Motion Planning and Dynamics in Relative Periodic Orbits

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joint work with

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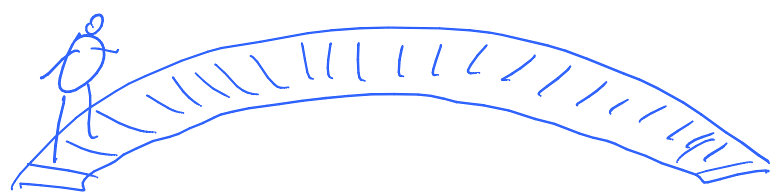
bridge



Dynamical Systems $\longleftrightarrow$ (Geometric) Control Theory

(geometric) reconstruction theory for systems with symmetry

- Linear control systems on Lie groups
- Robotic Locomotion Systems

Control Systems on Lie groups — (Robotic) Locomotion Systems

State space $G \times S$ — G Lie group "Configuration space"; $S \subseteq \mathbb{R}^m$ "Shape Space"

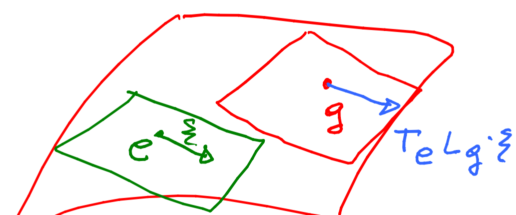
controls : (velocities of) curves in S $\quad (u \in \mathbb{R}^m)$

Brockett 1970's
Jurdjevic-Sussmann

$$\begin{cases} \dot{s} = u \\ \dot{g} = \sum_{i=1}^{m} T_e L_g \cdot \underline{A_i(s) u_i} \end{cases}$$

$A_i : S \to \mathfrak{g}$

Driftless affine control systems.
Control of mechanical systems by coordinates
(Aldo Bressan, C.R. Marle 1980's)



$T_e G \simeq \mathfrak{g}$

$L_g h = gh$

(In a matrix group
$$\dot{g} = \sum_i g\, A_i(s)\, u_i \quad)$$

Periodic control problem

given $T_1 > 0$, $g_0, g_1 \in G$

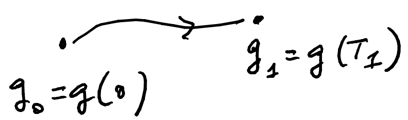
find T-periodic

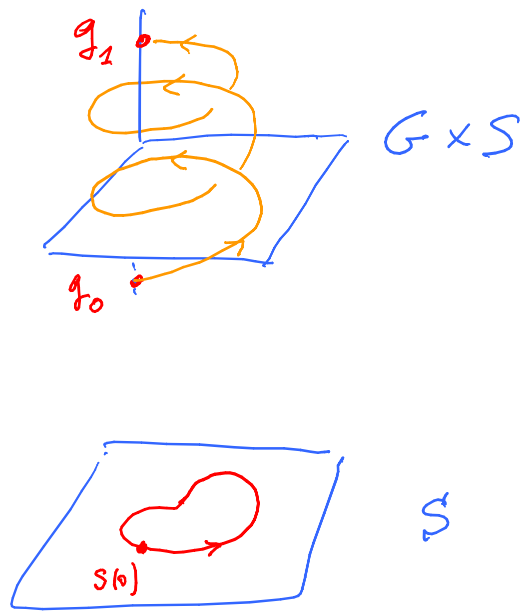
$$t \longmapsto s(t) \in S$$

s.t.

$$\dot g = \sum_i T_e L_g \cdot A_i(s(t))\, \dot s_i(t)$$

has solution s.t.





$\boxed{\text{Example}}$ "Car Robot"

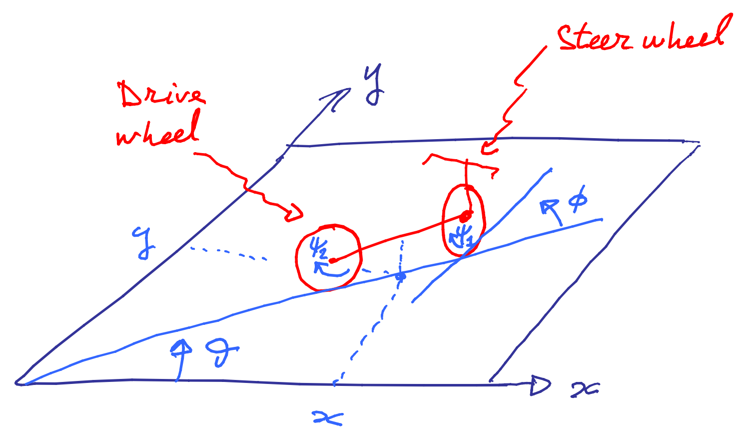
Configuration:

$$(\vartheta, (x,y), \psi_1, \psi_2, \phi) \in S^1 \times \mathbb{R}^2 \times S^1 \times S^1 \times (-\tfrac{\pi}{2}, \tfrac{\pi}{2})$$

+ no slip conditions $\Rightarrow$ rank 2 distribution

$$\dot\vartheta = \frac{\rho}{\lambda} \operatorname{tg}\phi \quad , \quad \dot x = \rho \dot\psi_2 \cos\vartheta$$

$$\dot y = \rho \dot\psi_2 \sin\vartheta, \quad \dot\psi_1 = \frac{\dot\psi_2}{\cos\phi}$$



Controls : $t \longmapsto \psi_2(t)$, $t \longmapsto \phi(t)$

Phase space (as a mechanical system)

$$\underbrace{S^1 \times \mathbb{R}^2 \times S^1}_{\substack{\vartheta \quad (x,y) \quad \psi_1 \\ \downarrow \\ G = SE(2)\times S^1 \ni g}} \times \underbrace{S^1 \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}_{\substack{\psi_2 \quad \phi \\ \downarrow \\ S \ni s}} \times \underbrace{\mathbb{R} \times \mathbb{R}}_{\substack{\dot\psi_2 \quad \dot\phi \\ \downarrow \\ \mathbb{R}^2 \ni \dot s}}$$

Control System:

$$\dot g = \text{no slip conditions} = A_{\psi_2}(s)\,\dot\psi_2 + A_\phi(s)\,\dot\phi$$

$$\dot g = (\dot\vartheta, \dot x, \dot y, \dot\psi_1), \qquad A_{\psi_2}(s) = \left(\frac{\rho}{d}\,\mathrm{tg}\,\phi,\ 1,\ 0,\ \frac{1}{\cos\phi}\right), \qquad A_\phi(s) = 0$$

Dynamical system with symmetry

$$\dot{z} = X(z), \qquad z \in M$$

Action Ψ of Lie group G on M

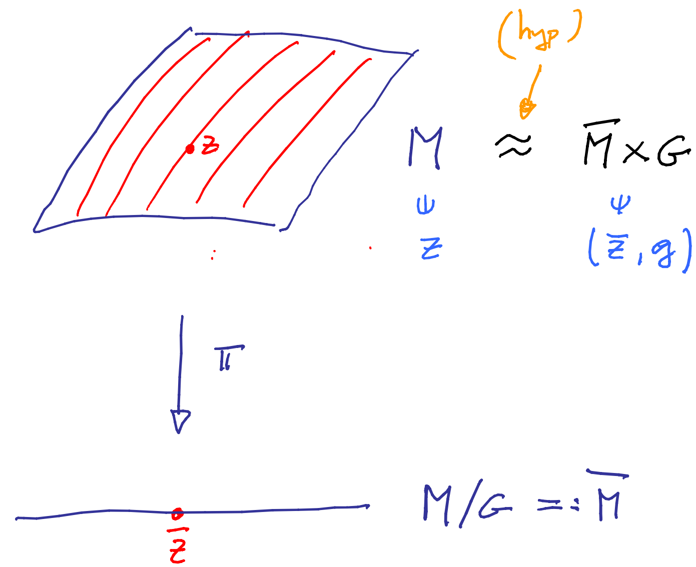
$$\Psi : G \times M \longrightarrow M$$
$$(g, z) \longmapsto \Psi_g(z)$$

$(\Psi_{gh} = \Psi_g \circ \Psi_h, \quad \Psi_e = id, \quad \Psi_g : M \to M \text{ diffeo})$

G-orbits : $\{\Psi_g(z) : g \in G\} \approx G$

Quotient space $\overline{M} := M/G$

Projection $\pi : M \longrightarrow \overline{M}$
$$z \longmapsto \overline{z}$$



Dynamical system on M

$$\dot{z} = X(z) \qquad z \in M$$

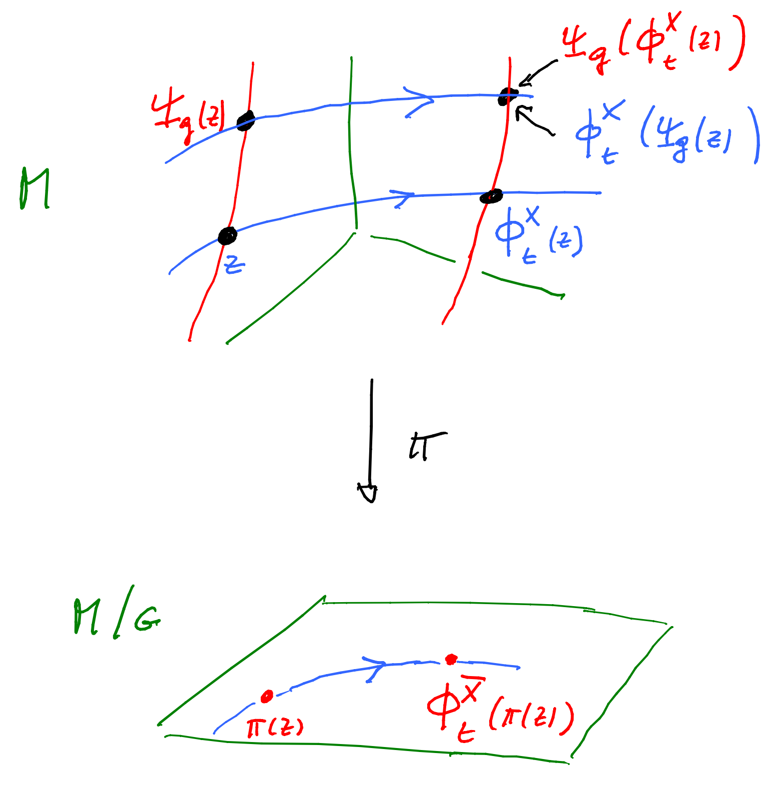
is G-invariant if

- $X(\Psi_g(z)) = T_z\Psi_g \cdot X(z) \qquad \forall g, z$
- $\phi_t^X(\Psi_g(z)) = \Psi_g(\phi_t^X(z)) \qquad \forall g, z, t$

$$\Downarrow$$

$\exists$ "reduced" system $\dot{\bar{z}} = \bar{X}(\bar{z})$ on $\bar{M}$

s.t. $\phi_t^{\bar{X}}(\pi(z)) = \pi(\phi_t^X(z)) \qquad \forall z, t$



Strategy:

- Reduce
  Solve (?) reduced system

- Reconstruct

How?

If

$$M = \overline{M} \times G \ni (\bar{z}, g)$$

then

$$X(\bar{z}, g) = \left(\overline{X}(\bar{z}),\ T_e L_g \cdot \xi(\bar{z}) \right)$$

with

$$\xi : \overline{M} \longrightarrow \mathfrak{g}$$

Hence

$$\dot{z} = X(z)$$

becomes

$$\dot{\bar{z}} = \overline{X}(\bar{z}) \quad \leftarrow \text{reduced eq.}$$

$$\dot{g} = T_e L_g \cdot \xi(\bar{z}) \quad \leftarrow \text{reconstruction eq.}$$

Compare to locomotion control system

$$t \longmapsto s(t)$$

$$\dot{g} = T_e L_g \cdot \sum_i A_i(s)\, \dot{s}_i$$

Locomotion control system

$S \times G \ni (s, g)$

$$\begin{cases} t \longmapsto s(t) \quad \text{given} \\ \dot{g} = T_e L_g \cdot \sum_i A_i(s)\, \dot{s}_i \end{cases}$$

Symmetric dynamical system

$\overline{M} \times G \ni (\overline{z}, g)$

$$\begin{cases} \dot{\overline{z}} = \overline{X}(\overline{z}) \\ \dot{g} = T_e L_g \cdot \xi(\overline{z}) \end{cases}$$

Techniques, ideas ...

Well known, but used mostly at a COMPUTATIONAL level

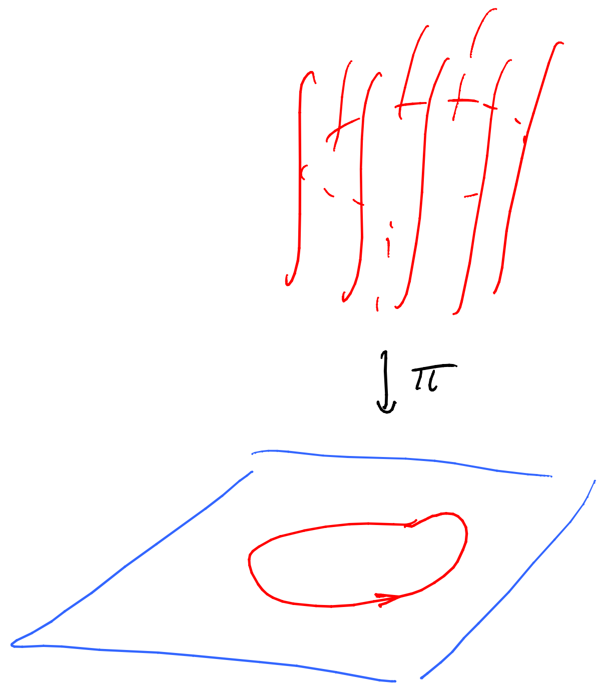
Here: QUALITATIVE informations on PERIODIC control $\longleftrightarrow$ Reduced system has a periodic orbit

Relative periodic orbit

$\dot{\bar{z}} = \bar{X}(\bar{z})$ has T-periodic solution $t \longmapsto \bar{z}(t)$

R.P.O: set of all points "over" the reduced periodic orbit:

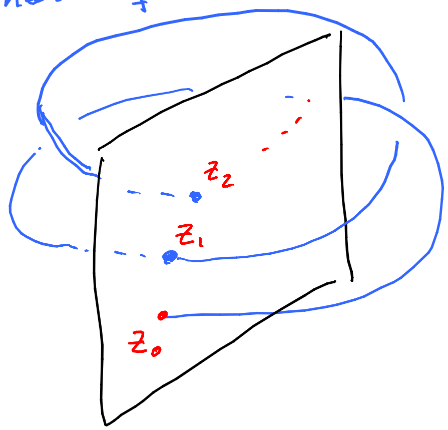
$$P := \pi^{-1}\left(\{\bar{z}(t) : 0 \le t \le T\}\right) \approx S^1 \times G$$



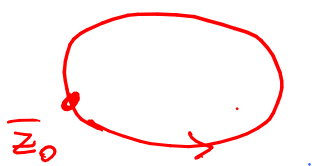
Dynamics in a R.P.O. ?

- G compact (Krupa, Field ~1980's)
- G not compact (Ashwin-Melbourne ~1997
  FF - SP - MZ)

Phase of RPO



$\Psi_G(z_0)$



$\exists!\ \gamma \in G$ s.t.

$$z_1 = \gamma z_0$$

and, by the invariance of X,

$$z_2 = \gamma z_1$$

$$z_3 = \gamma z_2$$

$$\vdots$$

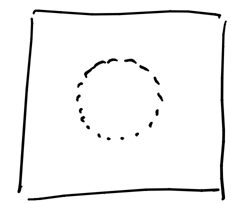
γ : Phase

Consider subgroup

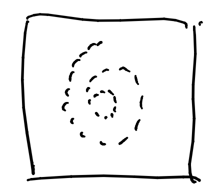
$$H(\gamma) = \overline{\{\dots, \gamma^{-1}, e, \gamma, \gamma^2, \dots\}}$$

2 cases

$H(\gamma)$ compact



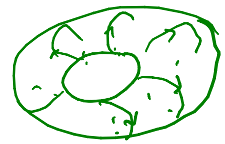
$H(\gamma)$ not compact



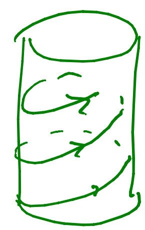
1. $H(\gamma)$ compact.

All motions in $\mathcal{P}$ conjugate to linear motions on a torus:

$$t \longmapsto \left(\frac{2\pi}{T} t, \omega_1 t, \dots, \omega_K t\right) \in \underbrace{S^1 \times \dots \times S^1}_{K+1} = \mathbb{T}^{K+1}$$



Quasi Periodic



Runaway

2. $H(\gamma)$ not compact

All motions in $\mathcal{P}$ conjugate to linear motions on a "cylinder"

$$t \mapsto \left(\frac{2\pi}{T} t, \omega_1 t, \dots, \omega_K t, t\right) \in \underbrace{S^1 \times \dots \times S^1}_{K+1} \times \mathbb{R} = \mathbb{T}^{K+1} \times \mathbb{R}$$

$\boxed{\text{In control problem non compactness is preferrable}}$

- $K, \omega_1, \dots, \omega_K$ depend on γ (the same in all of $\mathcal{P}$)
- $K \leq \text{rank}(G)$

  $SE(2) : K \leq 1$
  $SE(3) : K \leq 1$

- Compactness of $H(\gamma)$ depends on dynamics but also on G

<u>Fact</u> (Ashwin - Melbourne 1998)

$$G_{QP} := \{ \gamma \in G : H(\gamma) \text{ compact} \}$$

$$G_R := \{ \gamma \in G : H(\gamma) \text{ not compact} \}$$

In a given G, one of the two may be $\gg$ than the other

$\Rightarrow$ one behaviour is "preferred"

Ex:

$G = SE(2)$: $\boxed{G_{QP}}$ has codim. 0; G_R has codim 1

$G = SE(3)$: G_{QP} has codim. 1; $\boxed{G_R}$ has codim. 0

Often, in mechanical systems, products of $SE(2)$ and S^1

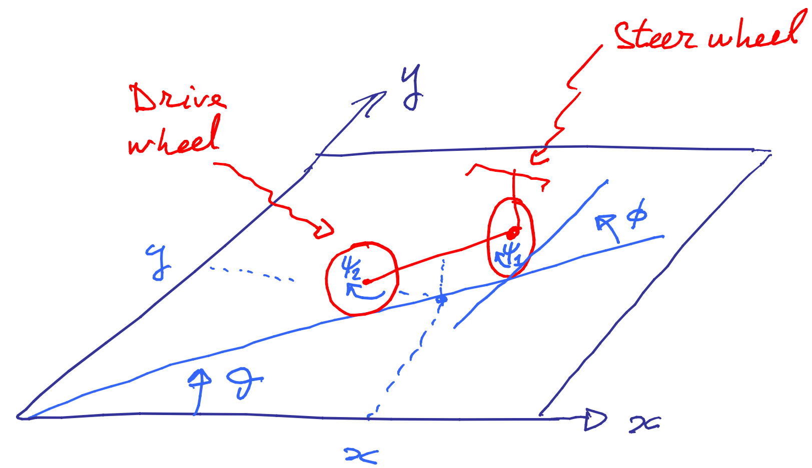
- Car robot
- Microswimmer

Ex: car robot

$$\dot{\vartheta} = \frac{\rho}{d}\,\dot{\varphi}_2(t)\,\mathrm{tg}\,\phi(t) \longrightarrow$$

$$\dot{x} = \rho\,\dot{\varphi}_2(t)\cos\vartheta$$

$$\dot{y} = \rho\,\dot{\varphi}_2(t)\sin\vartheta$$



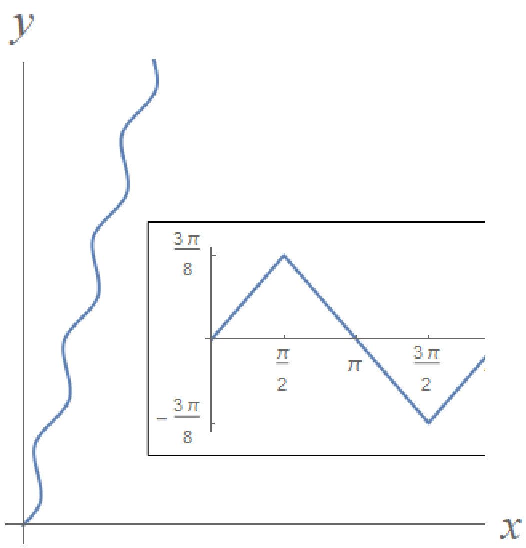
Choose $t \mapsto \varphi_2(t), \phi(t)$. Compute

$$\vartheta(t) = \frac{\rho}{d}\int_0^t \dot{\varphi}_2(\tau)\,\mathrm{tg}\,\phi(\tau)\,d\tau$$

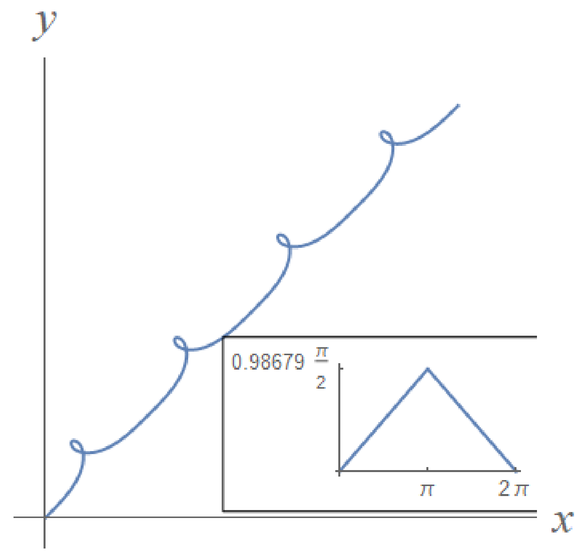
Then phase $\gamma \in G_R$ iff

$$\vartheta(T) \equiv 0 \mod 2\pi, \qquad \int_0^T \begin{pmatrix} \dot{\varphi}_2(t)\cos(\vartheta(t)) \\ \dot{\varphi}_2(t)\sin(\vartheta(t)) \end{pmatrix} dt \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

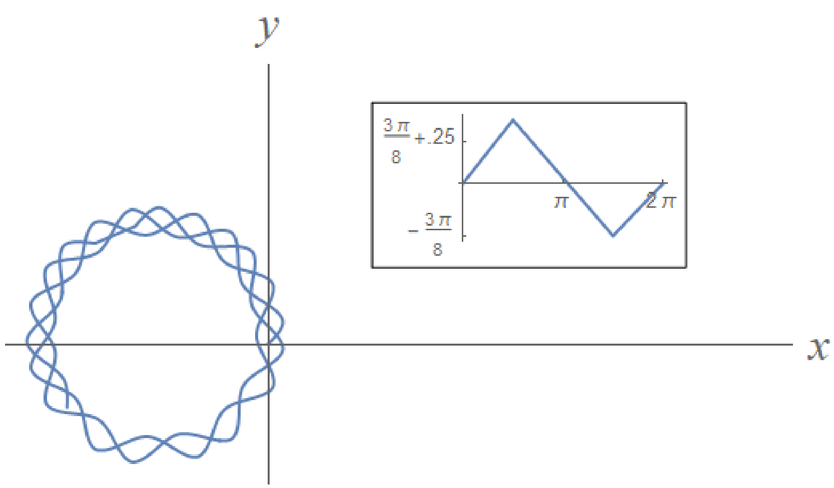
(a)


y
x
3π/8
−3π/8
π/2
π
3π/2

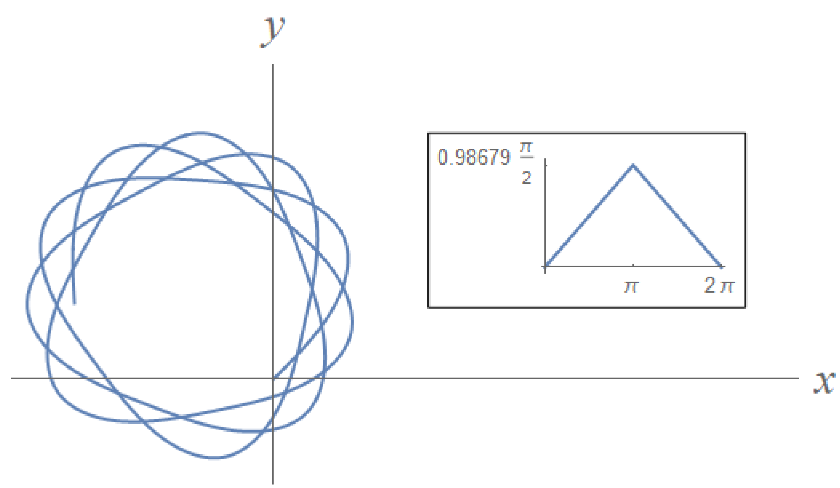
(b)


y
x
0.98679 π/2
π
2π

(c)


y
x
3π/8 +.25
−3π/8
π
2π

(b)


y
x
0.98679 π/2
π
2π