On the relation between the eigenvalues induced by a class of circulant graphs and the Dirichlet kernel

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Circulant matrix

\[ F = \text{circ}(\varpi) := \begin{bmatrix}
\varpi_0 & \varpi_1 & \ldots & \varpi_{n-2} & \varpi_{n-1} \\
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\vdots & \ddots & \ldots & \ddots & \vdots \\
\varpi_2 & \varpi_3 & \ldots & \varpi_0 & \varpi_1 \\
\varpi_1 & \varpi_2 & \ldots & \varpi_{n-1} & \varpi_0 
\end{bmatrix} \in \mathbb{R}^{n \times n} \]
Preliminaries: Circulant graphs & their applications

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Circulant matrix spectrum

\[ \lambda^F(j) = \sum_{k=0}^{n-1} \left[ \varpi_k \exp \left( -\frac{2k\pi i}{n} j \right) \right] \text{ for } j = 0 \ldots n - 1 \]
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Laplacian matrix relation + d-regularity

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L := D - A = d\mathcal{L} = d(I_n - \mathcal{R})
\]
Preliminaries: Circulant graphs & their applications

Circulant matrix

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Circulant matrix spectrum

\[ \lambda^\mathbf{F}(j) = \sum_{k=0}^{n-1} \left[ \varpi_k \exp \left( -\frac{2k\pi i}{n} j \right) \right] \quad \text{for } j = 0 \ldots n - 1 \]

Randić matrix relation + d-regularity

\[ \mathbf{F} := \mathbf{D}^{-1} \mathbf{A} = \mathbf{D}^{-1/2} \mathbf{A} \mathbf{D}^{-1/2} =: \mathbf{R} \]

Laplacian matrix relation + d-regularity

\[ \mathbf{L} := \mathbf{D} - \mathbf{A} = d\mathbf{L} = d(\mathbf{I}_n - \mathbf{R}) \]

Spectral equivalence between normalize Laplacian and Randić matrices

\[ \lambda^\mathbf{F}(j) = \lambda^\mathbf{R}(j) = 1 - \lambda^\mathbf{L}(j) \quad \text{for } j = 0 \ldots n - 1 \]
Preliminaries: Circulant graphs & their applications

Intelligent surveillance of public spaces

Tracking-by-Detection

Person detection
Preliminaries: Circulant graphs & their applications

Distributed Consensus-like algorithms
Preliminaries: $\kappa$-ring graphs

$\kappa$-ring graphs $C_n(1, \kappa)$ are a class of circulant graphs constructed by multiple circulant edge layers.

![Graphs C9(1, 1), C9(1, 2), C9(1, 3)]

<table>
<thead>
<tr>
<th>#Vertices</th>
<th>#Edges</th>
<th>Diameter</th>
<th>Radius</th>
<th>Girth</th>
<th>Regularity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>V</td>
<td>= n \geq 4$</td>
<td>$</td>
<td>E</td>
<td>= n\kappa$</td>
</tr>
<tr>
<td>$n$, if $\kappa = 1$</td>
<td>$3$, otherwise</td>
<td></td>
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Main results: Spectral characterization

- General aim: investigate stability, performances of graph-based protocols and the communication exchange over networks.
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\[ x(t + 1) = Fx(t) + u \]
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- We expect the spectrum of $F$ to be a real subset of the unit circle, since $F$ is row-stochastic and symmetric.

- The spectrum of $F$ is linked to the spectrum of the Laplacian $L$.

\[ x(t + 1) = Fx(t) + u \]

\[ \lambda^F_{n-1} \]

\[ \lambda^F_1 \lambda^F_0 \]

\[ \lambda^F_j = 1 - d^{-1} \lambda^L_j \]
Main results: Spectral characterization

Definition (Dirichlet kernel)

\[ D_\kappa : \mathbb{R} \to \mathbb{R} \] of order \( \kappa \in \mathbb{N} \) such that

\[
D_\kappa(x) := \begin{cases} 
\sin((\kappa + 1/2)x) / 2 \sin(x/2), & \text{if } x \neq 2\pi l, \forall l \in \mathbb{Z}; \\
\kappa + 1/2, & \text{otherwise}.
\end{cases}
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\end{cases}
\]

**Theorem (Spectral characterization of \( \kappa \)-ring graphs)**

The graph Laplacian of \( \kappa \)-ring graph \( C_n(1, \kappa), \theta := \pi/n \). Eigenvalues \( \lambda^L(j) \in \Lambda(L) \) can be expressed in function of the Dirichlet kernel as

\[
\lambda^L(j) = 1 + 2 (\kappa - D_\kappa(2\theta j)), \quad \text{for } j = 0 \ldots \lfloor n/2 \rfloor;
\]

\[
\lambda^L(n - j) = \lambda^L(j), \quad \text{for } j = 1 \ldots \lfloor n/2 \rfloor.
\]

\( \lambda^L(j) \in [0, 4\kappa], \forall j = 0 \ldots n - 1, \lambda^L_0 := \lambda^L(0) = 0 \text{ is simple and, if } \exists j^* \in \mathbb{N} \text{ s.t. } \lambda^L(j^*) = 4\kappa, j^* \in (0, n), \text{ then } \lambda^L(j^*) \text{ is simple.} \)
Main results: Spectral characterization

Proof. Exploiting the spectrum of the circulant matrices and setting

$$[\mathcal{A}]_i := \begin{cases} d^{-1}, & \text{if } e_{i_1} \in \mathcal{E}; \\ 0, & \text{otherwise}; \end{cases}$$
Main results: Spectral characterization

Proof. Exploiting the spectrum of the circulant matrices and setting

$$[\varpi]_i := \begin{cases} d^{-1}, & \text{if } e_{i1} \in \mathcal{E}; \\ 0, & \text{otherwise}; \end{cases}$$

eigenvalues of the Randić matrix $\mathcal{R}$ can be rewritten as

$$\lambda_{\mathcal{R}}(j) = \frac{1}{d} \sum_{k=1}^{d/2} [\exp(-i2k\theta j)] + \frac{1}{d} \sum_{k=n-d/2}^{n-1} [\exp(-i2k\theta j)]$$

$$= \frac{1}{d} \sum_{k=1}^{d/2} [\exp(-i2k\theta j)] + \frac{1}{d} \sum_{k=1}^{d/2} [\exp(i2k\theta j)]$$

$$= \frac{2}{d} \left( \frac{1}{2} \sum_{|k| \leq d/2} [\exp(i2k\theta j)] - \frac{1}{2} \right)$$

$$= \kappa^{-1} (D_\kappa(2\theta j) - 1/2)$$

protocol performances improve as $\kappa$ increases!
Main results: Spectral characterization

Proof. Exploiting the spectrum of the circulant matrices and setting

\[
[\omega]_i := \begin{cases} d^{-1}, & \text{if } e_{i1} \in \mathcal{E}; \\ 0, & \text{otherwise}; \end{cases}
\]

eigenvalues of the Randić matrix \( R \) can be rewritten as

\[
\lambda_R(j) = \frac{1}{d} \sum_{k=1}^{d/2} \left[ \exp(-i2k\theta_j) \right] + \frac{1}{d} \sum_{k=n-d/2}^{n-1} \left[ \exp(-i2k\theta_j) \right]
\]

\[
= \frac{1}{d} \sum_{k=1}^{d/2} \left[ \exp(-i2k\theta_j) \right] + \frac{1}{d} \sum_{k=1}^{d/2} \left[ \exp(i2k\theta_j) \right]
\]

\[
= \frac{2}{d} \left( \frac{1}{2} \sum_{|k| \leq d/2} \left[ \exp(i2k\theta_j) \right] - \frac{1}{2} \right)
\]

\[
= \kappa^{-1} \left( D_\kappa(2\theta_j) - 1/2 \right)
\]

protocol performances improve as \( \kappa \) increases!

Leveraging the \( d \)-regularity, the rest of the statement can be proven resorting to Landau H., Odlyzko A., 1981 “Bounds for Eigenvalues of Certain Stochastic Matrices”. □
Main results: Fiedler value

The previous theorem offers a deep insight on the connection between the Dirichlet kernel and the eigenvalues of $L$.

The analysis continues focusing on the extremal eigenvalues of the restricted spectrum $\Lambda_0(L) := \Lambda(L) \setminus \{\lambda_0^L\} \subseteq (0, 4\kappa]$, denoting the eigenvalues of $\Lambda(L)$ with $0 = \lambda_0^L < \lambda_1^L \leq \ldots \leq \lambda_{n-1}^L$. 

Corollary (Fiedler value of $\kappa$-ring graphs)

The smallest positive eigenvalue $\lambda_{L1}$ of the graph Laplacian $L$ associated to the $\kappa$-ring graph $C_n(1, \kappa)$ is given by

$$\lambda_{L1} := \min_{j=1}^{n-1} \lambda_j^L \in (0, 2\kappa).$$

Eigenvalue $\lambda_{L1}$ gives us information on the right limit $\lambda_F1$ of the unit circle allowing to determine protocol performances.
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Corollary (Fiedler value of $\kappa$-ring graphs)

The smallest positive eigenvalue $\lambda_1^L$ of the graph Laplacian $L$ associated to the $\kappa$-ring graph $C_n(1, \kappa)$ is given by

$$\lambda_1^L := \min_{j=1 \ldots n-1} \lambda^L(j) = \lambda^L(1) = \lambda^L(n-1) \in (0, 2\kappa).$$

Eigenvalue $\lambda_1^L$ gives us information on the right limit $\lambda_1^F$ of the unit circle allowing to determine protocol performances.
Main results: Spectral radius of the Laplacian

Corollary (Spectral radius of $\kappa$-ring graphs: properties)

For the largest eigenvalue $\lambda_{n-1}^L$ of the graph Laplacian $L$ associated to the $\kappa$-ring graph $C_n(1, \kappa)$ one has

1. $\lambda_{n-1}^L \in [n - 2\kappa, 4\kappa]$, with the equality for the upper bound holding iff $n$ is even and $\kappa = 1$;
2. $\lambda_{n-1}^L = \lambda^L(j^*) = \lambda^L(n - j^*)$, where $j^* \in \mathbb{N}$ belongs to $[j, \bar{j}] \subset \mathbb{N}$ with $j = 1 + \lfloor n/(2\kappa + 1) \rfloor$, $\bar{j} = \lceil (3n/(2\kappa + 1) - 1)/2 \rceil$;
3. $\lambda_{n-1}^L = \lambda^L(\lfloor n/2 \rfloor) = \lambda^L(\lceil n/2 \rceil)$ iff $\kappa = 1$;
4. $\lambda_{n-1}^L = \lambda^L(2) = \lambda^L(n - 2)$ if $\kappa \geq \kappa_n$ with $\kappa_n := 3n/10 - 1/2$.

Eigenvalue $\lambda_{n-1}^L$ gives us information on the left limit $\lambda_{n-1}^F$ of the unit circle allowing to determine, again, protocol performances.
Main results: Spectral radius of the Laplacian

- Recall that $\lambda^L_{n-1} = d(1 - \lambda^R_{n-1});$
- $\lambda^F_{n-1} = \lambda^R_{n-1}$ can be computed through a binary search;
- $\mathcal{D}_\kappa^\prime$ is crucial for the index selection;
- Complexity: $O(\log(n/\kappa))$ as $n/\kappa \to +\infty.$

```
1: set $(j, \bar{j})$
2: if $\mathcal{D}_\kappa(2\theta_j) \leq 0$ then $(j^*, \lambda^R_{n-1}) \leftarrow (\bar{j}, \lambda^R(\bar{j}));$
3: else if $\mathcal{D}_\kappa(2\theta_j) \geq 0$ then $(j^*, \lambda^R_{n-1}) \leftarrow (j, \lambda^R(j));$
4: else
5: found $\leftarrow$ false;
6: while $\bar{j} - j > 1$ and not found do
7: $j^* \leftarrow \lfloor (\bar{j} + j)/2 + 1/2 \rfloor$
8: if $\mathcal{D}_\kappa(2\theta_j) < 0$ then $j \leftarrow j^*$
9: else if $\mathcal{D}_\kappa(2\theta_j) > 0$ then $\bar{j} \leftarrow j^*$
10: else found $\leftarrow$ true; $\lambda^R_{n-1} \leftarrow \lambda^R(j^*)$
11: end if
12: end while
13: if not found then $(j^*, \lambda^R_{n-1}) \leftarrow \min_{j \in \{\bar{j}, j\}} \{\lambda^R(j)\};$
14: end if
15: end if
```
Main results: Stochastic spectral radius

Definition (Stochastic spectral radius)

The stochastic spectral radius (SSR) is defined as

\[ \lambda^F := \max_{\lambda \in \Lambda(F) \setminus \{\lambda^F_0\}} |\lambda| = \max \left\{ |\lambda_1^R|, |\lambda_{n-1}^R| \right\} =: |\lambda^F_m| \]
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Theorem (SSR of the $\kappa$-ring graphs: properties)

For the SSR $\lambda^R$ of the Randić matrix $R$ associated to the $\kappa$-ring graph $C_n(1, \kappa)$ one has

1. $\lambda^R > \max \left\{ (n\kappa)^{-1} \sqrt{n + 2\kappa n(n - 2\kappa)} - (2\kappa + 1)^2 - \lambda_1^R, \lambda_1^R \right\}$
2. $\lambda^R \leq 1$ with the equality holding iff $n$ is even and $\kappa = 1$;
3. $\lambda^R = |\lambda^R(j')| = |\lambda^R(n - j')|$, where $j' \in \mathbb{N}$ belongs to $\{1\} \cup [j, j] \subset \mathbb{N}$;
4. $\lambda^R = -\lambda^R(\lfloor n/2 \rfloor) = -\lambda^R(\lceil n/2 \rceil)$ iff $\kappa = 1$;
5. $\lambda^R = -\lambda^R(2) = -\lambda^R(n - 2)$ if $\kappa \geq \max \{\kappa_n, \kappa_\theta\}$. 

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Conjecture on the SSR characterization

Conjecture (SSR index characterization)

The SSR $\lambda^F$ for a $\kappa$-ring graph is equal to $\left|\lambda^F(j')\right|$ where $\theta := \frac{\pi}{n}$ and

$$j' = \begin{cases} 
\lfloor n/2 \rfloor, & \text{if } \kappa = 1; \\
3, & \text{if } n = 9 \text{ and } \kappa = 2; \\
1, & \text{if } \kappa \in [2, \kappa_\theta]; \\
2, & \text{if } \kappa > \kappa_\theta.
\end{cases}$$

Sparsity of $F$
Discussion: Eigenvalue distribution for $n = 4, 5$

$c_4(1, \kappa)$
\[
k_n = 0.7
\]
\[
k_\theta \simeq 0.2596
\]

$c_5(1, \kappa)$
\[
k_n = 1
\]
\[
k_\theta \simeq 0.6274
\]
Discussion: Eigenvalue distribution for $n = 6, 7$

For $C_6(1, \kappa)$:
- $\kappa_n = 1.3$
- $\kappa_\theta = 1$

For $C_7(1, \kappa)$:
- $\kappa_n \approx 1.6$
- $\kappa_\theta \approx 1.3773$
Discussion: Eigenvalue distribution for \( n = 8, 9 \)

\[ C_8(1, \kappa) \]
\[ \kappa_n = 1.9 \]
\[ \kappa_\theta \simeq 1.7589 \]

\[ C_9(1, \kappa) \]
\[ \kappa_n = 2.2 \]
\[ \kappa_\theta \simeq 2.1442 \]

\( \lambda^F = -\lambda^F(3) \) if \((n, \kappa) = (9, 2)\)
Discussion: Eigenvalue distribution for $n = 10, 11$

For $n = 10$, 11

$\lambda^F = -\lambda^F(3) = \lambda^F(1) = \sqrt{5}/4$

if $(n, \kappa) = (10, 2)$

$C_{10}(1, \kappa)$

$\kappa_n = 2.5$

$\kappa_\theta \simeq 2.5330$

$C_{11}(1, \kappa)$

$\kappa_n = 2.8$

$\kappa_\theta \simeq 2.9249$
Discussion: Eigenvalue distribution for $n = 10, 11$

For $n = 10$, we have:

- $\kappa_n = 2.5$
- $\kappa_\theta \simeq 2.5330$
- Conjecture: $\kappa_\theta > \kappa_n, \forall n \geq 10$

For $n = 11$, we have:

- $\kappa_n = 2.8$
- $\kappa_\theta \simeq 2.9249$
- Conjecture: $\forall n \geq 11$:
  - $\lambda_F = \lambda_F(1)$ if $\kappa \in (1, \kappa_\theta]$,
  - $\lambda_F = -\lambda_F(2)$ if $\kappa > \kappa_\theta$
Conclusions and future directions

- A class of circulant topologies defining $\kappa$-ring graphs has been examined providing results on consensus-like algorithm performances.

- Both the spectral and structural properties of $\kappa$-ring graphs are inherently linked to the Dirichlet kernel.

- The algebraic connectivity has been fully characterized.

- The stochastic spectral radius and the Laplacian spectral radius have been partially characterized.

- Additional investigations on this topology are envisaged.

- Further studies of new spectral properties related to the Dirichlet kernel might be considered.
Thanks for the attention
References

- Chung F.R., Graham F.C., 1997 “Spectral Graph Theory”
- Brunckner A.M., Brunckner J.B., Thomson B.S., 1997 “Real Analysis”
- Fiedler M., 1973 “Algebraic Connectivity of Graphs”
- Abramowitz M., Stegun I.A., 1972 “Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables”
Computation of threshold $\kappa_\theta$ relative to the SSR

**Lemma**

Let $R$ be the Randić matrix of a $\kappa$-ring graph $C_n(1, \kappa)$ and $\theta := \pi/n \in (0, \pi/4]$. There exists a real number $\kappa_\theta \in (0, n/2)$ such that if $\kappa \geq \kappa_\theta$ then $\lambda^R(1) + \lambda^R(2) \leq 0$, with the equality holding iff $\kappa = \kappa_\theta$. Moreover, letting $c_{2\theta} := \cos(2\theta)$, the value of $\kappa_\theta$ is yielded by

$$\kappa_\theta = \theta^{-1} \arcsin \left( \sqrt{x_\theta} \right),$$

where $x_\theta$ is the unique solution belonging to $(0, 1)$ of the polynomial equation

$$p_\theta(x) := x^3 + a_{\theta,2}x^2 + a_{\theta,1}x + a_{\theta,0} = 0,$$

with $a_{\theta,2} = -(c_{2\theta} + 5)/2$, $a_{\theta,1} = (4c_{2\theta}^2 + 7c_{2\theta} + 13)/8$, $a_{\theta,0} = -(3c_{2\theta} + 1)^2/16$. 