}(j), && \text{for } j = 1 \ldots \lfloor n/2 \rfloor. \end{aligned}$$

$\lambda^{\mathbf{L}}(j) \in [0, 4\kappa]$, $\forall j = 0 \ldots n-1$, $\lambda_0^{\mathbf{L}} := \lambda^{\mathbf{L}}(0) = 0$ is simple and, if $\exists j^\star \in \mathbb{N}$ s.t. $\lambda^{\mathbf{L}}(j^\star) = 4\kappa$, $j^\star \in (0, n)$, then $\lambda^{\mathbf{L}}(j^\star)$ is simple.

# Main results: Spectral characterization

*Proof*. Exploiting the spectrum of the circulant matrices and setting

$$[\boldsymbol{\varpi}]_i := \begin{cases} \mathrm{d}^{-1}, & \text{if } e_{i1} \in \mathcal{E}; \\ 0, & \text{otherwise}; \end{cases}$$

# Main results: Spectral characterization

*Proof.* Exploiting the spectrum of the circulant matrices and setting

$$[\boldsymbol{\varpi}]_i := \begin{cases} \mathrm{d}^{-1}, & \text{if } e_{i1} \in \mathcal{E}; \\ 0, & \text{otherwise}; \end{cases}$$

eigenvalues of the Randić matrix $\mathscr{R}$ can be rewritten as

$$\begin{aligned} \lambda^{\mathscr{R}}(j) &= \frac{1}{\mathrm{d}} \sum_{k=1}^{\mathrm{d}/2} [\exp(-\mathbf{i}2k\theta j)] + \frac{1}{\mathrm{d}} \sum_{k=n-\mathrm{d}/2}^{n-1} [\exp(-\mathbf{i}2k\theta j)] \\ &= \frac{1}{\mathrm{d}} \sum_{k=1}^{\mathrm{d}/2} [\exp(-\mathbf{i}2k\theta j)] + \frac{1}{\mathrm{d}} \sum_{k=1}^{\mathrm{d}/2} [\exp(\mathbf{i}2k\theta j)] \\ &= \frac{2}{\mathrm{d}} \left( \frac{1}{2} \sum_{|k| \le \mathrm{d}/2} [\exp(\mathbf{i}2k\theta j)] - \frac{1}{2} \right) \\ &= \kappa^{-1} (\mathcal{D}_\kappa(2\theta j) - 1/2) \end{aligned}$$

protocol performances improve as $\kappa$ increases!

# Main results: Spectral characterization

*Proof*. Exploiting the spectrum of the circulant matrices and setting

$$[\boldsymbol{\varpi}]_i := \begin{cases} \mathrm{d}^{-1}, & \text{if } e_{i1} \in \mathcal{E}; \\ 0, & \text{otherwise}; \end{cases}$$

eigenvalues of the Randić matrix $\mathscr{R}$ can be rewritten as

$$\begin{aligned}
\lambda^{\mathscr{R}}(j) &= \frac{1}{\mathrm{d}} \sum_{k=1}^{\mathrm{d}/2} [\exp(-\mathbf{i}2k\theta j)] + \frac{1}{\mathrm{d}} \sum_{k=n-\mathrm{d}/2}^{n-1} [\exp(-\mathbf{i}2k\theta j)] \\
&= \frac{1}{\mathrm{d}} \sum_{k=1}^{\mathrm{d}/2} [\exp(-\mathbf{i}2k\theta j)] + \frac{1}{\mathrm{d}} \sum_{k=1}^{\mathrm{d}/2} [\exp(\mathbf{i}2k\theta j)] \\
&= \frac{2}{\mathrm{d}} \left( \frac{1}{2} \sum_{|k| \leq \mathrm{d}/2} [\exp(\mathbf{i}2k\theta j)] - \frac{1}{2} \right) \\
&= \kappa^{-1} \left( \mathcal{D}_{\kappa}(2\theta j) - 1/2 \right)
\end{aligned}$$

protocol performances improve as $\kappa$ increases!

Leveraging the $d$-regularity, the rest of the statement can be proven resorting to Landau H., Odlyzko A., 1981 *"Bounds for Eigenvalues of Certain Stochastic Matrices"*. □

# Main results: Fiedler value

The previous theorem offers a deep insight on the connection between the Dirichlet kernel and the eigenvalues of $\mathbf{L}$.

The analysis continues focusing on the extremal eigenvalues of the restricted spectrum $\Lambda_0(\mathbf{L}) := \Lambda(\mathbf{L}) \setminus \{\lambda_0^{\mathbf{L}}\} \subseteq (0, 4\kappa]$, denoting the eigenvalues of $\Lambda(\mathbf{L})$ with $0 = \lambda_0^{\mathbf{L}} < \lambda_1^{\mathbf{L}} \leq \ldots \leq \lambda_{n-1}^{\mathbf{L}}$.

# Main results: Fiedler value

The previous theorem offers a deep insight on the connection between the Dirichlet kernel and the eigenvalues of $\mathbf{L}$.

The analysis continues focusing on the extremal eigenvalues of the restricted spectrum $\Lambda_0(\mathbf{L}) := \Lambda(\mathbf{L}) \setminus \{\lambda_0^{\mathbf{L}}\} \subseteq (0, 4\kappa]$, denoting the eigenvalues of $\Lambda(\mathbf{L})$ with $0 = \lambda_0^{\mathbf{L}} < \lambda_1^{\mathbf{L}} \leq \ldots \leq \lambda_{n-1}^{\mathbf{L}}$.

**Corollary (Fiedler value of $\kappa$-ring graphs)**

The smallest positive eigenvalue $\lambda_1^{\mathbf{L}}$ of the graph Laplacian $\mathbf{L}$ associated to the $\kappa$-ring graph $C_n(1, \kappa)$ is given by

$$\lambda_1^{\mathbf{L}} := \min_{j=1\ldots n-1} \lambda^{\mathbf{L}}(j) = \lambda^{\mathbf{L}}(1) = \lambda^{\mathbf{L}}(n-1) \in (0, 2\kappa).$$

Eigenvalue $\lambda_1^{\mathbf{L}}$ gives us information on the right limit $\lambda_1^{\mathbf{F}}$ of the unit circle allowing to determine protocol performances.

# Main results: Spectral radius of the Laplacian

**Corollary (Spectral radius of $\kappa$-ring graphs: properties)**

For the largest eigenvalue $\lambda^{\mathbf{L}}_{n-1}$ of the graph Laplacian $\mathbf{L}$ associated to the $\kappa$-ring graph $C_n(1,\kappa)$ one has

- i $\lambda^{\mathbf{L}}_{n-1} \in [n-2\kappa, 4\kappa]$, with the equality for the upper bound holding iff $n$ is even and $\kappa = 1$;
- ii $\lambda^{\mathbf{L}}_{n-1} = \lambda^{\mathbf{L}}(j^\star) = \lambda^{\mathbf{L}}(n-j^\star)$, where $j^\star \in \mathbb{N}$ belongs to $[\underline{j}, \overline{j}] \subset \mathbb{N}$ with

$$\underline{j} = 1 + \lfloor n/(2\kappa+1) \rfloor, \quad \overline{j} = \lceil (3n/(2\kappa+1) - 1)/2 \rceil;$$

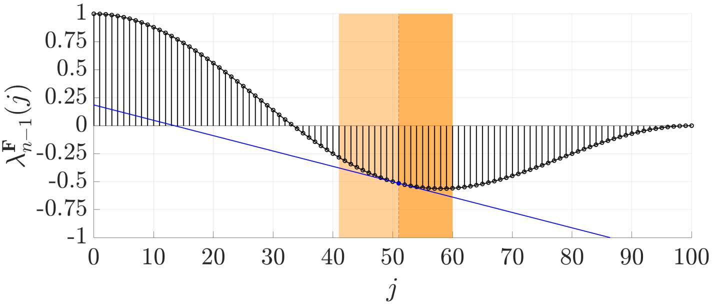
- iii $\lambda^{\mathbf{L}}_{n-1} = \lambda^{\mathbf{L}}(\lfloor n/2 \rfloor) = \lambda^{\mathbf{L}}(\lceil n/2 \rceil)$ iff $\kappa = 1$;
- iv $\lambda^{\mathbf{L}}_{n-1} = \lambda^{\mathbf{L}}(2) = \lambda^{\mathbf{L}}(n-2)$ if $\kappa \geq \kappa_n$ with $\kappa_n := 3n/10 - 1/2$.

Eigenvalue $\lambda^{\mathbf{L}}_{n-1}$ gives us information on the left limit $\lambda^{\mathbf{F}}_{n-1}$ of the unit circle allowing to determine, again, protocol performances.

# Main results: Spectral radius of the Laplacian

- Recall that $\lambda^{\mathbf{L}}_{n-1} = \mathrm{d}(1 - \lambda^{\mathscr{R}}_{n-1})$;
- $\lambda^{\mathbf{F}}_{n-1} = \lambda^{\mathscr{R}}_{n-1}$ can be computed through a binary search;
- $\mathcal{D}'_{\kappa}$ is crucial for the index selection;
- Complexity: $O(\log(n/\kappa))$ as $n/\kappa \to +\infty$.

**set** $(\underline{j}, \overline{j})$;
**if** $\mathcal{D}'_{\kappa}(2\theta\overline{j}) \leq 0$ **then** $(j^{\star}, \lambda^{\mathscr{R}}_{n-1}) \leftarrow (\overline{j}, \lambda^{\mathscr{R}}(\overline{j}))$;
**else if** $\mathcal{D}'_{\kappa}(2\theta\underline{j}) \geq 0$ **then** $(j^{\star}, \lambda^{\mathscr{R}}_{n-1}) \leftarrow (\underline{j}, \lambda^{\mathscr{R}}(\underline{j}))$;
**else**
  $found \leftarrow$ **false**;
  **while** $\overline{j} - \underline{j} > 1$ **and not** $found$ **do**
    $j^{\star} \leftarrow \lfloor(\underline{j} + \overline{j})/2 + 1/2\rfloor$;
    **if** $\mathcal{D}'_{\kappa}(2\theta j^{\star}) < 0$ **then** $\underline{j} \leftarrow j^{\star}$;
    **else if** $\mathcal{D}'_{\kappa}(2\theta j^{\star}) > 0$ **then** $\overline{j} \leftarrow j^{\star}$;
    **else** $found \leftarrow$ **true**; $\lambda^{\mathscr{R}}_{n-1} \leftarrow \lambda^{\mathscr{R}}(j^{\star})$;
    **end if**
  **end while**
  **if not** $found$ **then** $(j^{\star}, \lambda^{\mathscr{R}}_{n-1}) \leftarrow \min_{j \in \{\underline{j}, \overline{j}\}} \{\lambda^{\mathscr{R}}(j)\}$;
  **end if**
**end if**



# Main results: Stochastic spectral radius

## Definition (Stochastic spectral radius)

The stochastic spectral radius (SSR) is defined as

$$\lambda^{\mathbf{F}} := \max_{\lambda \in \Lambda(\mathbf{F}) \setminus \{\lambda_0^{\mathbf{F}}\}} |\lambda| = \max \left\{ |\lambda_1^{\mathscr{R}}|, |\lambda_{n-1}^{\mathscr{R}}| \right\} =: |\lambda_m^{\mathbf{F}}|$$

# Main results: Stochastic spectral radius

## Definition (Stochastic spectral radius)

The stochastic spectral radius (SSR) is defined as

$$\lambda^{\mathbf{F}} := \max_{\lambda \in \Lambda(\mathbf{F}) \setminus \{\lambda_0^{\mathbf{F}}\}} |\lambda| = \max\left\{|\lambda_1^{\mathscr{R}}|, |\lambda_{n-1}^{\mathscr{R}}|\right\} =: |\lambda_m^{\mathbf{F}}|$$

## Theorem (SSR of the $\kappa$-ring graphs: properties)

For the SSR $\lambda^{\mathscr{R}}$ of the Randić matrix $\mathscr{R}$ associated to the $\kappa$-ring graph $C_n(1,\kappa)$ one has

- i $\lambda^{\mathscr{R}} > \max\{(n\kappa)^{-1}\sqrt{n + 2\kappa n(n-2\kappa) - (2\kappa+1)^2} - \lambda_1^{\mathscr{R}}, \lambda_1^{\mathscr{R}}\}$
- ii $\lambda^{\mathscr{R}} \leq 1$ with the equality holding iff $n$ is even and $\kappa = 1$;
- iii $\lambda^{\mathscr{R}} = \left|\lambda^{\mathscr{R}}(j')\right| = \left|\lambda^{\mathscr{R}}(n-j')\right|$, where $j' \in \mathbb{N}$ belongs to $\{1\} \cup [\underline{j}, \overline{j}] \subset \mathbb{N}$;
- iv $\lambda^{\mathscr{R}} = -\lambda^{\mathscr{R}}(\lfloor n/2 \rfloor) = -\lambda^{\mathscr{R}}(\lceil n/2 \rceil)$ iff $\kappa = 1$;
- v $\lambda^{\mathscr{R}} = -\lambda^{\mathscr{R}}(2) = -\lambda^{\mathscr{R}}(n-2)$ if $\kappa \geq \max\{\kappa_n, \kappa_\theta\}$.

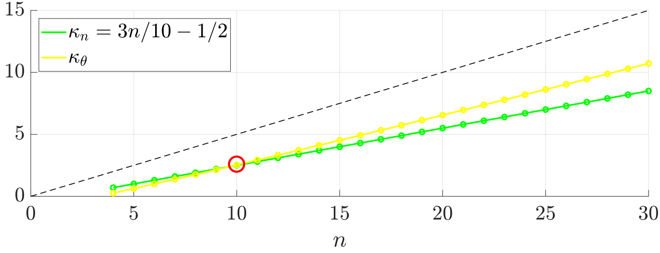
# Conjecture on the SSR characterization

## Conjecture (SSR index characterization)

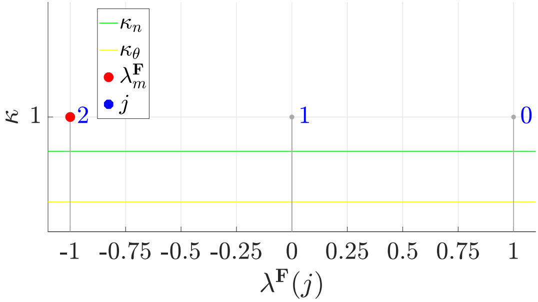
The SSR $\lambda^{\mathbf{F}}$ for a $\kappa$-ring graph is equal to $\left|\lambda^{\mathbf{F}}(j')\right|$ where $\theta := \dfrac{\pi}{n}$ and

$$j' = \begin{cases} \lfloor n/2 \rfloor, & \text{if } \kappa = 1; \\ 3, & \text{if } n = 9 \text{ and } \kappa = 2; \\ 1, & \text{if } \kappa \in [2, \kappa_\theta]; \\ 2, & \text{if } \kappa > \kappa_\theta. \end{cases}$$

↑ Sparsity of $\mathbf{F}$



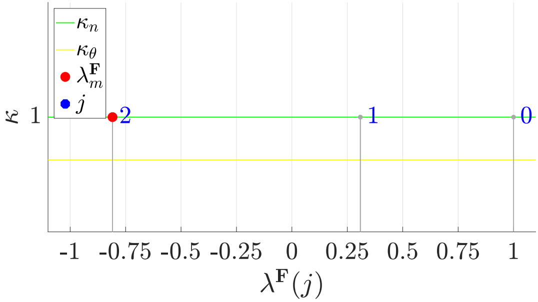
# Discussion: Eigenvalue distribution for $n = 4, 5$




$C_4(1, \kappa)$

$\kappa_n = 0.7$

$\kappa_\theta \simeq 0.2596$

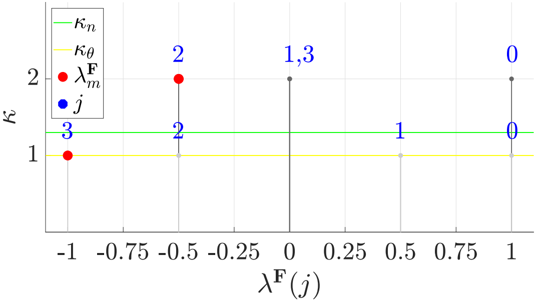



$C_5(1, \kappa)$

$\kappa_n = 1$

$\kappa_\theta \simeq 0.6274$

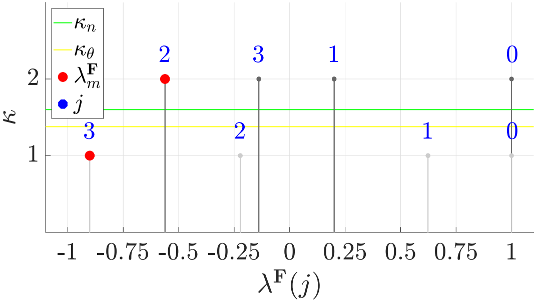
# Discussion: Eigenvalue distribution for $n = 6, 7$




$C_6(1, \kappa)$

$\kappa_n = 1.3$

$\kappa_\theta = 1$

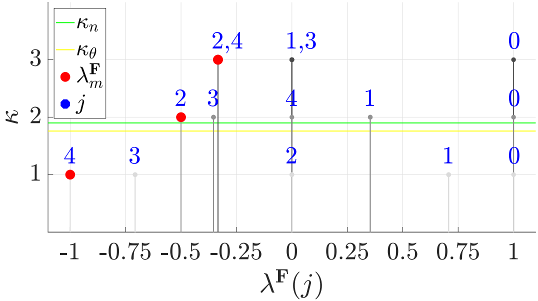



$C_7(1, \kappa)$

$\kappa_n = 1.6$

$\kappa_\theta \simeq 1.3773$

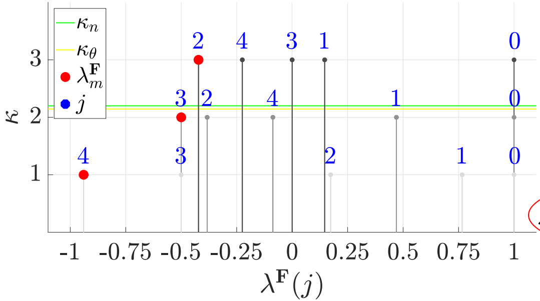
# Discussion: Eigenvalue distribution for $n = 8, 9$




$C_8(1, \kappa)$

$\kappa_n = 1.9$

$\kappa_\theta \simeq 1.7589$



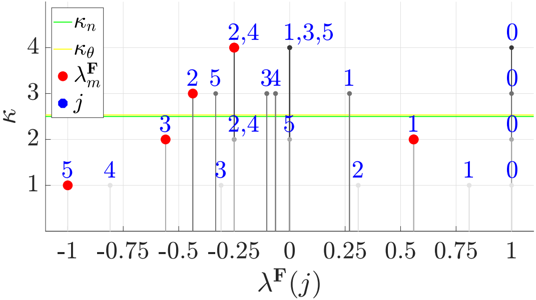

$C_9(1, \kappa)$

$\kappa_n = 2.2$

$\kappa_\theta \simeq 2.1442$

$\lambda^{\mathbf{F}} = -\lambda^{\mathbf{F}}(3)$ if $(n, \kappa) = (9, 2)$

# Discussion: Eigenvalue distribution for $n = 10, 11$

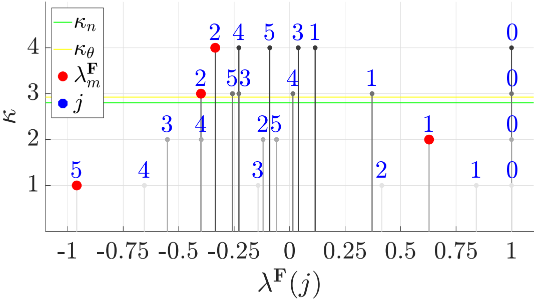



$C_{10}(1, \kappa)$

$\kappa_n = 2.5$

$\kappa_\theta \simeq 2.5330$

$\lambda^{\mathbf{F}} = -\lambda^{\mathbf{F}}(3) = \lambda^{\mathbf{F}}(1) = \sqrt{5}/4$
if $(n, \kappa) = (10, 2)$

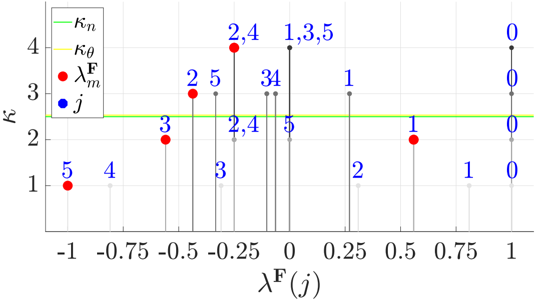



$C_{11}(1, \kappa)$

$\kappa_n = 2.8$

$\kappa_\theta \simeq 2.9249$

# Discussion: Eigenvalue distribution for $n = 10, 11$



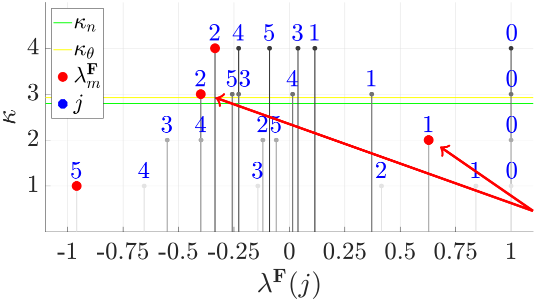

$C_{10}(1, \kappa)$

$\kappa_n = 2.5$

$\kappa_\theta \simeq 2.5330$

$\kappa_\theta > \kappa_n, \ \forall n \geq 10$

**Conjecture**




$C_{11}(1, \kappa)$

$\kappa_n = 2.8$

$\kappa_\theta \simeq 2.9249$

$\forall n \geq 11$:

$\lambda^{\mathbf{F}} = \lambda^{\mathbf{F}}(1)$ if $\kappa \in (1, \kappa_\theta]$

$\lambda^{\mathbf{F}} = -\lambda^{\mathbf{F}}(2)$ if $\kappa > \kappa_\theta$

# Conclusions and future directions

- A class of circulant topologies defining $\kappa$-ring graphs has been examined providing results on consensus-like algorithm performances
- Both the spectral and structural properties of $\kappa$-ring graphs are inherently linked to the Dirichlet kernel
- The algebraic connectivity has been fully characterized
- The stochastic spectral radius and the Laplacian spectral radius have been partially characterized
- Additional investigations on this topology are envisaged
- Further studies of new spectral properties related to the Dirichlet kernel might be considered

Thanks for the attention

# References


- Fabris M., Michieletto G., Cenedese A., 2019 *"On the Distributed Estimation from Relative Measurement: a Graph-based Convergence Analysis"*
- Andrade E., Freitas M.A.A., Robbiano M., Rodríguez J. 2018 *"New Lower Bounds for the Randić Spread"*
- Wiggins A., 2007 *"The Minimum of the Dirichlet kernel"*
- Chung F.R., Graham F.C., 1997 *"Spectral Graph Theory"*
- Brunckner A.M., Brunckner J.B., Thomson B.S., 1997 *"Real Analysis"*
- Landau H., Odlyzko A., 1981 *"Bounds for Eigenvalues of Certain Stochastic Matrices"*
- Fiedler M., 1973 *"Algebraic Connectivity of Graphs"*
- Abramowitz M., Stegun I.A., 1972 *"Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables"*

# Computation of threshold $\kappa_\theta$ relative to the SSR

## Lemma

Let $\mathscr{R}$ be the Randić matrix of a $\kappa$-ring graph $C_n(1,\kappa)$ and $\theta := \pi/n \in (0, \pi/4]$. There exists a real number $\kappa_\theta \in (0, n/2)$ such that if $\kappa \geq \kappa_\theta$ then $\lambda^{\mathscr{R}}(1) + \lambda^{\mathscr{R}}(2) \leq 0$, with the equality holding iff $\kappa = \kappa_\theta$. Moreover, letting $\mathrm{c}_{2\theta} := \cos(2\theta)$, the value of $\kappa_\theta$ is yielded by

$$\kappa_\theta = \theta^{-1} \arcsin\left(\sqrt{x_\theta}\right),$$

where $x_\theta$ is the unique solution belonging to $(0,1)$ of the polynomial equation

$$\mathrm{p}_\theta(x) := x^3 + a_{\theta,2}x^2 + a_{\theta,1}x + a_{\theta,0} = 0,$$

with $a_{\theta,2} = -(\mathrm{c}_{2\theta} + 5)/2$, $a_{\theta,1} = (4\mathrm{c}_{2\theta}^2 + 7\mathrm{c}_{2\theta} + 13)/8$, $a_{\theta,0} = -(3\mathrm{c}_{2\theta} + 1)^2/16$.