Stabilizability in Optimal Control

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The problem

We consider **a nonlinear control system**:

$$\begin{cases} \dot{x} = f(x,u), \quad u \in U \\ x(0) = z \in \mathbb{R}^n \end{cases} \tag{1}$$

and a **target** $\mathcal{C} \subset \mathbb{R}^n$.

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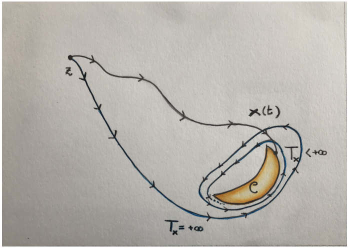
and a **target** $\mathcal{C} \subset \mathbb{R}^n$.

We say that (x,u) is **admissible** if there exists a time $T_x \leq +\infty$ such that $\mathbf{d}(x(t)) > 0$ for all $t \in [0, T_x)$ and $\lim_{t \to T_x^-} \mathbf{d}(x(t)) = 0$.



For any admissible (x, u), we define **the cost**:

$$x^0(t) := \int_0^t l(x(\tau), u(\tau))\, d\tau \quad \forall t \in [0, T_x),$$

where

$$l(x, u) \geq 0 \quad \forall (x, u).$$

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If $T_x < +\infty$, we extend continuously (x^0, x) to $[0, +\infty)$, by setting

$$(x^0, x)(t) = \lim_{t \to T_x^-} (x^0, x)(t) \qquad \forall t \geq T_x.$$

From now on, we will always consider admissible trajectories and associated costs defined on $[0, +\infty)$.

Goals

1) **To extend the concepts of sampling and Euler solutions for control systems associated to discontinuous feedbacks**

[Clarke, Ledyaev, Sontag, Subbotin, '97], [Clarke, Ledyaev, Rifford, Stern, '00]

by considering also the corresponding costs.

In particular, we introduce the notions of **Sample and Euler stabilizability to a closed target set $\mathcal{C}$ with W-regulated cost.**

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   In particular, we introduce the notions of **Sample and Euler stabilizability to a closed target set $\mathcal{C}$ with W-regulated cost.**

2) **To provide a closed-loop control strategy to achieve Sample and Euler stabilizability to a closed target set $\mathcal{C}$ with W-regulated cost.**

Results

- **We prove that the existence of a special semiconcave Control Lyapunov Function W, called here p_0-Minimum Restraint function, p_0-MRF, implies Sample and Euler stabilizability to $\mathcal{C}$ with W-regulated cost.**

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- **We prove that the existence of a special semiconcave Control Lyapunov Function W, called here p_0-Minimum Restraint function, p_0-MRF, implies Sample and Euler stabilizability to $\mathcal{C}$ with W-regulated cost.**
- **We show that, when dynamics and cost are Lipschitz continuous in the state variable, the semiconcavity of the p_0-MRF can be replaced by Lipschitz continuity.**

Motivations

Let us introduce the value function

$$V(x) := \inf_{(x,u)} \int_0^{T_x} l(x(\tau), u(\tau))\, d\tau.$$

Our results provide **an upper bound for** V, which in particular implies the **continuity of V on the target's boundary.**

- This continuity property is crucial to establish **comparison, uniqueness, and robustness properties for the associated Hamilton–Jacobi–Bellman equation** [Soravia '99, Cannarsa and Sinestrari, '04, Malisoff, '04, M.' 04, M. and Sartori, '15] [M. and Sartori, '15]

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- Our approach could be useful to **design approximated optimal closed-loop strategies**, or at least to **obtain "safe" performances, keeping the cost under the value** W. [Motta and Sartori, '15]

Assumption **(H0)**

- The sets $U \subseteq \mathbb{R}^m$, $\mathcal{C} \subseteq \mathbb{R}^n$ are closed and the boundary $\partial\mathcal{C}$ is compact.
- $f : (R^n \setminus \mathcal{C}) \times U \to \mathbb{R}^n$, $l : (R^n \setminus \mathcal{C}) \times U \to [0, +\infty)$ are continuous functions which are:

  - bounded on any compact subset $\mathcal{K} \subset \overline{\mathbb{R}^n \setminus \mathcal{C}}$, uniformly w.r.t. U,

  - uniformly continuous on $\mathcal{K} \times U$ for every compact subset $\mathcal{K} \subset \mathbb{R}^n \setminus \mathcal{C}$

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- When U is bounded, f, l continuous on $\mathbb{R}^n \times U$ satisfy (H0).
- For U unbounded, (H0) includes, e.g., the class of *control problems with saturation*:

$$f(x,u) = f_0(x) + \sum_{i=1}^m f_i(x)\sigma_i(u), \qquad l(x,u) = l_0(x) + l_1(x)|\sigma_0(u)|,$$

where l_0, l_1, $f_0, \dots, f_m \in C(\mathbb{R}^n)$ and $\sigma_0, \dots, \sigma_m$ are bounded, uniformly continuous maps on U [Bao, Lin, '00], [Chitour, '01], [Chitour, Liu, Sontag, '96], [Sussmann, Sontag, Yang, '94].

Sampling trajectory and sampling cost

- Given a **partition** $\pi = (t^j)$ of $[0, +\infty)$, we call **diam**$(\pi) := \sup_{j\geq 1}(t^j - t^{j-1})$ the **diameter** or the **sampling time** of π.

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- A **descent rate** $\beta(d, t)$ is a continuous, nonnegative function s.t.:

(1) $\beta(0, t) = 0$ and $\beta(\cdot, t)$ is strictly increasing and unbounded for each $t \geq 0$;

(2) $\beta(d, \cdot)$ is decreasing for each $d \geq 0$; (3) $\lim_{t \to +\infty} \beta(d, t) = 0 \ \forall d \geq 0$.

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- A **feedback** for (1) is any locally bounded function $K : \mathbb{R}^n \setminus \mathcal{C} \to U$.

We allow discontinuous feedbacks which may be unbounded approaching the target.

Sampling trajectory and sampling cost

Definition 1.

Given a feedback K, a partition π, and a point $z \in \mathbb{R}^n \setminus \mathcal{C}$, a π**-sampling trajectory** for (1) is a continuous function x defined by recursively solving

$$\dot{x}(t) = f(x(t), K(x(t_{k-1})) \qquad t \in [t_{k-1}, t_k], \ (x(t) \in \mathbb{R}^n \setminus \mathcal{C})$$

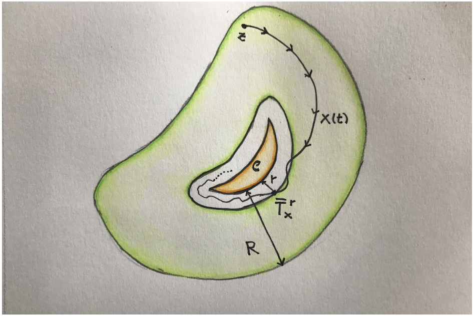
where $x(0) = z$. Let $[0, t^-)$ denote it maximal interval of definition. Accordingly, set

$$u(t) := K(x(t_{k-1})) \quad \forall t \in [t_{k-1}, t_k) \cap [0, t^-), \quad k \geq 1.$$

The **sampling cost** associated to (x, u) is given by

$$x^0(t) := \int_0^t l(x(\tau), u(\tau))\, d\tau \quad t \in [0, t^-).$$

Sample stabilizability with W-regulated cost



System (1) is **sample stabilizable** to $\mathcal{C}$ for some feedback K, if there is a descent rate β such that for each pair $0 < r < R$, $\exists \delta > 0$ such that **any** π-sampling trajectory x starting from z with $\mathbf{d}(z) \leq R$ and $\mathrm{diam}(\pi) \leq \delta$ verifies for all t,

$$\mathbf{d}(x(t)) \leq \max\{\beta(\mathbf{d}(z), t), r\}.$$

The cost is *W*-regulated if any associated sampling cost verifies:

$$x^0(\bar{T}_x^r) = \int_0^{\bar{T}_x^r} l(x(\tau), u(\tau))\, d\tau \leq \frac{W(z)}{p_0},$$

where

$$\bar{T}_x^r := \inf\{t > 0 : \ \mathbf{d}(x(\tau)) \leq r \ \ \forall \tau \geq t\}.$$

This W has to be continuous, zero on the target, proper and positive definite outside the target.

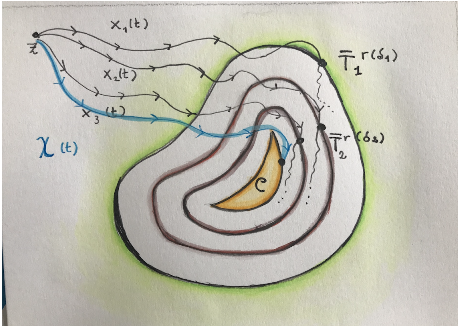
Definition 2.

Let (π_i) be a sequence of partitions of $[0, +\infty)$ such that $\delta_i := diam(\pi_i) \to 0$ as $i \to \infty$. For every i, let (x_i, u_i) be an admissible π_i-sampling trajectory-control pair of (1) and let x_i^0 be the corresponding cost. If there exists a pair $(\mathcal{X}^0, \mathcal{X}) : [0, +\infty) \to \mathbb{R}^n$, verifying

$$(x_i^0, x_i) \to (\mathcal{X}^0, \mathcal{X}) \qquad \text{locally uniformly in } [0, +\infty)$$

we call $\mathcal{X}$ an **Euler trajectory** of (1) with **Euler cost** $\mathcal{X}^0$.

Euler stabilizability with W-regulated cost



The system (1) is **Euler stabilizable** to $\mathcal{C}$ if there exists a descent rate β such that for any z, **every** Euler solution $\mathcal{X}$ verifies for all t,

$$\mathbf{d}(\mathcal{X}(t)) \leq \beta(\mathbf{d}(z), t).$$

(1) is **Euler stabilizable with W regulated cost** if any Euler cost $\mathcal{X}^0$ verifies:

$$\lim_{t \to T_{\mathcal{X}}^-} \mathcal{X}^0(t) \leq \frac{W(z)}{p_0}.$$

CONTROL LYAPUNOV-TYPE FUNCTIONS

and

SAMPLE AND EULER STABILIZABILITY with REGULATED COST

(Semiconcave) p_0-Minimum Restraint Function

Definition 3.

We say that a continuous function $W : \overline{\mathbb{R}^n \setminus \mathcal{C}} \to [0, +\infty)$ is a p_0**-Minimum Restraint Function** –in short, p_0**-MRF**– for some $p_0 \geq 0$ if W is

- *locally semiconcave*,
- positive definite, and
- proper

on $\mathbb{R}^n \setminus \mathcal{C}$, and verifies the following **decrease condition**:

$$\inf_{u \in U} \{\langle D^* W(x) \, , \, f(x,u) \rangle + p_0 \, l(x,u)\} \leq -\gamma(W(x)) \quad \forall x \in \mathbb{R}^n \setminus \mathcal{C}$$

for some continuous, strictly increasing function $\gamma : (0, +\infty) \to (0, +\infty)$.

$D^* W(x)$ denotes the set of reachable gradients of W at x:

$D^* W(x) \doteq \Big\{ p : \; p = \lim_k \nabla W(x_k), \; x_k \in DIFF(W) \setminus \{x\}, \; \lim_k x_k = x \Big\}.$

Remark 1

- **A p_0-MRF W is a Control Lyapunov Function.** Indeed, from

$$\inf_{u\in U}\{\langle D^* W(x)\,,\, f(x,u)\rangle + p_0\, l(x,u)\} \leq -\gamma(W(x)) \text{ and } l(x,u)\geq 0$$

we get

$$\inf_{u\in U}\langle D^* W(x)\,,\, f(x,u)\rangle \leq -\gamma(W(x) < 0.$$

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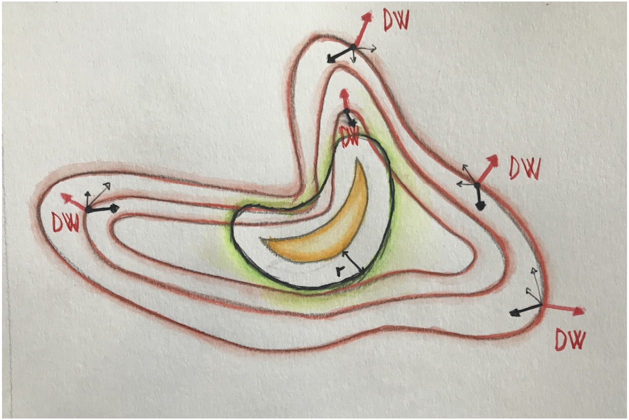
we get

$$\inf_{u \in U} \langle D^* W(x) \, , \, f(x,u) \rangle \leq -\gamma(W(x) < 0.$$

- It is well-known that the existence of a CLF characterizes asymptotic controllability and stabilizability [Sontag, '83], [Sontag and o., '97, '04].

Level sets of a (smooth) p_0-MRF



$$\inf_{u\in U}\langle DW(x)\,,\,f(x,u)\rangle < 0 \quad \Longrightarrow \quad \exists u(x)\in U;\ \ \langle DW(x)\,,\,f(x,u(x))\rangle < 0$$

$$\Big(\Longrightarrow \exists x(t) \text{ such that } \lim_{t\to+\infty}\mathbf{d}(x(t))=0\Big)$$

Remark 2

- **When** $l(x,a) \geq \mu > 0$ **the decrease condition includes classical Petrov-type controllability conditions**.

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- **When** $l(x,a) \geq \mu > 0$ **the decrease condition includes classical Petrov-type controllability conditions**.

- For instance, for the minimum time problem where $l \equiv 1$, the Petrov condition
$$\inf_{u \in U} \{\langle D^* \mathbf{d}(x), f(x,u)\rangle\} \leq -\mu,$$
setting $W := \mathbf{d}$ can be stated as
$$\inf_{u \in U} \{\langle D^* W(x), f(x,u)\rangle + p_0\, 1\} \leq -(\mu - p_0) < 0$$
for any $p_0 \in (0, \mu)$.

  In this case it well-known that the minimum time function T verifies $T(z) \leq C\,\mathbf{d}(z)$
  [Cannarsa, Sinestrari '04].

HOWEVER,

- **Our decrease condition is not a mere application of the usual Lyapunov-type condition** to the extended dynamics obtained by adding the equation $\dot{x}^0 = l(x, u)$, with the extended target $[0, +\infty) \times \mathcal{C}$.

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- **Since** $l(x, u) \geq 0$ **and may be zero on an arbitrary set, an upper estimate for the cost in terms of** W **is not an immediate consequence of the decrease condition**, in that the first order PDE

$$\inf_{u \in U} \{\langle Dw(x)\,,\ f(x, u)\rangle + p_0\, l(x, u)\} = 0$$

does not verify any comparison principle.

Main result

Theorem 4.

Assume hypothesis (H0) *and let W be a p_0-MRF with $p_0 > 0$ for $(f, l, \mathcal{C})$.*

Then there exists a locally bounded feedback $K : \mathbb{R}^n \setminus \mathcal{C} \to U$ that **sample and Euler stabilizes system** (1) **to** $\mathcal{C}$ **with** W**-regulated cost.**

Hints on the proof

The proof of the sample stabilizability with W-regulated cost relies on

1. the construction of a discontinuous feedback control law;
2. the use of the semiconcavity property of the p_0-MRF W in the spirit of feedback stabilization;
3. sharp uniform estimates of the cost, until the time when the sampling trajectory definitively enters a r-neighborhood of the target;

The proof of the Euler stabilizability with W-regulated cost relies on

1. the uniform estimate, given a sampling time $\delta > 0$ small enough, of a radius $r < \mathbf{d}(z)$ such that any π-sampling trajectory with $\text{diam}(\pi) = \delta$ definitively enters a r-neighborhood of the target;
2. a uniform lower bound of the time needed to any π-sampling trajectory with $\text{diam}(\pi) = \delta$ to definitively enter the r-neighborhood of the target;
3. a limit procedure.

Lipschitz continuous p_0-MRF

Assumption (H1)

- The sets $U \subset \mathbb{R}^m$, $\mathcal{C} \subset \mathbb{R}^n$ are closed and the boundary $\partial\mathcal{C}$ is compact.
- $f : \overline{(\mathbb{R}^n \setminus \mathcal{C})} \times U \to \mathbb{R}^n$, $l : \overline{(\mathbb{R}^n \setminus \mathcal{C})} \times U \to [0, +\infty)$ are continuous functions such that for every compact subset $\mathcal{K} \subset \overline{\mathbb{R}^n \setminus \mathcal{C}}$ there exist M_f, M_l, L_f, $L_l > 0$ such that

$$\begin{cases} |f(x,u)| \leq M_f, \quad l(x,u) \leq M_l \qquad \forall (x,u) \in \mathcal{K} \times U, \\ |f(x_1,u) - f(x_2,u)| \leq L_f |x_1 - x_2|, \\ |l(x_1,u) - l(x_2,u)| \leq L_l |x_1 - x_2| \qquad \forall (x_1,u),\ (x_2,u) \in \mathcal{K} \times U. \end{cases}$$

Lipschitz continuous p_0-MRF

We say that a map W is a **Lipschitz continuous p_0-MRF** when it verifies all the properties of a p_0-MRF, except that local semiconcavity is replaced by local Lipschitz continuity (and D^*W is replaced by $\partial_P W$).

Theorem 5.

Assume hypothesis (H1) *and let W be a Lipschitz continuous p_0-MRF with $p_0 > 0$ for $(f, l, \mathcal{C})$.*

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Then there exists a locally bounded feedback $K : \mathbb{R}^n \setminus \mathcal{C} \to U$ that **sample and Euler stabilizes system** (1) **to $\mathcal{C}$ with W-regulated cost.**

Theorems 4 and 5 extend the results of [M., Rampazzo, '13], concerning asymptotic controllability in optimal control.

Example

Stabilization of the non-holonomic integrator control system with regulated cost

Set $U := \{u = (u_1, u_2) \in \mathbb{R}^2 : \ \ u_1^2 + u_2^2 \leq 1\}$, $\mathcal{C} := \{0\} \subset \mathbb{R}^3$ and consider the non-holonomic integrator control system:

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ \dot{x}_3 = x_1 u_2 - x_2 u_1, \qquad u(t) = (u_1, u_2)(t) \in U, \\ x(0) = (x_1, x_2, x_3)(0) = z \in \mathbb{R}^3 \setminus \{0\}. \end{cases} \tag{2}$$

Given a nonnegative, continuous Lagrangian $l(x, u)$, let us associate to (2) a cost

$$\int_0^{T_x} l(x(t), u(t))\, dt.$$

The following map, introduced by [Malisoff, Rifford, Sontag, '97],

$$W_1(x) := \left(\sqrt{x_1^2+x_2^2} - |x_3|\right)^2 + x_3^2 \qquad \forall x \in \mathbb{R}^3,$$

is proper, positive definite, locally semiconcave in $\mathbb{R}^3 \setminus \{0\}$, and verifies

$$\min_{u\in U}\langle p, f(x,u)\rangle = -\sqrt{V(x)} \qquad \forall x \in \mathbb{R}^3 \setminus \{0\},\ \forall p \in D^* W_1(x),$$

where

$$V(x) := \left(\sqrt{x_1^2+x_2^2} - |x_3|\right)^2 + \left(\sqrt{x_1^2+x_2^2} - 2|x_3|\right)^2 (x_1^2+x_2^2) \quad \forall x \in \mathbb{R}^3.$$

Therefore, as soon as l verifies

$$0 \le l(x,u) \le C\sqrt{V(x)} \qquad \forall (x,u) \in (\mathbb{R}^3 \setminus \{0\}) \times U,$$

W_1 is a p_0-MRF for every $p_0 \in (0, 1/C)$.

However, W_1 cannot be a p_0-MRF when

$$\lim_{x\to 0} \frac{\inf_{u\in U} l(x,u)}{\sqrt{V(x)}} = +\infty.$$

Since $V(x)$ tends to 0^+ as $x \to 0$, this is the case, for instance, of the minimum time problem, where $l \equiv 1$.

Instead, the following map W_2, introduced by [Rifford, '00]:

$$W_2(x) := \max\left\{\sqrt{x_1^2+x_2^2},\, |x_3| - \sqrt{x_1^2+x_2^2}\right\} \qquad \forall x \in \mathbb{R}^3$$

is a Lipschitz continuous, not semiconcave p_0-MRF for $l = 1$.

Final remarks

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Final remarks

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- In a forthcoming paper we address the question of *stabilizability with regulated cost for optimization problems with unbounded data,* including *control-polynomial systems with unbounded controls* [Lai, M. and Rampazzo, '16].
- Other interesting research directions include:

  - *the relation between p_0-MRFs and input-to-state stability* [Malisoff, Rifford, Sontag, '97];

  - *the study of a possible inverse Lyapunov theorem for p_0-MRFs*, [Sontag, '83]

Thank you for your attention!