Controllability of the Hydro-Chaplygin sleigh

Marta Zoppello joint work with Nicola Sansonetto
within the GNFM project: "Controllo Geometrico e Pianificazione di Traiettorie di Sistemi Dinamici con Simmetria su Fibrati Principali".

Control days 2019
Padova 9-10 May

Outline

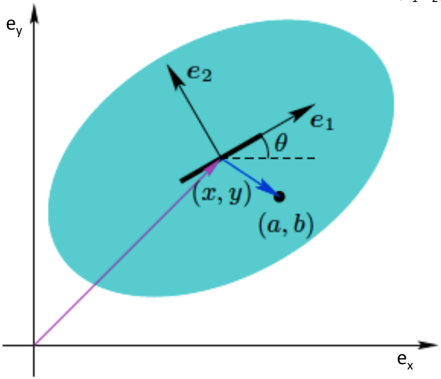
Hydro-Chaplygin sleigh with a moving mass

- Hydrodynamics
- The non-holonomic constraint
- Equation of motion
- Controllability

Hydro-Chaplygin sleigh with a moving mass

(e_x, e_y)= external frame

(e_1, e_2)= body frame



Hydrodynamics

We suppose that the Chaplygin sleigh is immersed in a potential fluid: $u = \nabla\Phi$. Morover we assume the fluid to be incompressible

$$\begin{cases} \Delta\Phi = 0 \quad x \in \mathbb{R}^2 \setminus \mathcal{B} \\ \frac{\partial\Phi}{\partial n} = (V + \Omega \times x) \cdot n \quad x \in \partial\mathcal{B} \\ |\Phi| \to 0 \quad |x| \to \infty \end{cases}$$

with

V	Body translation velocity
Ω	Body angular velocity

$$L = T = T^b + T^f$$

with

$$T^B = \frac{1}{2}\mathbf{v}^T \begin{pmatrix} I + m(a^2+b^2) & -mb & ma & -mb & ma \\ -mb & m+M & 0 & m & 0 \\ ma & 0 & m+M & 0 & m \\ -mb & m & 0 & m & 0 \\ ma & 0 & m & 0 & m \end{pmatrix} \mathbf{v} \qquad (1)$$

$$T^f = \frac{1}{2}\mathbf{v}^T \begin{pmatrix} \frac{\pi\rho}{4}(A^2 - B^2) & 0 & 0 & 0 & 0 \\ 0 & \pi\rho B^2 & 0 & 0 & 0 \\ 0 & 0 & \pi\rho A^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \mathbf{v} \qquad (2)$$

where $\mathbf{v}^T = (\Omega, V, v_a, v_b)$.

Symmetry and reduction

The Hydro-Chaplygin sleigh is invariant under the action of the group $SE(2)$. We have the following

$$\dot{g} = \begin{pmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{pmatrix} = g\xi = g \begin{pmatrix} \Omega \\ v_1 \\ v_2 \end{pmatrix}$$

Which link the velocities in the external frame with the ones in the body one

The Non-Holonomic constraint

The constraint of non-sliding of the blade is **Non-holonomic** $\Rightarrow$ velocity constraint that cannot be derived by a position constraint.

$$-\dot{x}\sin\theta + \dot{y}\cos\theta = 0$$

or in body coordinate

$$v_2 = 0$$

Observation

The constraint is invariant under the action of $SE(2)$.

Equation of motion

General form for a non-holonomic system

$$\begin{cases} \dot{g} = g(J(s)p + A(s)u) \\ \dot{p} = \langle M(s)p, p\rangle + \langle N(s)p, u\rangle + \langle C(s)u, u\rangle \\ \dot{s} = u \end{cases} \quad (3)$$

$\dot{s} = u \Rightarrow$ we are able to assign the velocity of some coordinates as function of time.

u are called controls

For the Chaplygin sleigh

Assumption

The center of mass remains in the contact point and the body frame is aligned with the principal inertia axes

Writing the Lagrangian in the body frame, let

$$p_\Omega = \frac{\partial L}{\partial \Omega} \qquad\qquad p_1 = \frac{\partial L}{\partial v_1}$$

be the conjugate momenta to Ω and v_1.
Solving this equations with respect to Ω and v_1 and taking into account the constraint $v_2 = 0$

$$\Omega = \frac{m(p_1 - mv_a)b + (p_\Omega + mv_b a - mv_a b)(m + M + B^2\pi\rho)}{m^2 b^2 - (m + M + B^2\pi\rho)(I + m(a^2 + b^2) + \frac{(A^2 - B^2)^2\pi\rho}{4})}$$

$$v_1 = \frac{((p_1 - mv_a)(I + ma^2) + m(p_\Omega - mv_b a)b + mp_1 b^2) + (A^2 - B^2)^2\pi\rho(p_1 - mv_a)}{2m^2 b^2 - (m + M + B^2\pi\rho)(I + m(a^2 + b^2) + \frac{(A^2 - B^2)^2\pi\rho}{4})}$$

$$\dot{a} = v_a = u_1$$

$$\dot{b} = v_b = u_2$$

The associated momentum equations are

$$\dot{p}_\Omega = -v_1 m(\frac{u_2}{2} + a\Omega)$$
$$\dot{p}_1 = \Omega m(\frac{u_2}{2} + a\Omega) \quad (4)$$

After solving these last two equations the evolution of the group configuration variables can be obtained from the reconstruction equations together with the constraint

$$\dot{g} = \begin{pmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{pmatrix} = \begin{matrix} \Omega \\ v_1 \cos\theta \\ v_1 \sin\theta \end{matrix} \quad (5)$$

Controllability

We now analyze the kind of trajectories attainable depending on the values of the controls u_i.

Uncontrolled case

First of all let us consider the uncontrolled case, i.e $u = 0$. The reduced equations $(3)_{2,3}$ on Q/G are

$$\begin{cases} \dot{p} = \langle M(s)p, p \rangle \\ \dot{s} = 0 \end{cases} \tag{6}$$

We analyze in first place what happens above an equilibrium solution (p^*, s^*).

Definition

The orbit of (g, p, s) is a relative equilibrium point if $\pi(g, p, s) = (p, s)$ is an equilibrium of the reduced vector field

Proposition (P. Ashwin and I. Melbourne [2])

Let ξ *be the infinitesimal generator of the relative equilibrium* (g, p^*, s^*) *and consider the set*

$$K(\xi) := clos\{\exp(t\xi)|t \in \mathbb{R}\}$$

Then

i) *If the group* G *is compact* $K(\xi)$ *is a torus and therefore the flow of the equations* (3) *above the equilibrium* (p^*, s^*) *is quasiperiodic with at most* $rank(G)$ *frequencies.*

ii) *If* G *is non compact* $K(\xi)$ *is a subgroup of* G *isomorphic either to a torus or to* $\mathbb{R}$*, therefore the flow of the equations* (3) *above the equilibrium* (p^*, s^*) *is either quasiperiodic or a spiral flow.*

one of the two behaviors (quasiperiodic or spiral) is called *generic* depending on the *codimension* of the two sets

$$\mathfrak{g}_{\mathfrak{T}} := \{\xi \in \mathfrak{g} \quad | \quad K(\xi) \text{ is a torus}\}$$
$$\mathfrak{g}_{\mathfrak{R}} := \{\xi \in \mathfrak{g} \quad | \quad K(\xi) \text{ is a subgroup isomorphic to } \mathbb{R}\}$$

The group G acting on the system of the hydro-Chaplygin sleigh is the euclidean group $SE(2)$ therefore we are in the non compact case and according to [2] and taking into account the nonholonomic constraint

$$\mathfrak{g}_{\mathfrak{T}} := \{(v_1, \Omega) \in \mathbb{R} \times \mathbb{R} \quad | \quad \Omega \neq 0\} \cup \{(0,0)\}$$
$$\mathfrak{g}_{\mathfrak{R}} := \{(v_1, \Omega) \in \mathbb{R} \times \mathbb{R} \quad | \quad \Omega = 0, v_1 \neq 0\}$$

The relative equilibria are of two types

i) $a = 0$ and $\Omega \neq 0$,
i.e $a = 0$ and $(m + M + B^2\pi\rho)p_\Omega + mbp_1 \neq 0$

ii) $\Omega = 0$,
i.e $(m + M + B^2\pi\rho)p_\Omega + mbp_1 = 0$

In the case i) the infinitesimal generator of the equilibrium belong to $\mathfrak{g}_{\mathfrak{T}}$ therefore the system moves along a circle. Instead in the case ii) $\xi \in \mathfrak{g}_{\mathfrak{R}}$ and the system moves along a straight line.

Controlled case

Let us assume now to be able to control the velocities of the two coordinates a and b

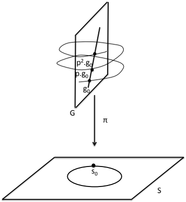
Definition

A nonholonomic mechanical shape control system is *configuration controllable*, if, for every g_0 , g_1. there exist a finite time $T > 0$ and an admissible control $\mathbf{u} : t \in [0, T] \longrightarrow U$ such that a solution $(g(t), p(t))$ of (4) (5) satisfies $g(0) = g_0$ and $g(T) = g_1$.

- Use periodic controls which produce a periodic solutions of the reduced equation whose reconstructed trajectory is either quasiperiodic or a spiral flow depending on the target.

Definition

Let X be a smooth vector field on Q invariant under a proper and free action of a Lie group G on Q. A **relative periodic orbit** for X is the set of the X-orbits in Q which project on a periodic orbit on Q/G. Moreover we will call a periodic orbit on Q/G a **loop**.



Proposition

The phase p is

- *a smooth map;*
- *constant along the X-orbits, i.e. $p \circ \varphi_t^X = p, \forall t$;*
- *equivariant with respect to the conjugation, i.e $p(h \cdot \gamma) = hp(\overline{\gamma})h^{-1}, \quad \forall h \in G, \forall \gamma \in \mathcal{F}$.*

Theorem

Given the dynamical system (3) *after a relative periodic orbit* $\mathcal{F}$*:*

i) *If the group G is compact, the flow over* $\mathcal{F}$ *is quasiperiodic with at most* $rank G + 1$ *frequencies;*

ii) *if G is non compact the flow over* $\mathcal{F}$ *can be either quasiperiodic, or a spiral flow.*

Also in this case if ξ is the infinitesimal generator of the phase p associated to a relative periodic orbit $\mathcal{F}$, we can introduce the semialgebric sets

$$\mathfrak{g}_{\mathfrak{T}} := \{\xi \in \mathfrak{g} \quad | \quad K(\xi) \text{ is a torus}\}$$
$$\mathfrak{g}_{\mathfrak{R}} := \{\xi \in \mathfrak{g} \quad | \quad K(\xi) \text{ is a subgroup isomorphic to } \mathbb{R}\}$$

and, as before, depending of their codimension have a *generic* or *special* reconstructed trajectory.

We can use the non compact statement for the Chaplygin sleigh

$$\mathfrak{g}_{\mathfrak{T}} := \{(v_1, \Omega) \in \mathbb{R} \times \mathbb{R} \quad | \quad \Omega \neq 0\} \cup \{(0,0)\}$$
$$\mathfrak{g}_{\mathfrak{R}} := \{(v_1, \Omega) \in \mathbb{R} \times \mathbb{R} \quad | \quad \Omega = 0, v_1 \neq 0\}$$

and

$$codim(\mathfrak{g}_{\mathfrak{T}}) = 0 \qquad codim(\mathfrak{g}_{\mathfrak{R}}) = 1$$

The hydrodynamic Chaplygin sleigh momentum reduced equations for some choice of periodic controls are

$$\dot{p} = A(t)p + b(t) \tag{7}$$

with A and b $T-$periodic.

Proposition

If one is not an eigenvalue of the monodromy matrix of the T–periodic homogeneous system $\dot{p} = A(t)p$, then (7) has at least one T–periodic solution.

Thus the generic reconstructed behavior after a periodic loop is quasiperiodic and the special one is the spiral flow. Given a loop which of the two trajectories we will have is determined by the value of Ω and v_1 .

- If $\Omega \neq 0$ the hydro Chaplygin sleigh will move along a circle
- If $\Omega = 0$ it will move spiraling along a certain direction.

Numerical Simulations

Using the following parameters: $A = 2$, $B = \frac{2}{\sqrt{3}}$, $\rho = 1$, $M = 1$, $m = 0.01$ and $J = A^2 + B^2 = \frac{16}{3}$. Chosing the following periodic controls

$$u_1(t) = 3\cos t\,, \qquad u_2(t) = 10\sin t\,.$$

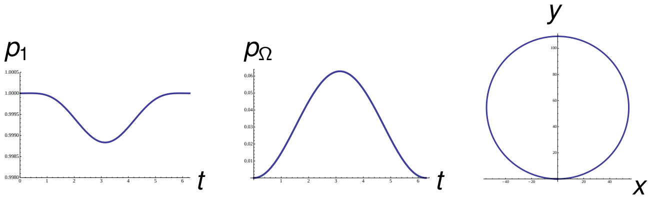


Figure: Reduced periodic solutions of (4) on the left and reconstructed periodic trajectory on the right.

Using instead the following periodic controls

$$u_1(t) = \cos\left(t + \frac{\pi}{2}\right) , \qquad u_2(t) = \sin 2t ,$$

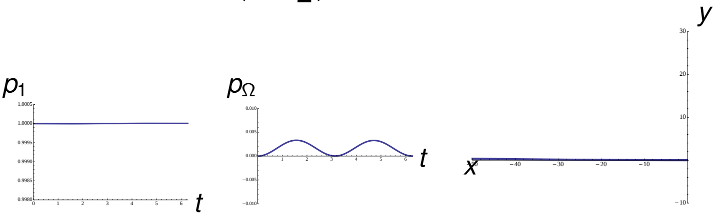


Figure: Reduced periodic solutions of (4), that give rise to a *special* reconstructed behaviour.

Conclusions and perspectives

- The hydro-Chaplygin sleigh is configuration controllable using a moving mass
- Use reconstruction techniques to prove configuration controllability of other non-holonomic mechanical shape control systems

Bibliography

- Y. N. Fedorov, L. C. Garcia-Naranjo
  The hydrodynamic Chaplygin sleigh,
  J. Phys. A: Math. Theor. 43 (2010) 434013
- P. Ashwin and I. Melbourne
  Noncompact drift for relative equilibria and relative periodic orbits
  IOP Publishing Ltd and LMS Publishing Ltd, 1997.
- F. Fassò. S. Passarella, and M. Zoppello, *Motion planning via reconstruction theory.*
  Preprint
- N. Sansonetto and M. Zoppello, *Controllability and trajectory generation for the hydrodynamic Chaplygin sleigh.*
  Under revision.