Modeling the Root Growth: an Optimal Control Approach

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Control Days 2019
Padova

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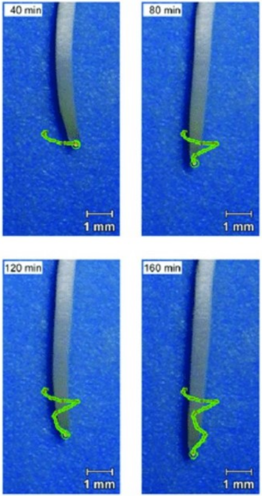
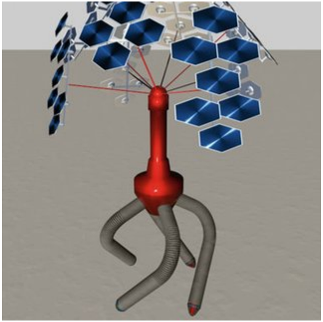
CIRCUMNUTATION



Circumnutation is an elliptical, circular or pendulum like movement performed by plant organs during growth

Why plants perform circumnutation is biologically not well understood

ROOTS

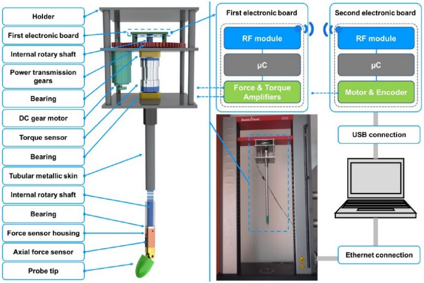


Plant roots have been recently used as a paradigm to design and build robotic technologies

Plant roots perform circumnutation too while they grow

It has been conjectured that circumnutation plays an important role in facilitating soil penetration

ROOT-LIKE MECHANISM IN REAL SOIL



[1] compares efficiency of circumnutation with respect to straight penetration when a robotic root tip with a parabolic shape is pushed into soil at different densities

E. Del Dottore, A. Mondini, A. Sadeghi, V. Mattoli, and B. Mazzolai. An efficient soil penetration strategy for explorative robots inspired by plant root circumnutation movements. *Bioinspiration & biomimetics*, 13(1):015003, 2017.

DECISION-MAKING FRAMEWORK

In this study, we want to build up a control-based framework aiming at modeling the robotic root.

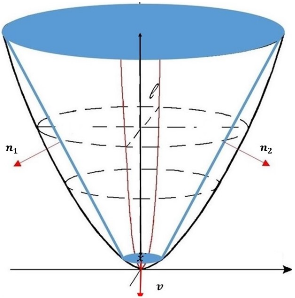
Why are we interested in this?

The study [1] shows that *circumnutation* perfoms better than straight-line penetration (in some specific cases)

Here, we want to compare circumnutation with *any* possible behavior

Is circumnutation still more efficient compare to any possible behavior?

MODEL VARIABLES



- x is the position of the center of the robotic device
- v is the velocity with direction n_v coinciding with the parabolic axis
- $k(x)$ is the strength of the soil
- i_{n_v} is the axial contribution of the friction along the direction n_v
- $i_{n_1}, \ i_{n_2}$ represent the lateral contribution of the friction along the directions $n_1, \ n_2$

We develope a model able to reproduce the settings of the experiment in [1] and that allows to extend results to a more general framework

$$\begin{cases} \dot{x} = v \\ \dot{v} = u - k(x)\Big(\big| \langle u, i_{n_v} \rangle \big| n_v + \big| \langle u, i_{n_1} \rangle \big| n_1 + \big| \langle u, i_{n_2} \rangle \big| n_2 \Big) \\ (x(T), v(T)) \in \mathcal{T} \\ (x(0), v(0)) = (x_0, v_0) \in \mathbb{R}^{3+3}, \\ u \in [-1, 1]^2 \times [-1, -u_{\min}], \;\; t \in [0, T] \end{cases}$$

ENGINEERING FRAMEWORK

The cost function

$$W(T,v,u) = \left(\int_0^T \left\{ \langle u, i_{n_v} \rangle^2 + \langle u, i_{n_1} \rangle^2 + \langle u, i_{n_2} \rangle^2 \right\} ds \right)^{1/2}$$

measures the energy dissipation due to either the penetration friction (along the direction n_v) or due to the lateral friction (along the directions n_1, n_2).

$k(x)$, i_{n_v}, i_{n_1}, i_{n_2} are our modeling variables

MODEL CALIBRATION

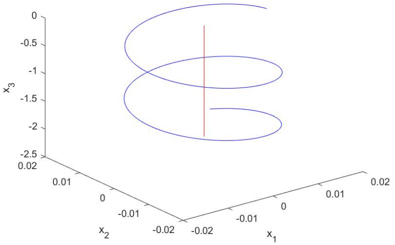
Experimental data are taken in the following settings:

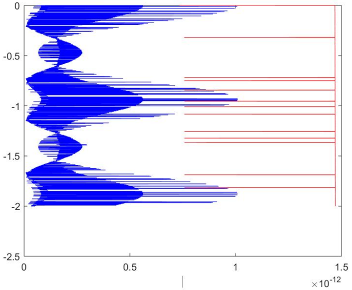
1) Constant speed, straight descent path;
2) Constant speed, helical descent path.

We then calibrate the model parameters using a control u which minimizes the cost functional W *and* satisfies one of the following conditions:

1) $v(t) = (0, 0, c)$, $\dot{v}(t) = (0, 0, 0)$ in the case in which straight penetration is considered
2) $|v(t)| = c$, $\dot{v}(t) = \frac{c\omega}{\sqrt{2}}(-\cos(\omega t), -\sin(\omega t), 0)$, in the case in which helical circumnutation is considered

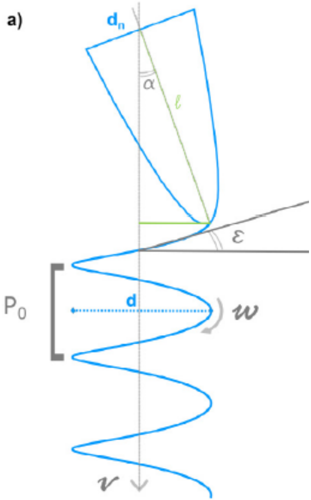
A computation of the instantaneous cost function W with special choices of u shows an higher efficiency when $\dot{v}$ is forced to be periodic





EXPERIMENTAL DATA

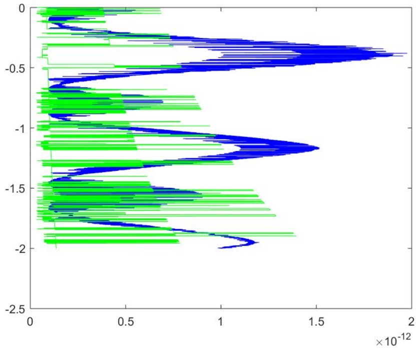
The model has to fit the following experimental data:



a) The amplitude of the helical α
b) The lead angle of the helical ε
c) The period of circumnutation T_0
d) The energy efficiency between straight and helical paths

The relation between all a), b), c) and d) has to be carefully addressed

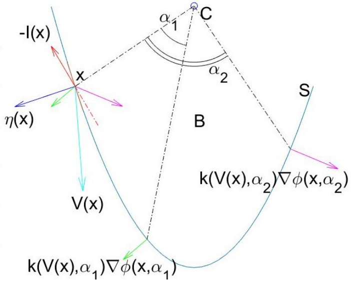
SENSITYVITY



An example of instantaneous energy dissipation when

- soil density $k(x)$ increases linearly with respect to depth
- one considers examples of dynamics generating two different helical paths

A vector field $V(x)$ is applied to a rigid body $\mathcal{B}$ at x. Suppose that an "averaged" dynamic friction $I(x)$ affects the vector field $V(x)$ at x, whose total effect depends on the surface of friction of the rigid body $\mathcal{B}$.



- $\ddot{x}(t) = V(x) - I(x)$
- $I(x) = \sum_i k(V(x), \alpha_i)\eta(x) \cdot Q(\alpha_i)$
- $k(V(x), \alpha_i) = \lambda_i > 0, \ \ i = 1, 2$

GENERAL SETTINGS

- μ measure such that $\mathrm{supp}(\mu) \subseteq A$
- μ chooses relevant points in A responsible for friction
- $\varphi(x, \alpha)$ a convex function with respect to x

We are computing an "average friction"

$$\dot{x} \in g(t,x) - \int_A k(g(t,x),\alpha)\partial_x\varphi(x,\alpha)\mu(d\alpha)$$

If both the vector field g and the friction strength k depend on a feedback control u

$$\dot{x} \in F(t,x,u) := g(t,x,u) - \int_A k(t,x,u,\alpha)\partial_x\varphi(x,\alpha)\mu(d\alpha)$$

Consider the optimal control problem

$$
(P_{t_0,x_0})\begin{cases}\text{Minimize } W(T,x(T))\\ \text{over } T>t_0,\ (x,u)\in\ AC([t_0,T];\mathbb{R}^n)\times\mathcal{U}\\ \dot{x}(t)\in F(t,x(t),u(t)),\ \text{a.e } t\in[t_0,T]\\ u(t)\in U\subseteq\mathbb{R}^m,\ \text{a.e } t\in[t_0,T]\\ x(t_0)=x_0\in\mathbb{R}^n\\ (T,x(T))\in\mathcal{T}\end{cases}
$$

$$
V(t_0,x_0)=\inf\{W(T,x(T))|\ (T,x(T)) \text{ solution of } (P_{t_0,x_0})\}
$$

ASSUMPTIONS ON THE DYNAMICS

Assume

H_1: The maps $(t,x,u,\alpha) \mapsto k(t,x,u,\alpha) \geq 0$, $(t,x,u) \mapsto g(t,x,u)$, $(x,\alpha) \mapsto \varphi(x,\alpha)$ and $(t,x) \mapsto W(t,x)$ are continuous.

H_2: There exist constants $L, C > 0$ such that

$$|g(t,x,u) - g(s,y,u)| \leq L\left(|t-s| + |x-y|\right)$$
$$|\varphi(x,\alpha) - \varphi(y,\alpha)| \leq L|x-y|$$
$$|k(t,x,u,\alpha) - k(s,y,u,\alpha)| \leq L\left(|t-s| + |x-y|\right)$$
$$|k(t,x,u,\alpha)|\,,\ |g(t,x,u)| \leq C$$

for every $x, y \in \mathbb{R}^n$, $t, s \in [t_0, T]$, $u \in U$ and $\alpha \in A$.

H_3: for each $\alpha \in A$, the mapping $x \mapsto \varphi(x,\alpha)$ is convex.

H_4: the set-valued map $\bar{F}(t,x) := \cup_{u \in U} F(t,x,u)$ takes convex values for each $(t,x) \in \mathbb{R}^{1+n}$.

PROPERTIES OF THE DYNAMICS

Proposition

Assume $H_1 - H_3$. Then

$$\bar{F}(t,x) = \bigcup_u F(t,x,u)$$

is non-empty, compact and upper semi-continuous. Furthermore, $\bar{F}$ is Lipschitz continuous w.r.t. t and One Sided Lipschitz[a] (OSL) w.r.t. x, uniformly w.r.t. t.

[a] T. Donchev, V. Rios, P. Wolenski, *"Strong invariance and one-sided Lipschitz multifunctions"*, Nonlinear Anal.

For any control and initial condition, there exists a unique solution to

$$\begin{cases} \dot{x}(t) \in F(t, x(t), u(t)), & \text{a.e. } t \in [t_0, +\infty) \\ x(t_0) = x_0 \end{cases}$$

ASSUMPTIONS ON THE O.C.P.

H_5: Given $\tilde{\mathcal{T}} : \mathbb{R} \rightsquigarrow \mathbb{R}^n$ with closed graph, $\mathcal{T} = \mathrm{Gr}\, \tilde{\mathcal{T}}$

H_6: Inward pointing condition:
For any compact $G \subseteq \mathbb{R}^{1+n}$ there exists $\rho > 0$ such that, for all $(t, x) \in \partial\mathcal{T} \cap G$

$$\min_{\xi \in \bar{F}(t,x)} \left\{ l^0 + \langle l, \xi \rangle \right\} \leq -\rho \quad \textit{for all} \quad (l^0, l) \in N_{\mathcal{T}}(t, x)$$

H_7: the function $W : \mathbb{R}^{1+n} \to \mathbb{R}$ is Lipschitz continuous

H_8: GROWTH CONDITION

Define the reachable set

$$\mathcal{R}(T;t,x) := \left\{x(T) : \ \dot{x}(s) \in \bar{F}(s,x(s)), \ s \in [t,T], \ x(t) = x\right\}$$

and the set of admissible trajectories with initial condition $x(t) = x$:

$$\mathcal{A}(t,x) := \{(T,x(T)) \in \mathcal{T} : \ x(T) \in \mathcal{R}(T;t,x), \ T \geq t\}.$$

Then we impose the following growth condition:

(GC) Fix any $(t,x) \in \mathbb{R}^{1+n}$. For every $(T_k, x_k) \in \mathcal{A}(t,x)$ such that $T_k \to +\infty$, one has that $W(T_k, x_k) \to +\infty$.

GROWTH CONDITIONS

If one replaces **(GC)** with the following condition:

LGC) For any $K \subset \mathbb{R}^{1+n}$ compact, there exists $\gamma > 0$ such that

$$W(t', x') \geq W(t, x) + \gamma(t' - t),$$

whenever $(t, x) \in K$, $(t', x') \in \mathcal{A}(t, x)$.

Then the related optimal solution stops *as soon as* it reaches the target.

However, (GC) does not imply that the optimal trajectory stops when it reaches the target...

PROPERTIES OF THE VALUE FUNCTION

$$\mathcal{D} = \{(t,x) \in \mathbb{R}^{1+n} : \; V(t,x) < +\infty\}$$

Theorem

Assume $H_1 - H_8$. Then:

- $\mathcal{D}$ is open (Inward Pointing Condition)
- For any $(t_0, x_0) \in \mathcal{D}$, there exists a minimizer for the optimal control problem (P_{t_0,x_0}) (Growth Condition)
- V is locally Lipschitz in $\mathcal{D}$; (Inward Pointing Condition)
- $V(s_k, x_k) \to +\infty$ for all $(s_k, x_k) \to (s_0, x_0)$ such that $(s_k, x_k) \in \mathcal{D}$ and $(s_0, x_0) \in \partial\mathcal{D}$ (Inward Pointing Condition)
- $V(s_k, x_k) \to +\infty$ for all $(s_k, x_k) \in \mathcal{D}$ such that $s_k \to \infty$ (Growth Condition)

RELATED WORKS

- M. Bardi, I. Capuzzo Dolcetta, *"Optimal Control and Viscosity Solutions of Hamilto-Jacobi-Bellman Equations"*, (1997)
- J. J. Ye, Q. Zhu, *"Hamilton-Jacobi Theory for a Generalized Optimal Stopping Problem"*, Nonlinear Anal. (1998)
- J. J. Ye, *"Discontinuous Solutions of the Hamiltin-Jacobi Equation for Exit Time Problems"*, SICON (2000)
- M. Malisoff, *"Viscosity Solutions of the Bellman Equations for Exit Time Optimal Control Problems with Vanishing Lagrangians"*, SICON (2002)
- M. Motta, C. Sartori, *"The Value Function of an Asymptotic Exit Time Optimal Control Problem"*, NoDea (2015)

INVARIANCE PRINCIPLES

Consider the augmented differential inclusion dynamics:

$$\begin{cases} (\tau(t), x(t), a(t)) \in \Gamma(\tau, x) = \{1\} \times \bar{F}(\tau, x) \times \{0\} \\ (\tau(0), x(0), a(0)) = (t_0, x_0, a_0) \in \mathbb{R}^{1+n+1} \\ t \in [0, +\infty) \end{cases}$$

it is possible to link

- the dynamic programming principle for the value function V
- and the invariance properties of the dynamics Γ:
  - $(\Gamma, \mathrm{epi}(V))$ is weakly invariant in a given open set A
  - $(\Gamma, \mathrm{hypo}(V))$ is strongly invariant

HAMILTON-JACOBI-BELLMAN INEQUALITIES

$$\Omega = \{(t,x) \in \mathcal{T} : \ V(t,x) = W(t,x)\}$$

(No Characteristics start from Ω)

$$h(t,x,\eta) = \eta_t + \inf_{v \in \bar{F}(t,x)} \langle v, \eta_x \rangle \qquad H(\tau,x,\eta) = \eta_t + \sup_{v \in \bar{F}(t,x)} \langle v, \eta_x \rangle$$

Are equivalent:

- $(\mathrm{epi}(V), \Gamma)$ is weakly invariant in Ω^c
- For all $(t,x,a) \in \mathrm{epi}(V) \cap \Omega^c$,
  $h(t,x,\eta) \leq 0$ for all $\eta = (\eta_t, \eta_x) \in N^P_{\mathrm{epi}(V)}(t,x,a)$

Are equivalent:

- $(\mathrm{hypo}(V), \Gamma)$ is strongly invariant
- For all $(t,x,a) \in \mathrm{hypo}(V)$,
  $\limsup_{(\tau,y) \to_\eta (t,x)} H(\tau,y,\eta) \leq 0$ for all
  $\eta = (\eta_t, \eta_x) \in N^P_{\mathrm{hypo}(V)}(\tau,x,a)$

HAMILTON-JACOBI-BELLMAN INEQUALITIES

Theorem

Assume H_1-H_8. Then V is the unique, loc. Lipschitz viscosity solution:

i) $V(t,x) \leq W(t,x)$ for each $(t,x) \in \mathcal{T}$;

ii) $V(t,x) = +\infty$ for all $(t,x) \notin \mathcal{D}$;

iii) $V(t_k,x_k) \to +\infty$ for all $(t_k,x_k) \in \mathcal{D}$ s.t. $(t_k,x_k) \to (t,x) \in \partial\mathcal{D}$;

iv) For every $(t_k,x_k) \in \mathcal{D}$ such that $t_k \to \infty$, then $V(t_k,x_k) \to \infty$;

v) for every $(t,x) \in \mathcal{D} \cap \Omega^c$ one has, for all $p = (p_t, p_x) \in \partial_P V(t,x)$

$$p_t + \min_{v \in \bar{F}(t,x)} v \cdot p_x \leq 0 \tag{0.1}$$

vi) for every $(t,x) \in \mathcal{D}$, one has, for all $q = (q_t, q_x) \in \partial^P V(t,x)$

$$q_t + \liminf_{(t',x') \to_q (t,x)} \left\{ \min_{v \in \bar{F}(t',x')} v \cdot q_x \right\} \geq 0 \tag{0.2}$$

Theorem

Assume H_1-H_8. Then V is the unique, loc. Lipschitz viscosity solution satisfying, if $(t,x) \in \mathcal{T}^c$,

$$\partial_t V + \liminf_{(t',x') \to_p (t,x)} \left[\min_{v \in \bar{F}(t',x')} v \cdot \nabla_x V \right] = 0, \tag{0.3}$$

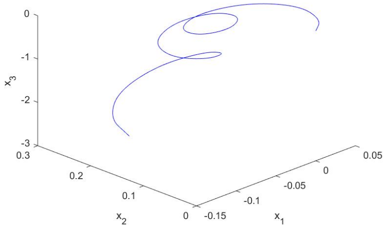
and, if $(t,x) \in \mathcal{T}$,

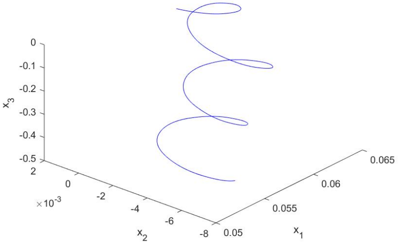
$$\min \left\{ W(t,x) - V(t,x),\ \partial_t V + \liminf_{(t',x') \to_p (t,x)} \left[\min_{v \in \bar{F}(t',x')} v \cdot \nabla_x V \right] \right\} = 0, \tag{0.4}$$

i) $V(t,x) = +\infty$ for all $(t,x) \notin \mathcal{D}$;

ii) $V(t_k,x_k) \to +\infty$ for all $(t_k,x_k) \in \mathcal{D}$ s.t. $(t_k,x_k) \to (t,x) \in \partial\mathcal{D}$;

iii) For every $(t_k,x_k) \in \mathcal{D}$ such that $t_k \to \infty$, then $V(t_k,x_k) \to \infty$.





Solving the related optimal control problem using the direct method, it seems the optimality is reached through rotations and helical trajectories instead of going deeper in a straight way

CONCLUSIONS AND OPEN QUESTIONS

In this talk, we have presented a model for the **root growth**.

The model fits some **experimental data**, which have been acquired measuring the energy dissipated by a mechanical root-like robot.

The model we consider aims at providing some insights in the **decision making process** that a root-like robot has to take into account in order to *efficiently* work.

This results can be helpful in understanding the decision-making process of plant (specifically, for what concerns roots).

Proving that the value function synthesis of the related optimal control problem provides a helical circumnutation is a possible strategy to show why circumnutation occurs in plants.

THANK YOU FOR YOUR ATTENTION