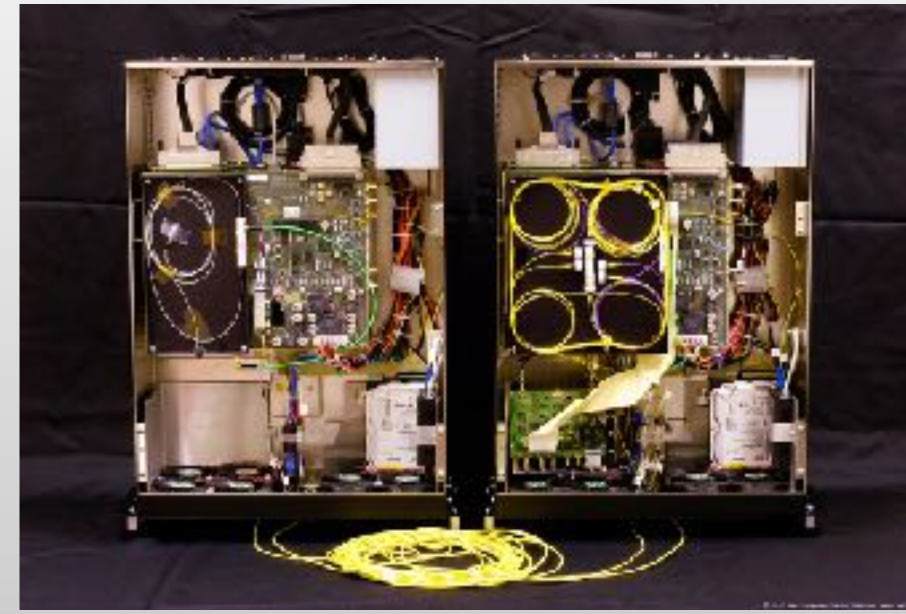
The Whole From The Parts: Three Inequivalent Ways To Obtain *Pure Quantum States From Local Information.*

Francesco Ticozzi
Dept. of Information Engineering, University of Padua
Dept. of Physics and Astronomy, Dartmouth College

In collaboration with
S. Karuvade (was PhD student@Dartmouth)
P.D.Johnson (Dartmouth, Zapata Comp.)
L. Viola (Dartmouth College)

Information and Quantum Physics today



- **Quantum cryptography:**
  Intrinsically “secure”. On the market (e.g. MagiQ).

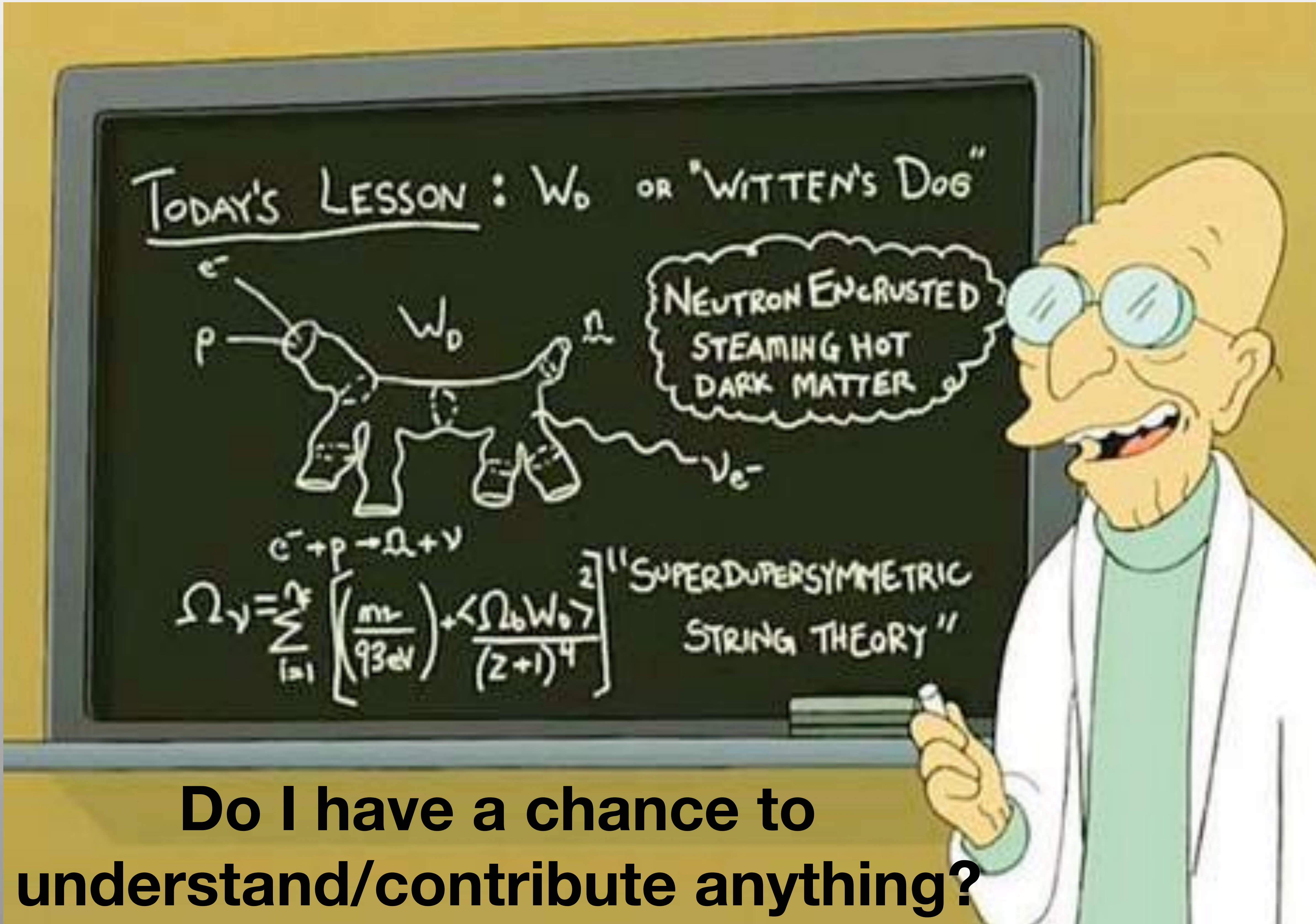
- **Full-fledged Quantum Computer (QC):**
  Theory: QC has advantage over state-of-the-art classical algorithms for factoring (exp), DFT (exp), search (quad).
  Practice: About 50 qubits (IBM)/
  72 qubits Google Bristlecone; scalability issues.

- **Dedicated quantum processors:**
  Simulated annealing/adiabatic processors are on the market (D-Wave); Google-NASA, Lockheed-Martin bought them (M$)!

- **Quantum simulation:** see QC - but needs exponentially less resources to be competitive.
  [original Feynman’s task,1960’s]

Also:
Metrology, Spectroscopy,
Controlled Q. chemistry,
Q. biology.

Ok, if they buy it, I am sold too. But...



Do I have a chance to understand/contribute anything?

Quantum is Non-commutative (Finite) Probability

✓ CLASSICAL PROBABILITY (finite Ω)

- Events, σ-algebra:

$$\omega_i \to (0,\dots,0,1,0,\dots 0)^T$$
$$e \to (0,1,1,0,\dots)^T$$

- Probability Distribution:

$$\mathbb{P}(\omega_i) = p_i \to (p_1,\dots,p_n)^T$$

- Random variables:

$$x(\omega_i) = x_i \to (x_1,\dots x_n)^T$$

- Probability and expectation:

$$\mathbb{P}(e) = \sum_i p_i e_i \to \langle \vec{\mathbb{P}}, \vec{e} \rangle$$
$$\mathbb{E}(x) = \sum_i p_i x_i \to \langle \vec{\mathbb{P}}, \vec{x} \rangle$$

- Conditioning: $\mathbb{P}(\cdot|e) \to \dfrac{\vec{e}\cdot\vec{\mathbb{P}}}{\mathbb{P}(e)}$

$\Longrightarrow$

✓ QUANTUM PROBABILITY (finite dim.) $\mathcal{H}_\Omega$

- **Orth. Projections:**

$$\{\Pi \mid \Pi = \Pi^2 = \Pi^\dagger\}$$

- **Density matrices:** *(states)*

$$\rho = \sum_i p_i \Pi_i,$$

- **Hermitian matrices:**

$$X = \sum_i x_i \Pi_i;$$

- **Probability and expectation:**

$$\mathbb{P}_\rho(\Pi) \to \langle \rho, \Pi \rangle = \mathrm{trace}(\rho\Pi)$$
$$\mathbb{E}_\rho(X) \to \langle \rho, X \rangle = \mathrm{trace}(\rho X)$$

- **Conditioning:** $\rho_{|\Pi} = \dfrac{\Pi\rho\Pi}{\mathrm{trace}(\rho\Pi)}$

Dynamics & Control for Quantum Information

- **Gate design**
- **Dissipation for QIP**
- **State stabilization**
- **Entanglement Generation**
- **Open System simulation**

nature physics
LETTERS
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Quantum computation and quantum-state engineering driven by dissipation

Frank Verstraete[1]*, Michael M. Wolf[2] and J. Ignacio Cirac[3]*

ARTICLE
doi:10.1038/nature09801

An open-system quantum simulator with trapped ions

Julio T. Barreiro[1]*, Markus Müller[2,3]*, Philipp Schindler[1], Daniel Nigg[1], Thomas Monz[1], Michael Chwalla[1,2], Markus Hennrich[1], Christian F. Roos[1,2], Peter Zoller[2,3] & Rainer Blatt[1,2]

Entanglement Generated by Dissipation and Steady State Entanglement of Two Macroscopic Objects

Hanna Krauter,[1] Christine A. Muschik,[2] Kasper Jensen,[1] Wojciech Wasilewski,[1,*] Jonas M. Petersen,[1] J. Ignacio Cirac,[2] and Eugene S. Polzik[1,†]

LETTER
doi:10.1038/nature12802

Autonomously stabilized entanglement between two superconducting quantum bits

S. Shankar[1], M. Hatridge[1], Z. Leghtas[1], K. M. Sliwa[1], A. Narla[1], U. Vool[1], S. M. Girvin[1], L. Frunzio[1], M. Mirrahimi[1,2] & M. H. Devoret[1]

LETTER
doi:10.1038/n

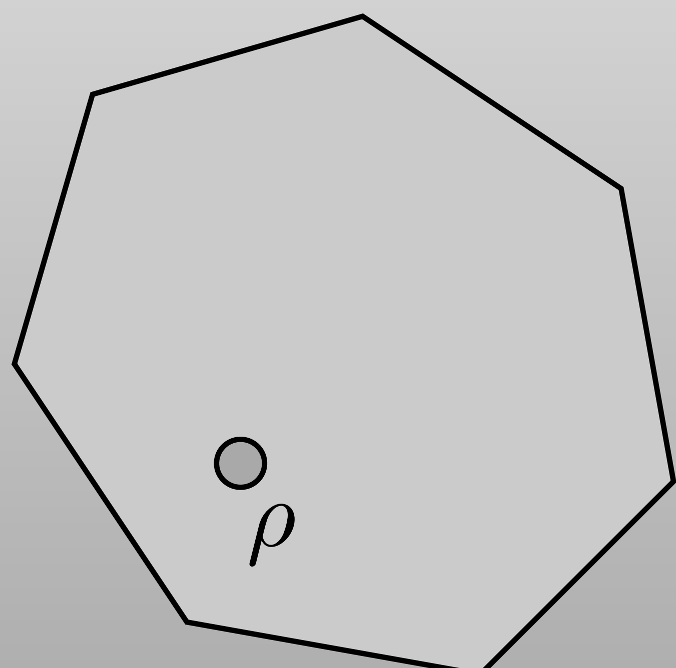
Deterministic entanglement of superconducting qubits by parity measurement and feedback

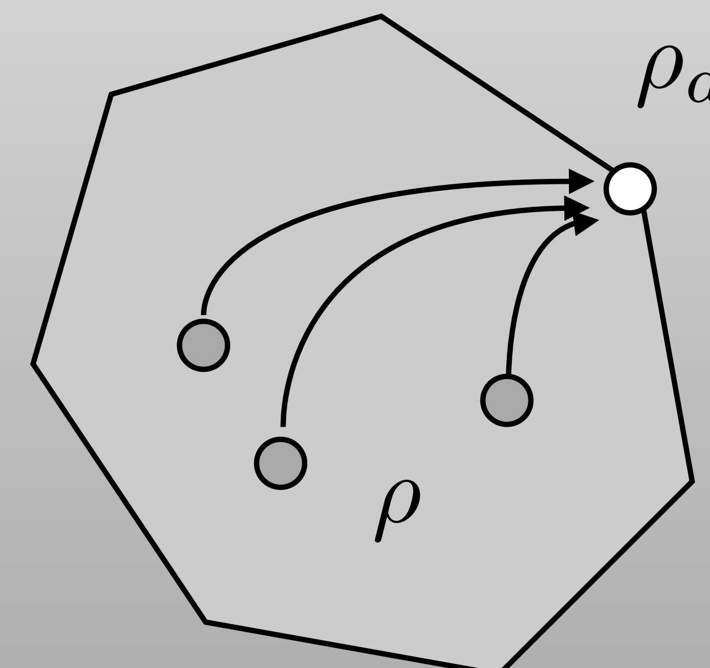
D. Ristè[1], M. Dukalski[1], C. A. Watson[1], G. de Lange[1], M. J. Tiggelman[1], Ya. M. Blanter[1], K. W. Lehnert[2], R. N. Schouten[1] & L. DiCarlo[1]

Pure States Stabilization for Quantum Systems

Consider a finite-dimensional quantum system;
General states (*density operators*) form a convex set,
extreme points are **pure states** (*rank-one orth. projections*):

$$\rho \in \mathfrak{D}(\mathcal{H}) := \{\rho = \rho^\dagger > 0,\, \mathrm{trace}(\rho) = 1\}$$



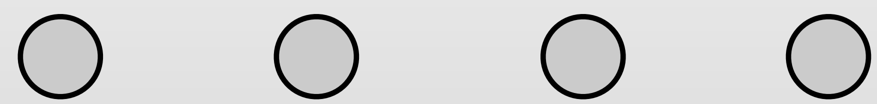


Stabilization Task:

Design dynamics that
1) prepare a given state from any initial condition,
(asymptotycally or in finite time) and
2) leave it *invariant*.

Multipartite Systems and Locality

- Consider *n* finite-dimensional systems, indexed:

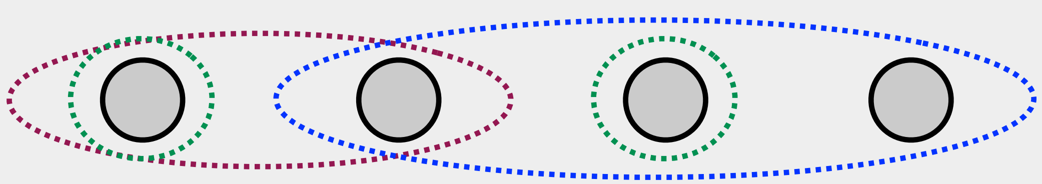


$a = 1 \quad 2 \quad 3 \quad \cdots$

$$\mathcal{H}_\mathcal{Q} = \bigotimes_{a=1}^{n} \mathcal{H}_a$$

- **Locality notion:** from the start, we specify *subsets of indexes,* or ***neighborhoods,*** corresponding to group of subsystems:

$$\mathcal{N}_1 = \{1, 2\}$$
$$\mathcal{N}_2 = \{1, 3\}$$
$$\mathcal{N}_3 = \{2, 3, 4\}$$



...on which "**we can act simultaneously**": how?

- **Neighborhood operator:** $M_k = M_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k}$
- A **Hamiltonian** respects locality if:

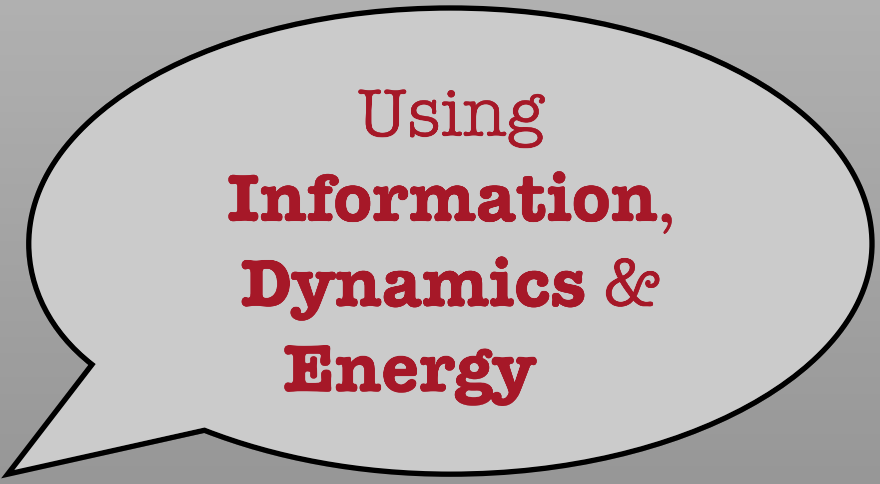
$$H = \sum_k H_k, \quad H_k = H_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k}$$

Hyper-graphs include: graph-induced locality, N-body locality, etc...

Neighborhood operators will model the allowed interactions.

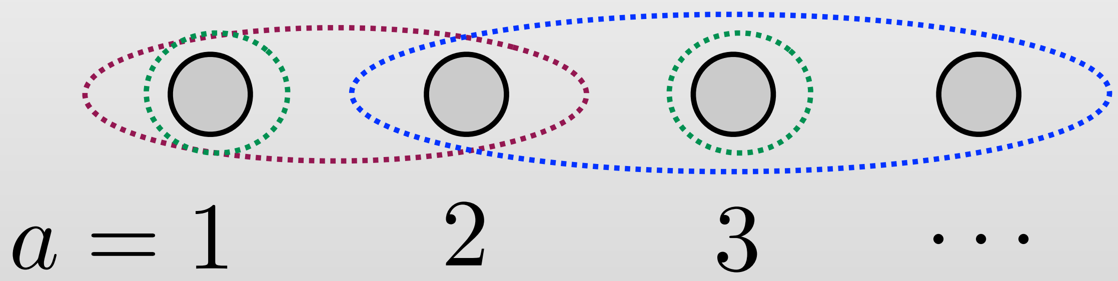
How can I describe a global pure state
using only locality-constrained means?
Which states allow for that?
How can I create them using control?

We'll review three (well-known!) classes…
and their relations



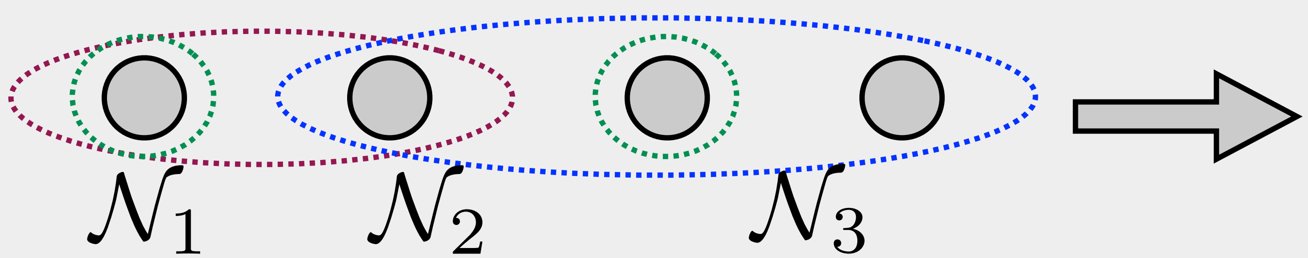
I. States Uniquely Determined By Their Marginals

- Consider *n* finite-dimensional systems, and a *fixed* locality notion.



$\mathcal{N}_1 = \{1,2\}$ $\mathcal{N}_2 = \{1,3\}$
$\mathcal{N}_3 = \{2,3,4\}$ $\cdots$

- Consider a pure state $\rho_d = |\psi\rangle\langle\psi|$ on the whole systems;
- Compute its marginal states $\rho_{\mathcal{N}_i} = \mathrm{Tr}_{\overline{\mathcal{N}}_i}(\rho_d)$ (*reduced density operator*):

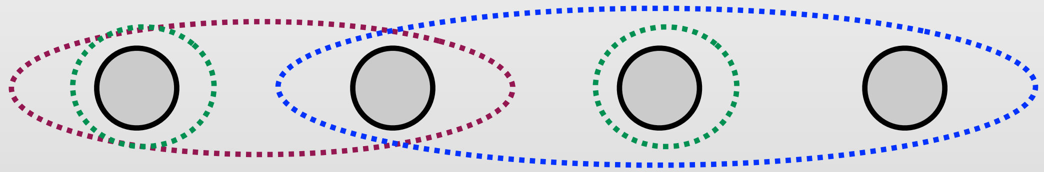


$\rho_{\mathcal{N}_1},\ \rho_{\mathcal{N}_2},\ \rho_{\mathcal{N}_3}$

- If $\rho_d = |\psi\rangle\langle\psi|$ is the unique state that has those marginals, it is said **Uniquely Determined** by its marginals **among All states (UDA)**.
- I can unambiguously identify a UDA state using **local information**; Interest in determining when UDA are generic `[Linden-Wooters]`;
- Practical Interest**: locality-constrained tomography!** `[LaFlamme group, PRL17]`

II. Unique Invariant States for Markov Dynamics

- Consider *n* finite-dimensional systems, and a *fixed* locality notion.



$a = 1 \quad 2 \quad 3 \quad \cdots$

$\mathcal{N}_1 = \{1,2\} \ \mathcal{N}_2 = \{1,3\}$

$\mathcal{N}_3 = \{2,3,4\} \quad \cdots$

- Quantum dynamical semigroup dynamics (Markov, Time-independent), GKS-L generator:

$$\dot{\rho}_t = \mathcal{L}(\rho) = -i[H, \rho_t] + \sum_{k=1}^{p} L_k \rho_t L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \, \rho_t\}$$

- **Quasi-Local (QL)** if

$$\mathcal{L} = \sum_k \mathcal{L}_{\mathcal{N}_k} \otimes \mathcal{I}_{\bar{\mathcal{N}}_k}$$

Sum of neighborhood generators!

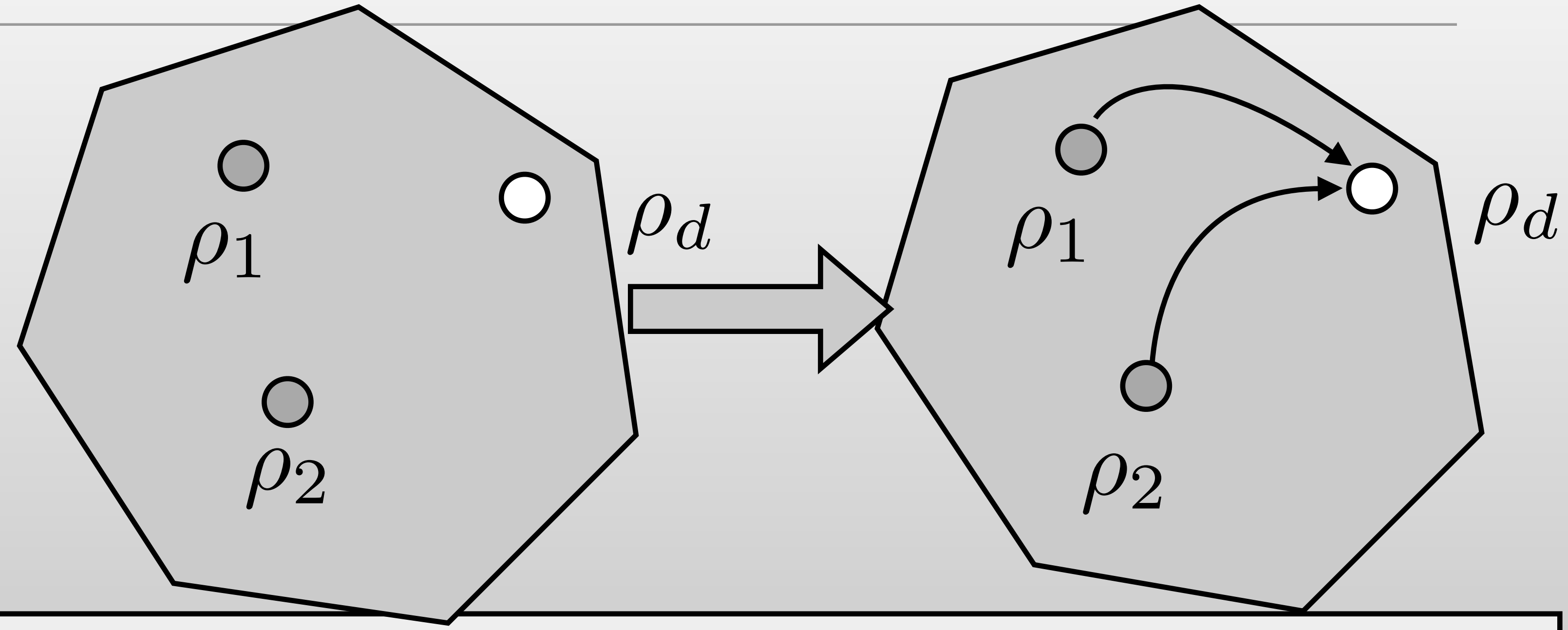
or, explicitly:

$$H = \sum_k H_k, \quad H_k = H_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k} \quad L_{k,j} = L_{\mathcal{N}_k(j)} \otimes I_{\bar{\mathcal{N}}_k}$$

- **We consider unique steady-states.**

II. Unique Invariant States for Markov Dynamics

Fact: a state is it is the **unique equilibrium** for a QDS *if and only if* it is **attracting**



Define: ρ_d is **Quasi-Locally Stabilizable (QLS)** if it is

1) **Invariant:** $\mathcal{L}(\rho_d) = 0$

2) **Attracting:** $\forall \rho \in \mathfrak{D}(\mathcal{H}), \quad \lim_{t \to +\infty} e^{\mathcal{L}t}(\rho) = \rho_d$

for **some quasi-local QDS dynamics.**

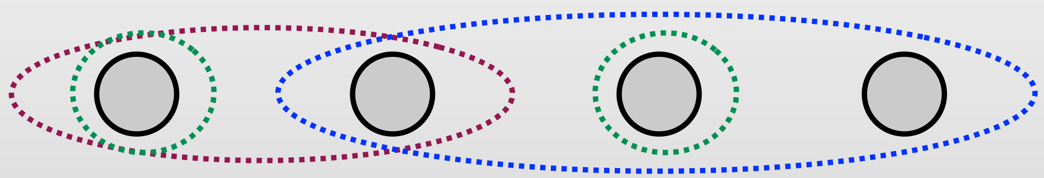
Constraints!

Practical Interest: **Basic task of QIP; Cooling to ground state; *Entanglement generation and preservation;* One-way computing; Metropolis-type sampling**

`[Cirac-Wolf; Kraus-Zoller;… T-Viola and collaborators,2012-19]`

III. Unique Ground States

- Consider *n* finite-dimensional systems, and a *fixed* locality notion.



$a = 1 \quad 2 \quad 3 \quad \cdots$

$\mathcal{N}_1 = \{1,2\}\ \mathcal{N}_2 = \{1,3\}$
$\mathcal{N}_3 = \{2,3,4\} \quad \cdots$

- Consider an **Hamiltonian**: $H = \sum_k H_k, \quad H_k = H_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k}$

- Ground (sub)space: eigenspace of minimum eigenvalue;

$$\lambda_{\rm gs} = \min_{\langle\psi|\psi\rangle=1} (\langle\psi|H|\psi\rangle) \qquad \mathcal{H}_{\rm gs} = \{|\psi\rangle : H|\psi\rangle = \lambda_{\rm gs}\}$$

- If one-dimensional, the ground subspace supports a unique state:
  $\rho_d = \rho_{\rm gs} = |\psi\rangle\langle\psi|$ is called a **Unique Ground State (UGS) for** H **.**

- Unique solution of a (global) minimum-energy variational problem with QL functional;

- Practical Interest: **Cooling; solid-state physics; annealing…**

Three classes of states… relations?

1st aim of this Talk: clarify and challenge intuition!

(INTUITIONS -> "FOLKLORE")

- **UDA vs. QLS (Information vs. Dynamics):**
  both seem to "reconstruct" the final state as the output of local processing/dynamics; are they equivalent?

- **UDA vs UGS (Information vs Energy):**
  we'll see that both solve variational problems with "local structure": UDA can be reformulated as SDP/MEP… are they equivalent?

- **UGS vs QLS (Energy vs Dynamics):**
  we *expect* that a UGS of a local Hamiltonian to be obtainable as the output of *local cooling*.
  UGS => QLS? are they equivalent?

Special Case: Frustration-Free UGS

- Let $\rho = |\psi\rangle\langle\psi|$ be a UGS of a QL Hamiltonian:

$$H = \sum_k H_k, \quad H_k = H_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k}$$

If $|\psi\rangle$ is an eigenvector of minimal energy for ***both the global and neighborhood Hamiltonians***, namely:

$$\langle\psi|H|\psi\rangle = \min \sigma(H) \quad \Longrightarrow \quad \langle\psi|H_k|\psi\rangle = \min \sigma(H_k), \quad \forall k.$$

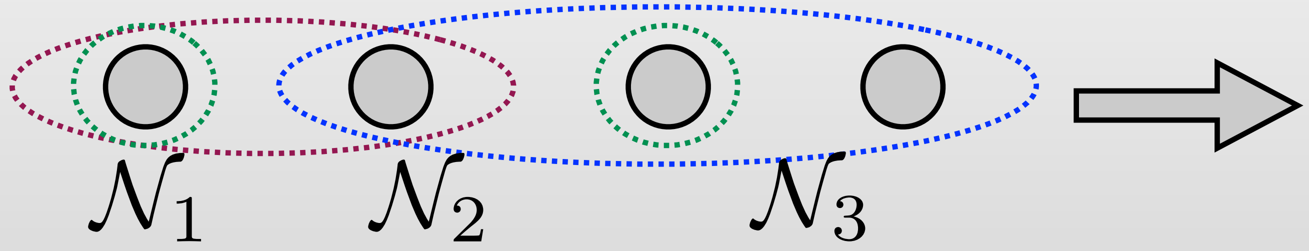
it is said to be a **frustration-free UGS (FF-UGS).**

- If the global ground state is unique, we can obtain it by simultaneously looking for minimal energy on each neighborhood, **and it does not change if we scale the neighborhood terms (no fine-tuning):**

$$H = \sum_k \alpha_k H_k, \ \alpha_1, \ldots, \alpha_k \in \mathbb{R},$$

Investigating FF-UGS

- For each *neighborhood* compute the reduced states;



$$\rho_{\mathcal{N}_1},\ \rho_{\mathcal{N}_2},\ \rho_{\mathcal{N}_3}$$
$$\rho_{\mathcal{N}_k} = \mathrm{trace}_{\bar{\mathcal{N}}_k}(\rho),$$

- For each neighborhood calculate the ***support* of the reduced state**, times the identity on the rest:

$$\mathcal{H}_{\mathcal{N}_k} = \mathrm{supp}(\rho_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k})$$

  - ***Theorem*** [T.-Viola, 2012]***:*** *The following are equivalent:*

i) $\bigcap_k \mathcal{H}_{\mathcal{N}_k} = \mathrm{span}(|\psi\rangle)$ **(implies UDA);**

ii) ρ **is QLS with dissipation only (DQLS):**
no Hamiltonian needed in the stabilizing $\mathcal{L}(\rho)$ **;**

iii) ρ **is FF-UGS, for** $H = \sum_k H_k, \quad H_k = (I - \Pi_{\mathcal{H}_{\mathcal{N}_k}} \otimes I_{\overline{\mathcal{N}}_k})$.

Is Frustration-Free Enough for Pure States?

- **Which states are FF-UGS? Using our test, it turns out that** [T-Viola12,14] **...**
  - All product states are FFS.
  - ***GHZ states (maximally entangled) and W states are not DQLS***
    Unless we have neighborhoods that cover the whole network/nonlocal interactions;
    $$\rho_{\mathrm{GHZ}} = |\Psi\rangle\langle\Psi|, \quad |\Psi\rangle \equiv |\Psi_{\mathrm{GHZ}}\rangle = (|0000\rangle + |1111\rangle)/\sqrt{2}.$$
  - ***Any graph state is DQLS* with respect to the locality induced by the graph;**
    $$U_G|00\ldots0\rangle = |\varphi_{\mathrm{graph},0}\rangle$$
    To each node is assigned a neighborhood, which contain all the nodes connected by edges.
  - ***Generic (injective) MPS/PEPS are DQLS*** for **some** locality definition...
    Neighborhood size may be big! `[see work by Peres-Garcia, Wolf, Cirac and co-workers]`
  - Some **Dicke states** *that are not graph* can be stabilized!
    E.g. on linear graph with NN interaction:
    $$\frac{1}{\sqrt{6}}(|1100\rangle + |1010\rangle + |0110\rangle + |0101\rangle + |0011\rangle + |1001\rangle)$$

UDA vs UGS: General Case

- Let $\Phi_{\mathcal{N}}(\sigma) = \sum_{k,j} \mathrm{Tr}(\sigma B_{\mathcal{N}_k,j}) B_{\mathcal{N}_k,j}.$ the projection on neighborhood terms.
- A pure state is **UDA iff the following has optimum** $\mathrm{Tr}(\rho\sigma_{\mathrm{opt}}) = 1$

$$\begin{aligned} \text{minimize} &: \mathrm{Tr}(\rho\sigma), \\ \text{subject to} &: \Phi_{\mathcal{N}}(\sigma) = \Phi_{\mathcal{N}}(\rho), \\ & \sigma \geq 0. \end{aligned}$$

Imposes marginals

- If I compute the **Lagrange dual** (optimal bound on above), I get:

$$\begin{aligned} \text{maximize} &: -\mathrm{Tr}(H\rho), \\ \text{subject to} &: H + \rho \geq 0 \\ & H = \Phi_{\mathcal{N}}(H), \\ & H = H^{\dagger}. \end{aligned}$$

If optimal value is 1
Equivalent to UGS!
$H_{\mathrm{gs}} := H + I$

- Refined Slater's Condition for Linear Programming:
Strong duality holds - optimal values for both are the same, but…

UDA vs UGS: General Case

- No guarantee that either problem **attain (finite) solutions!**

$$\begin{aligned} \text{minimize} : & \ \mathrm{Tr}(\rho\sigma), \\ \text{subject to} : & \ \Phi_{\mathcal{N}}(\sigma) = \Phi_{\mathcal{N}}(\rho), \\ & \ \sigma \geq 0. \end{aligned}$$

$$\begin{aligned} \text{maximize} : & \ -\mathrm{Tr}(H\rho), \\ \text{subject to} : & \ H + \rho \geq 0 \\ & \ H = \Phi_{\mathcal{N}}(H), \\ & \ H = H^{\dagger}. \end{aligned}$$

Might "explode"!

- **Fact 1:**
  UGS implies UDA, direct simple proof.

- **Fact 2 [details in arXiv:1902.09481]**
  UDA does not imply UGS!
  Counterexample provided by suitably constructed state:

$$|\Psi_6\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_6 + |\overline{\mathrm{D}}\rangle_6\right)$$

$$|\overline{\mathrm{D}}\rangle_6 = \frac{1}{3}\big(|13\rangle_6 + |14\rangle_6 + |15\rangle_6 + |24\rangle_6 + |25\rangle_6 + |26\rangle_6 + |35\rangle_6 + |36\rangle_6 + |46\rangle_6\big).$$

2-Dicke without NN terms;
No UGS via symmetry

QLS vs DQLS: General Case

- Key result to establish relations -
  characterize QLS (but not DQLS, which we know) states

- Define **DQLS subspace** as the smallest subspace that contains and that can be stabilized by dissipation alone:

$$\mathcal{H}_\psi := \bigcap_k \mathcal{H}_{\mathcal{N}_k} \qquad \mathcal{H}_{\mathcal{N}_k} = \text{supp}(\rho_{\mathcal{N}_k} \otimes I_{\overline{\mathcal{N}}_k})$$

- **THM:** $\rho_d = |\psi\rangle\langle\psi|$ not DQLS, **is QLS if and only if there exists an Hamiltonian** H **such that:**

$$H|\psi\rangle = 0$$

$$H|\phi\rangle \notin \mathcal{H}_\psi, \ \forall |\phi\rangle \in \mathcal{H}_\psi$$

Necessary conditions [T-Viola, QIP 2014]; "iff" forthcoming

- **Idea:** invariance is the hard part; Hamiltonian takes care of ,
  then dissipator stabilizes $\mathcal{H}_\psi$.

UGS (and UDA) but not QLS: W states

- W states are UGS, but not QLS

$$|\Psi_{\mathrm{W}}\rangle = (|100\ldots0\rangle + |010\ldots0\rangle + \ldots + |000\ldots1\rangle)/\sqrt{n}.$$

- It is known that W states can be described as UGS of XX-antiferromagnetic type Hamiltonian with transverse magnetic field for NN interactions [D. Bruß, et al.PRA 72,014301(2005)]
- In **[arXiv:1902.09481]** we prove it is *UDA for any locality notion.*

- Consider in same NN interactions/neighborhoods as above.

$$\mathcal{H}_{\mathrm{W}} = \mathrm{span}\{|00\ldots0\rangle, |\mathrm{W}\rangle\}$$

- But it cannot be QLS **for n>5:** we prove H s.t.

$$H|\psi\rangle = 0$$

$$H|\phi\rangle \notin \mathcal{H}_{\psi},\ \forall|\phi\rangle \in \mathcal{H}_{\psi}$$

does not exists [T-Kuravade-Viola, forthcoming].

Recall: necessary conditions for QLS

QLS but not UGS or UDA: GHZ States

- **GHZ states are never DQLS for non-trivial locality:**

$$\rho_{\rm GHZ} = |\Psi\rangle\langle\Psi|, \quad |\Psi\rangle \equiv |\Psi_{\rm GHZ}\rangle = (|000\ldots0\rangle + |111\ldots1\rangle)/\sqrt{2}.$$

By symmetry, $\mathcal{H}_{\Psi_{\rm GHZ}}$ **must contain** $|000\ldots0\rangle, |111\ldots1\rangle$.

Hence the following orthogonal states **must remain** stable for the QL dynamics.

$$\begin{aligned} |\Psi_{\rm GHZ^+}\rangle &= (|000\ldots0\rangle + |111\ldots1\rangle)/\sqrt{2}; \\ |\Psi_{\rm GHZ^-}\rangle &= (|000\ldots0\rangle - |111\ldots1\rangle)/\sqrt{2}; \end{aligned}$$

We need to "select" the right one How?

Hamiltonian making GHZ QLS:

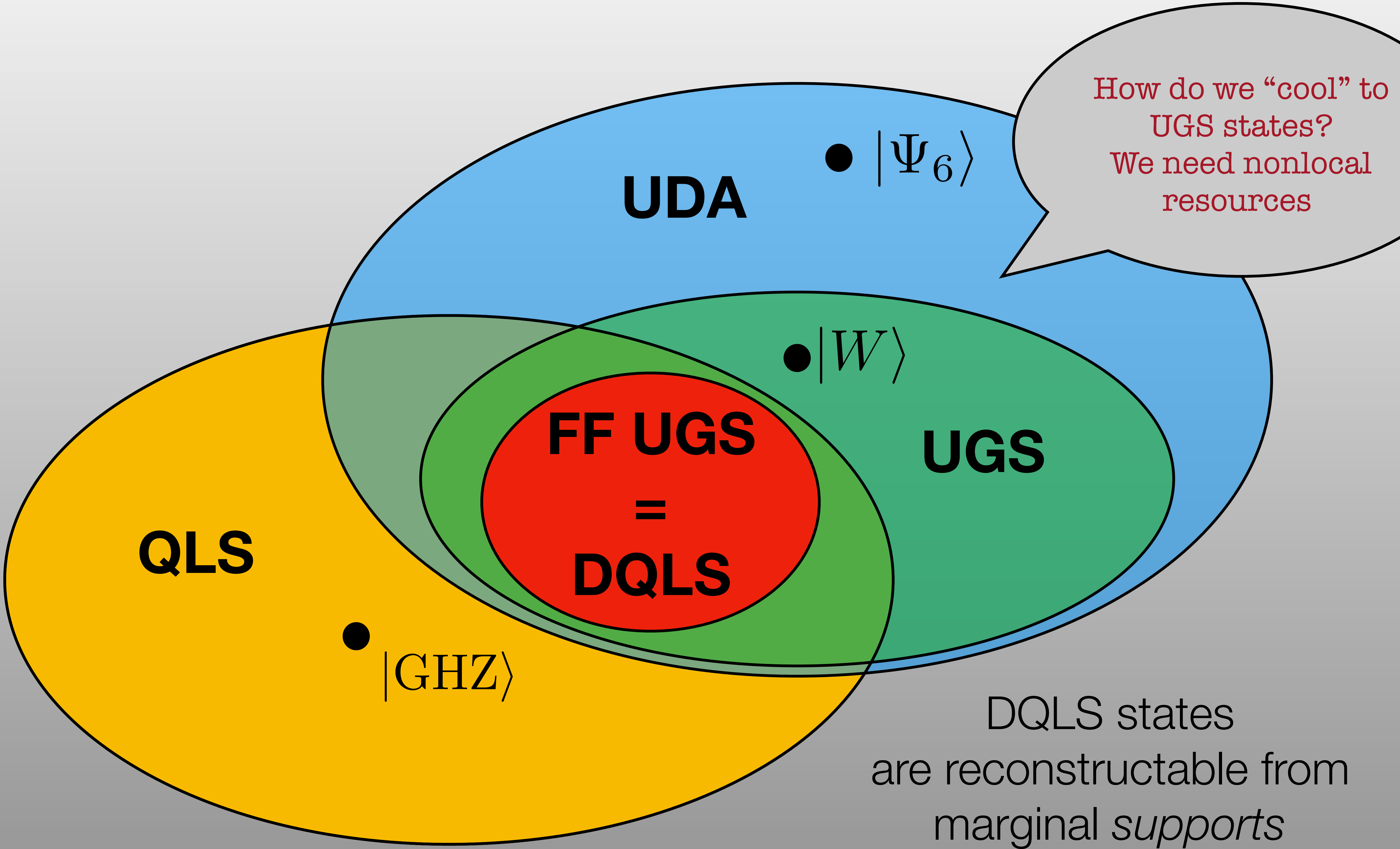
$$\begin{aligned} H|000\ldots0\rangle &= |1\ldots10\ldots0\rangle - |0\ldots01\ldots1\rangle)/\sqrt{2}, \\ H|111\ldots1\rangle &= -|1\ldots10\ldots0\rangle + |0\ldots01\ldots1\rangle)/\sqrt{2}, \end{aligned}$$

$$H|\Psi_{\rm GHZ}\rangle = 0, \quad H|\Psi^\perp_{\rm GHZ}\rangle = \frac{2}{\sqrt{2}}(|1\ldots10\ldots0\rangle - |0\ldots01\ldots1\rangle) \notin \mathcal{H}_{\Psi_{\rm GHZ}}.$$

- **However these are provably never UDA, hence never UGS.**

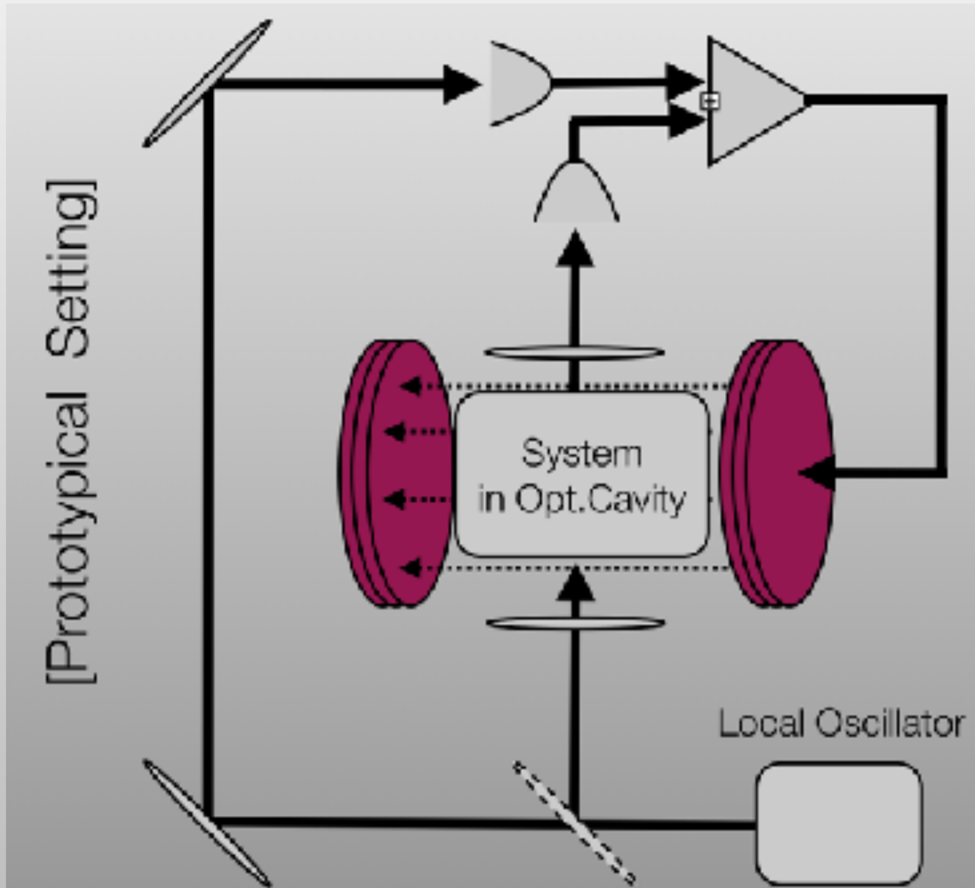
[Walck-Lyons, PRA 79, 2009].

Visual Conclusions



DQLS states
are reconstructable from
marginal *supports*

Recovering Intuition: Feedback Cooling of UGS



Measurements of global Hamiltonian

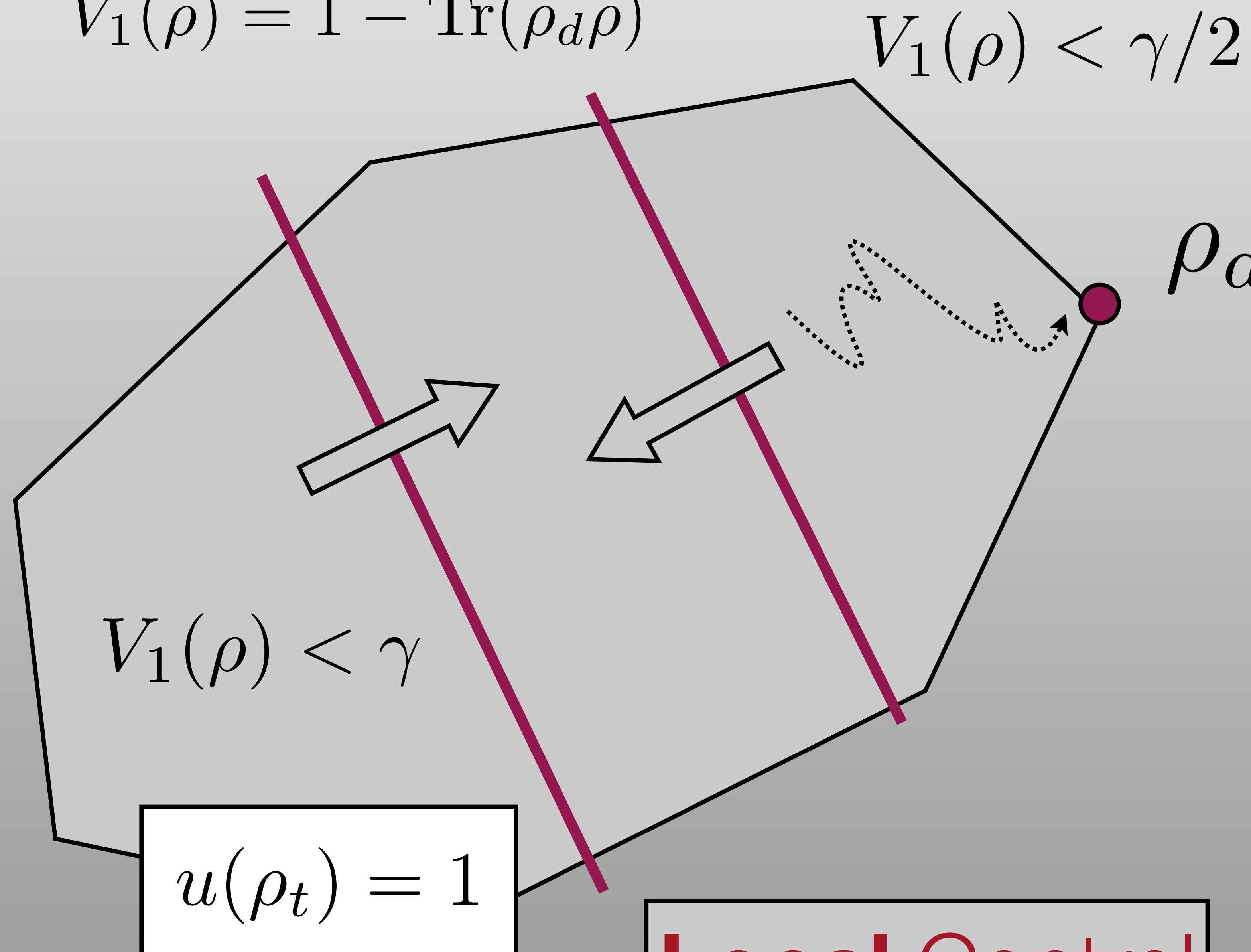
$$M = H_{\mathrm{gs}}$$

$$H = H_{\mathrm{cont}}\, u(\rho_t)$$

$$d\rho_t = \big(-i[H,\rho_t] + \mathcal{D}(M,\rho_t)\big)dt + \mathcal{G}(M,\rho_t)dW_t,$$

$$u(\rho_t) = -\mathrm{tr}(i[H_f,\rho_t]\rho_d)$$

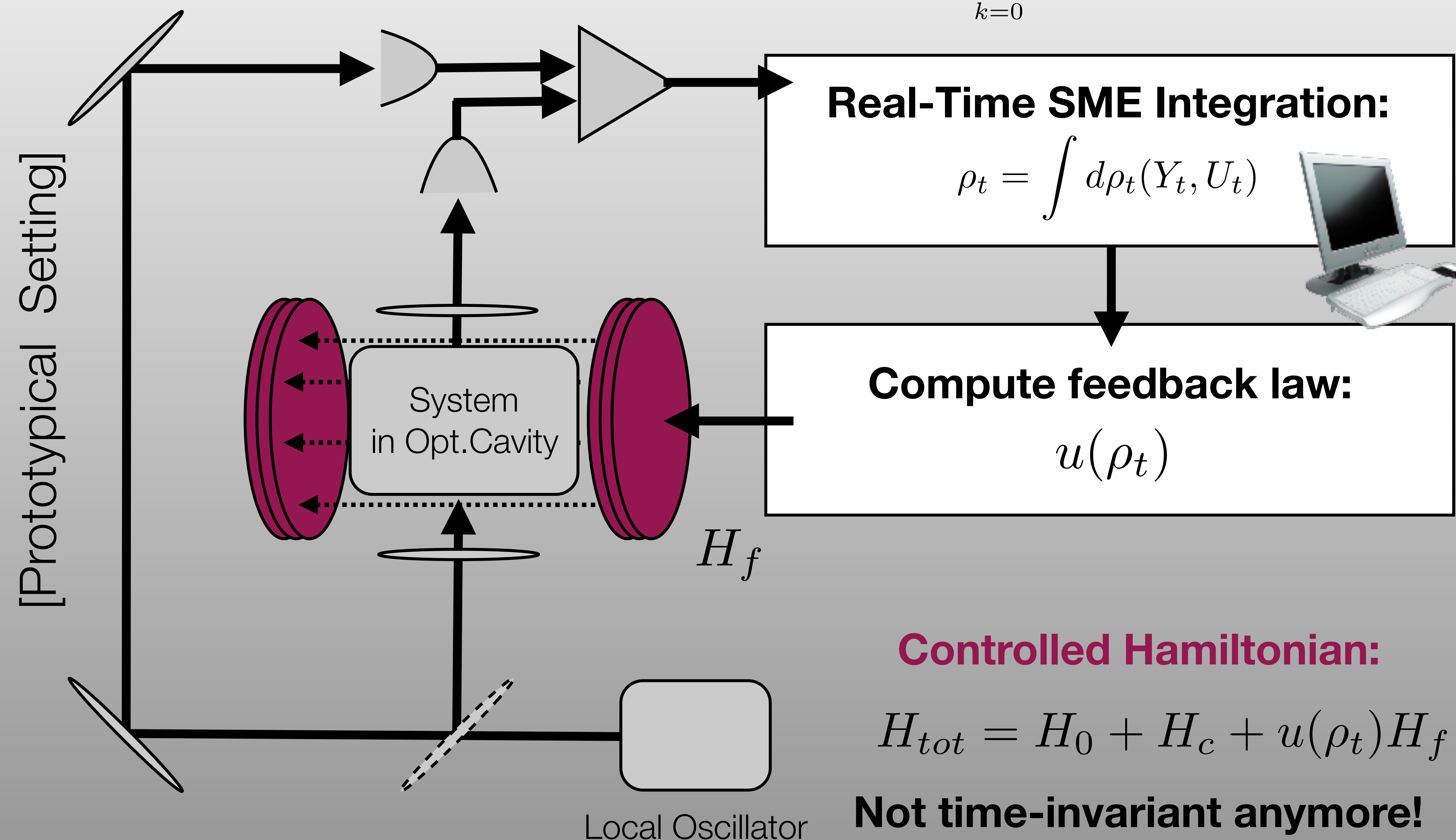
$$V_1(\rho) = 1 - \mathrm{Tr}(\rho_d\rho)$$



Local Control

Estimation-Based Feedback

$$dy_t = \sqrt{\eta}\frac{1}{2}\mathrm{Tr}(\rho_t(L_0 + L_0^\dagger))dt + dW_t, \Rightarrow d\rho_t = \Big(-i[H,\rho_t] + \sum_{k=0}^{K}\mathcal{D}(L_k,\rho_t)\Big)dt + \mathcal{G}(L_0,\rho_t)dW_t,$$



Controlled Hamiltonian:

$$H_{tot} = H_0 + H_c + u(\rho_t)H_f$$

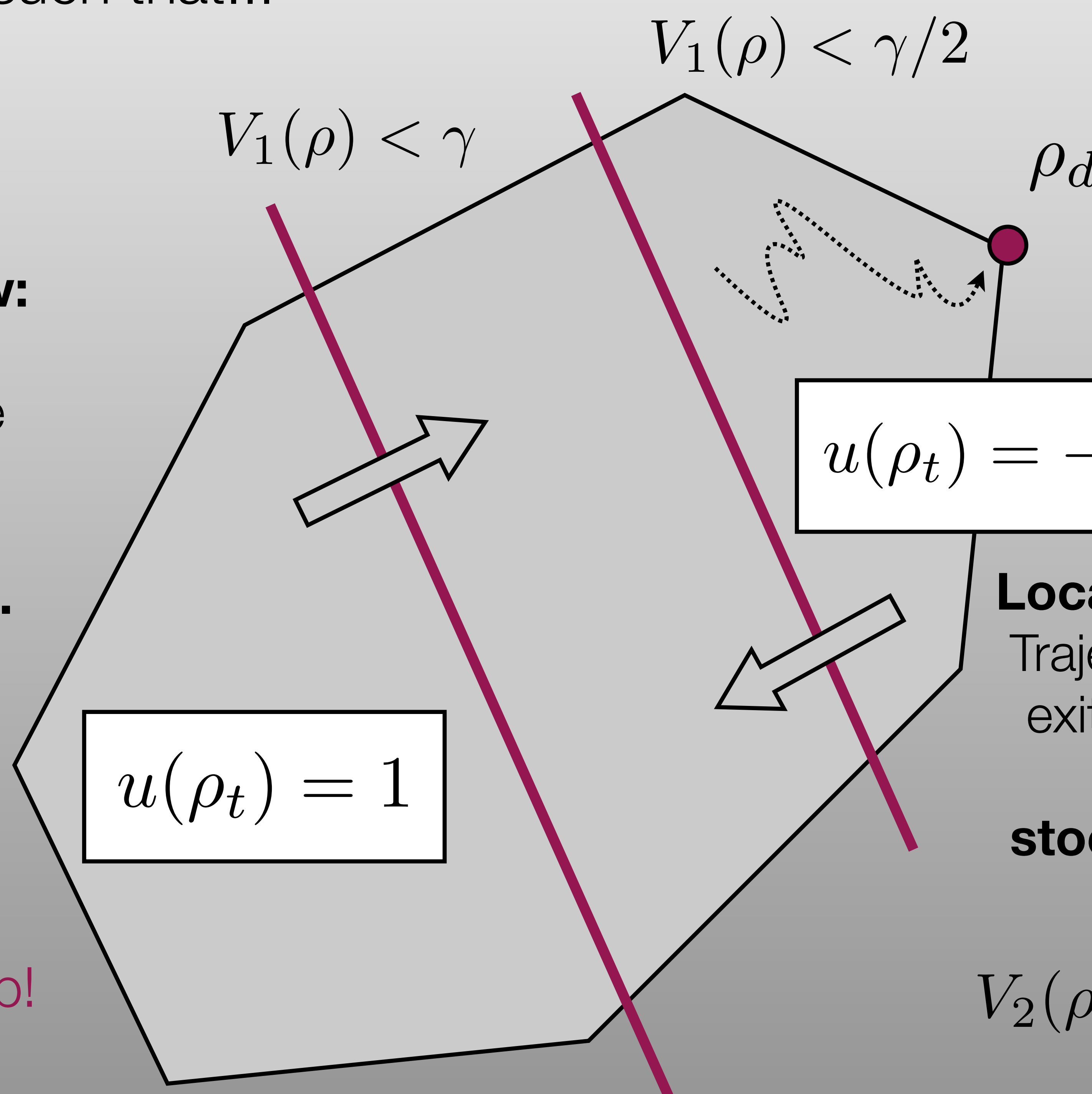
Not time-invariant anymore!

Switching Feedback Controller

Following [Mirrahimi-van Handel SIAM Cont. Opt. 2007]

Define: $V_1(\rho) = 1 - \mathrm{Tr}(\rho_d \rho)$

Exists $\gamma > 0$ such that...



$$u(\rho_t) = -\mathrm{tr}(i[H_f, \rho_t]\rho_d)$$

De-Stabilizing Law:
Trajectories exit the set in finite time in expectation.
Proved using Support Theorem.

Local Stabilizing Law:
Trajectories that do not exit the set converge.
Proved using **stochastic Lyapunov** method, with:

$$V_2(\rho) = 1 - \mathrm{Tr}(\rho_d \rho)^2$$

Open loop control does most of the job!

Advertising: Talk @ DEI

Tuesday June 11
11:30
ROOM 201
DEI

Operator Entropies, Quantum Statistical Mechanics & Large Deviations

Prof. Vojkan Jaksic
McGill Univ.
Mathematics Department