Modeling the Root Growth: an Optimal Control Approach

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Circumnutation is an elliptical, circular or pendulum-like movement performed by plant organs during growth.

Why plants perform circumnutation is biologically not well understood.
Plant roots have been recently used as a paradigm to design and build robotic technologies.

Plant roots perform circumnutation too while they grow.

It has been conjectured that circumnutation plays an important role in facilitating soil penetration.
[1] compares efficiency of circumnutation with respect to straight penetration when a robotic root tip with a parabolic shape is pushed into soil at different densities.

In this study, we want to build up a control-based framework aiming at modeling the robotic root.

Why are we interested in this?

The study [1] shows that *circumnutation* performs better than straight-line penetration (in some specific cases)

Here, we want to compare circumnutation with *any* possible behavior

Is circumnutation still more efficient compared to any possible behavior?
MODEL VARIABLES

- $x$ is the position of the center of the robotic device
- $v$ is the velocity with direction $n_v$ coinciding with the parabolic axis
- $k(x)$ is the strength of the soil
- $i_{n_v}$ is the axial contribution of the friction along the direction $n_v$
- $i_{n_1}, i_{n_2}$ represent the lateral contribution of the friction along the directions $n_1, n_2$
We develop a model able to reproduce the settings of the experiment in [1] and that allows to extend results to a more general framework

\[\begin{align*}
\dot{x} &= v \\
\dot{v} &= u - k(x) \left( |\langle u, i_{n_v} \rangle| n_v + |\langle u, i_{n_1} \rangle| n_1 + |\langle u, i_{n_2} \rangle| n_2 \right) \\
(x(T), v(T)) &\in \mathcal{T} \\
(x(0), v(0)) &= (x_0, v_0) \in \mathbb{R}^{3+3}, \\
u &\in [-1, 1]^2 \times [-1, -u_{\text{min}}], \quad t \in [0, T]
\end{align*}\]
The cost function

\[ W(T, v, u) = \left( \int_0^T \left\{ \langle u, i_{nv}\rangle^2 + \langle u, i_{n1}\rangle^2 + \langle u, i_{n2}\rangle^2 \right\} ds \right)^{1/2} \]

measures the energy dissipation due to either the penetration friction (along the direction \( n_v \)) or due to the lateral friction (along the directions \( n_1, n_2 \)).

\( k(x), i_{nv}, i_{n1}, i_{n2} \) are our modeling variables.
Experimental data are taken in the following settings:

1) Constant speed, straight descent path;
2) Constant speed, helical descent path.

We then calibrate the model parameters using a control $u$ which minimizes the cost functional $W$ and satisfies one of the following conditions:

1) $\mathbf{v}(t) = (0, 0, c)$, $\dot{\mathbf{v}}(t) = (0, 0, 0)$ in the case in which straight penetration is considered
2) $|\mathbf{v}(t)| = c$, $\dot{\mathbf{v}}(t) = \frac{c\omega}{\sqrt{2}}(-\cos(\omega t), -\sin(\omega t), 0)$, in the case in which helical circumnutation is considered
A computation of the instantaneous cost function $W$ with special choices of $u$ shows an higher efficiency when $\dot{v}$ is forced to be periodic.
The model has to fit the following experimental data:

a) The amplitude of the helical $\alpha$

b) The lead angle of the helical $\varepsilon$

c) The period of circumnutation $T_0$

d) The energy efficiency between straight and helical paths

The relation between all a), b), c) and d) has to be carefully addressed
An example of instantaneous energy dissipation when
- soil density \( k(x) \) increases linearly with respect to depth
- one considers examples of dynamics generating two different helical paths
MATHEMATICAL FRAMEWORK

A vector field $V(x)$ is applied to a rigid body $B$ at $x$. Suppose that an “averaged” dynamic friction $I(x)$ affects the vector field $V(x)$ at $x$, whose total effect depends on the surface of friction of the rigid body $B$.

\[ \ddot{x}(t) = V(x) - I(x) \]

\[ I(x) = \sum_i k(V(x), \alpha_i) \eta(x) \cdot Q(\alpha_i) \]

\[ k(V(x), \alpha_i) = \lambda_i > 0, \ i = 1, 2 \]
μ measure such that \( \text{supp}(\mu) \subseteq A \)

μ chooses relevant points in \( A \) responsible for friction

\( \varphi(x, \alpha) \) a convex function with respect to \( x \)

We are computing an “average friction”

\[
\dot{x} \in g(t, x) - \int_A k(g(t, x), \alpha) \partial_x \varphi(x, \alpha) \mu(d\alpha)
\]

If both the vector field \( g \) and the friction strength \( k \) depend on a feedback control \( u \)

\[
\dot{x} \in F(t, x, u) := g(t, x, u) - \int_A k(t, x, u, \alpha) \partial_x \varphi(x, \alpha) \mu(d\alpha)
\]
Consider the optimal control problem

\[
(P_{t_0, x_0}) \quad \begin{cases} 
\text{Minimize} & W(T, x(T)) \\
\text{over} & T > t_0, \ (x, u) \in AC([t_0, T]; \mathbb{R}^n) \times U \\
& \dot{x}(t) \in F(t, x(t), u(t)), \ \text{a.e} \ t \in [t_0, T] \\
& u(t) \in U \subseteq \mathbb{R}^m, \ \text{a.e} \ t \in [t_0, T] \\
& x(t_0) = x_0 \in \mathbb{R}^n \\
& (T, x(T)) \in T 
\end{cases}
\]

\[V(t_0, x_0) = \inf \{ W(T, x(T)) | (T, x(T)) \text{ solution of } (P_{t_0, x_0}) \}\]
ASSUMPTIONS ON THE DYNAMICS

Assume

$H_1$: The maps $(t, x, u, \alpha) \mapsto k(t, x, u, \alpha) \geq 0$, $(t, x, u) \mapsto g(t, x, u)$, $(x, \alpha) \mapsto \varphi(x, \alpha)$ and $(t, x) \mapsto W(t, x)$ are continuous.

$H_2$: There exist constants $L, C > 0$ such that

\[
|g(t, x, u) - g(s, y, u)| \leq L(|t - s| + |x - y|)
\]
\[
|\varphi(x, \alpha) - \varphi(y, \alpha)| \leq L|x - y|
\]
\[
|k(t, x, u, \alpha) - k(s, y, u, \alpha)| \leq L(|t - s| + |x - y|)
\]
\[
|k(t, x, u, \alpha)|, |g(t, x, u)| \leq C
\]

for every $x, y \in \mathbb{R}^n$, $t, s \in [t_0, T]$, $u \in U$ and $\alpha \in A$.

$H_3$: for each $\alpha \in A$, the mapping $x \mapsto \varphi(x, \alpha)$ is convex.

$H_4$: the set-valued map $\bar{F}(t, x) := \bigcup_{u \in U} F(t, x, u)$ takes convex values for each $(t, x) \in \mathbb{R}^{1+n}$. 
Proposition

Assume $H_1 - H_3$. Then

$$\bar{F}(t, x) = \bigcup_u F(t, x, u)$$

is non-empty, compact and upper semi-continuous. Furthermore, $\bar{F}$ is Lipschitz continuous w.r.t. $t$ and One Sided Lipschitz\(^a\) (OSL) w.r.t. $x$, uniformly w.r.t. $t$.


For any control and initial condition, there exists a unique solution to

$$\begin{cases} 
\dot{x}(t) \in F(t, x(t), u(t)), & \text{a.e. } t \in [t_0, +\infty) \\
x(t_0) = x_0 
\end{cases}$$
**H₅:** Given $\tilde{T}: \mathbb{R} \rightsquigarrow \mathbb{R}^n$ with closed graph, $\mathcal{T} = \text{Gr} \tilde{T}$

**H₆:** Inward pointing condition:

For any compact $G \subseteq \mathbb{R}^{1+n}$ there exists $\rho > 0$ such that, for all $(t, x) \in \partial \mathcal{T} \cap G$

$$\min_{\xi \in \tilde{F}(t, x)} \{l^0 + \langle l, \xi \rangle\} \leq -\rho \quad \text{for all} \quad (l^0, l) \in N_{\mathcal{T}}(t, x)$$

**H₇:** The function $W: \mathbb{R}^{1+n} \rightarrow \mathbb{R}$ is Lipschitz continuous
Define the reachable set

$$\mathcal{R}(T; t, x) := \{x(T) : \dot{x}(s) \in \bar{F}(s, x(s)), \ s \in [t, T], \ x(t) = x\}$$

and the set of admissible trajectories with initial condition $x(t) = x$:

$$\mathcal{A}(t, x) := \{(T, x(T)) \in \mathcal{T} : x(T) \in \mathcal{R}(T; t, x), \ T \geq t\}.$$ 

Then we impose the following growth condition:

**(GC)** Fix any $(t, x) \in \mathbb{R}^{1+n}$. For every $(T_k, x_k) \in \mathcal{A}(t, x)$ such that $T_k \to +\infty$, one has that $W(T_k, x_k) \to +\infty$. 

If one replaces \((\text{GC})\) with the following condition:

\[ (\text{LGC}) \quad \text{For any } K \subset \mathbb{R}^{1+n} \text{ compact, there exists } \gamma > 0 \text{ such that} \]

\[ W(t', x') \geq W(t, x) + \gamma(t' - t), \]

whenever \((t, x) \in K, (t', x') \in A(t, x)\).

Then the related optimal solution stops as soon as it reaches the target.

However, \((\text{GC})\) does not imply that the optimal trajectory stops when it reaches the target...
\[ \mathcal{D} = \{(t, x) \in \mathbb{R}^{1+n} : V(t, x) < +\infty\} \]

**Theorem**

Assume \( H_1 - H_8 \). Then:

- \( \mathcal{D} \) is open (Inward Pointing Condition)
- For any \((t_0, x_0) \in \mathcal{D}\), there exists a minimizer for the optimal control problem \((P_{t_0, x_0})\) (Growth Condition)
- \( V \) is locally Lipschitz in \( \mathcal{D} \); (Inward Pointing Condition)
- \( V(s_k, x_k) \to +\infty \) for all \((s_k, x_k) \to (s_0, x_0)\) such that \((s_k, x_k) \in \mathcal{D} \) and \((s_0, x_0) \in \partial \mathcal{D}\) (Inward Pointing Condition)
- \( V(s_k, x_k) \to +\infty \) for all \((s_k, x_k) \in \mathcal{D} \) such that \(s_k \to \infty\) (Growth Condition)
Consider the augmented differential inclusion dynamics:

\[
\begin{cases}
(\tau(t), x(t), a(t)) \in \Gamma(\tau, x) = \{1\} \times \bar{F}(\tau, x) \times \{0\} \\
(\tau(0), x(0), a(0)) = (t_0, x_0, a_0) \in \mathbb{R}^{1+n+1} \\
t \in [0, +\infty)
\end{cases}
\]

it is possible to link

- the dynamic programming principle for the value function \( V \)
- and the invariance properties of the dynamics \( \Gamma \):
  - \((\Gamma, \text{epi}(V))\) is weakly invariant in a given open set \( A \)
  - \((\Gamma, \text{hypo}(V))\) is strongly invariant
\[ \Omega = \{(t, x) \in \mathcal{T} : V(t, x) = W(t, x)\} \]

(No Characteristics start from \(\Omega\))

\[ h(t, x, \eta) = \eta_t + \inf_{v \in \bar{F}(t, x)} \langle v, \eta_x \rangle \quad H(\tau, x, \eta) = \eta_t + \sup_{v \in \bar{F}(t, x)} \langle v, \eta_x \rangle \]

Are equivalent:

- \((\text{epi}(V), \Gamma)\) is weakly invariant in \(\Omega^c\)
- For all \((t, x, a) \in \text{epi}(V) \cap \Omega^c\),
  \[ h(t, x, \eta) \leq 0 \quad \text{for all} \quad \eta = (\eta_t, \eta_x) \in N^P_{\text{epi}(V)}(t, x, a) \]

Are equivalent:

- \((\text{hypo}(V), \Gamma)\) is strongly invariant
- For all \((t, x, a) \in \text{hypo}(V)\),
  \[ \limsup_{(\tau, y) \to (t, x)} H(\tau, y, \eta) \leq 0 \quad \text{for all} \quad \eta = (\eta_t, \eta_x) \in N^P_{\text{hypo}(V)}(\tau, x, a) \]
Theorem

Assume $H_1-H_8$. Then $V$ is the unique, loc. Lipschitz viscosity solution:

i) $V(t,x) \leq W(t,x)$ for each $(t,x) \in \mathcal{T}$;

ii) $V(t,x) = +\infty$ for all $(t,x) \notin \mathcal{D}$;

iii) $V(t_k,x_k) \to +\infty$ for all $(t_k,x_k) \in \mathcal{D}$ s.t. $(t_k,x_k) \to (t,x) \in \partial \mathcal{D}$;

iv) For every $(t_k,x_k) \in \mathcal{D}$ such that $t_k \to \infty$, then $V(t_k,x_k) \to \infty$;

v) for every $(t,x) \in \mathcal{D} \cap \Omega^c$ one has, for all $p = (p_t,p_x) \in \partial_P V(t,x)$

$$p_t + \min_{v \in \overline{F}(t,x)} v \cdot p_x \leq 0$$

(0.1)

vi) for every $(t,x) \in \mathcal{D}$, one has, for all $q = (q_t,q_x) \in \partial_P V(t,x)$

$$q_t + \liminf_{(t',x') \to q(t,x)} \left\{ \min_{v \in \overline{F}(t',x')} v \cdot q_x \right\} \geq 0$$

(0.2)
Theorem

Assume $H_1$-$H_8$. Then $V$ is the unique, loc. Lipschitz viscosity solution satisfying, if $(t, x) \in T^c$,

$$\partial_t V + \liminf_{(t', x') \rightarrow p(t, x)} \left[ \min_{v \in \bar{F}(t', x')} v \cdot \nabla_x V \right] = 0,$$

(0.3)

and, if $(t, x) \in T$,

$$\min \left\{ W(t, x) - V(t, x), \partial_t V + \liminf_{(t', x') \rightarrow p(t, x)} \left[ \min_{v \in \bar{F}(t', x')} v \cdot \nabla_x V \right] \right\} = 0,$$

(0.4)

i) $V(t, x) = +\infty$ for all $(t, x) \notin D$;

ii) $V(t_k, x_k) \rightarrow +\infty$ for all $(t_k, x_k) \in D$ s.t. $(t_k, x_k) \rightarrow (t, x) \in \partial D$;

iii) For every $(t_k, x_k) \in D$ such that $t_k \rightarrow \infty$, then $V(t_k, x_k) \rightarrow \infty$. 

Michele Palladino (GSSI)
Solving the related optimal control problem using the direct method, it seems the optimality is reached through rotations and helical trajectories instead of going deeper in a straight way.
In this talk, we have presented a model for the root growth.

The model fits some experimental data, which have been acquired measuring the energy dissipated by a mechanical root-like robot.

The model we consider aims at providing some insights in the decision making process that a root-like robot has to take into account in order to efficiently work.

This results can be helpful in understanding the decision-making process of plant (specifically, for what concerns roots).

Proving that the value function synthesis of the related optimal control problem provides a helical circumnutation is a possible strategy to show why circumnutation occurs in plants.
THANK YOU FOR YOUR ATTENTION