Transmission Eigenvalues for Maxwell's Equations

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Research supported by grants from AFOSR and NSF

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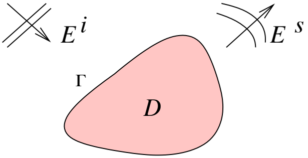
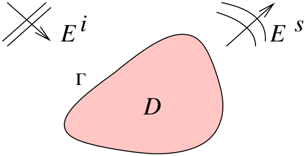
Scattering by an Inhomogeneous Media



$$
\begin{aligned}
\operatorname{curl} E^s - i\omega\mu_0 H^s &= 0 && \text{in } \mathbb{R}^3 \setminus \overline{D} \\
\operatorname{curl} H^s - i\omega\epsilon_0 E^s &= 0 && \text{in } \mathbb{R}^3 \setminus \overline{D} \\
\operatorname{curl} E - i\omega\mu(x) H &= 0 && \text{in } D \\
\operatorname{curl} H - (i\omega\epsilon(x) - \sigma(x) E &= 0 && \text{in } D \\
\nu \times E &= \nu \times (E^s + E^i) && \text{on } \partial D \\
\nu \times H &= \nu \times (H^s + H^i) && \text{on } \partial D \\
\lim_{|x|\to\infty} \left(\sqrt{\mu_0} H^s \times x - \sqrt{\epsilon_0} |x| E^s\right) &= 0 \\
\lim_{|x|\to\infty} \left(\sqrt{\epsilon_0} E^s \times x - \sqrt{\mu_0} |x| H^s\right) &= 0
\end{aligned}
$$

- E^i, H^i incident electro-magnetic field (satisfy the equations in the vacuum).
- ϵ_0 and μ_0 electric permittivity and magnetic permeability in the vacuum.
- $\epsilon(x)$, $\mu(x)$ and $\sigma(x)$ electric permittivity, magnetic permeability and conductivity in the homogeneity.

Scattering by an Inhomogeneous Media



$$
\begin{aligned}
\text{curl curl } E^s - k^2 E^s &= 0 && \text{in } \mathbb{R}^3 \setminus \overline{D} \\
\text{curl } A \, \text{curl } E - k^2 N E &= 0 && \text{in } D \\
\nu \times E &= \nu \times (E^s + E^i) && \text{on } \partial D \\
\nu \times A \, \text{curl } E &= \nu \times (\text{curl } E^s + \text{curl } E^i) && \text{on } \partial D \\
\lim_{|x| \to \infty} (\text{curl } E^s \times x - ik|x|E^s) &= 0
\end{aligned}
$$

- $k = \omega\sqrt{\epsilon_0 \mu_0}$ is the wave number.
- $N = \frac{\epsilon(x)}{\epsilon_0} + i\frac{\sigma(x)}{\omega\epsilon_0}$ (relative permittivity plus conductivity), positive definite 3×3 matrix valued function in $L^\infty(D)$
- $A = \frac{\mu(x)}{\mu_0}$ (relative permeability), positive definite 3×3 matrix valued function in $L^\infty(D)$

How does the

Transmission Eigenvalue Problem

appear ?

Transmission Eigenvalue Problem

Question

Is there an incident electromagnetic field E^i and H^i that does not scatter by the given inhomogeneity?

If so, the total fields E H, and $E_0 := E^i|_D$ and $H_0 := H^i|_D$ satisfy the transmission eigenvalue problem

$$\begin{cases} \text{curl } E = i\omega\mu H \\ \text{curl } H = -i(\omega\epsilon - \sigma)E \end{cases} \text{in } D, \qquad \begin{cases} \text{curl } E_0 = i\omega\mu_0 H_0 \\ \text{curl } H_0 = -i\omega\epsilon_0 E_0 \end{cases} \text{in } D,$$

$$(E_0 - E) \times \nu = 0 \text{ on } \partial D, \quad \text{and} \quad (H_0 - H) \times \nu = 0 \text{ on } \partial D$$

Frequencies ω for which this problem has non-trivial solution are called transmission eigenvalues.

One Field Transmission Eigenvalue Problem

Expressed only in terms of the electric field the transmission eigenvalue problem reads:

$$
\begin{aligned}
\text{curl}\,\text{curl}\, E_0 - k^2 E_0 &= 0 && \text{in} \quad D \\
\text{curl}\, A\, \text{curl}\, E - k^2 N E &= 0 && \text{in} \quad D \\
\nu \times E &= \nu \times E_0 && \text{on} \quad \partial D \\
\nu \times A\, \text{curl}\, E &= \nu \times \text{curl}\, E_0 && \text{on} \quad \partial D
\end{aligned}
$$

Values of $k \in \mathbb{C}$ for which the transmission eigenvalue problem has non trivial solution are called transmission eigenvalues.

TE and Non-Scattering Frequencies

If ω is a transmission eigenvalue and the eigenfunction E_0, H_0 can be extended outside D, $\tilde{E}_0$, $\tilde{H}_0$ as solutions of the same equations, then the scattered field due to $\tilde{E}_0, \tilde{H}_0$ as an incident wave is identically zero.

In general such an extension of E_0, H_0 is not possible. For example if ∂D contains a corner than the inhomogeneity scatters.

- BLASTEN-LIU-XIAO, *Analysis & PDEs. (2021)*

In the scalar case this question is better studied

- BLÅSTEN-PÄIVÄRINTA-SYLVESTER, *Comm. Math. Phys. (2013)*
- VOGELIUS-XIAO, *SIAM Math Analysis (2021)*
- CAKONI-XIAO, *Comm. PDEs (2021)*
- CAKONI-VOGELIUS, *preprint*

Transmission Eigenvalue Problem

Important questions in the context of inverse scattering:

- Study of the resolvent of the transmission eigenvalue problem. It arises in important questions such as uniqueness of inverse problems for inhomogenuous media or justification of linear sampling methods.
- Discreteness of transmission eigenvalues. Methods for solving the inverse problem for inhomogeneous media such as the linear sampling method and factorization method fail at a transmission eigenvalue.
- Existence of transmission eigenvalues
  - Real transmission eigenvalues can be determined from the scattering data.
  - Transmission eigenvalues carry information about material properties.
- Location of transmission eigenvalues in the complex plane is closely related to time depend inverse scattering

F. Cakoni, D. Colton, H. Haddar, Inverse Scattering Theory and Transmission Eigenvalues, CBMS-SIAM (2016)

TE for Dielectric Media

From now on we consider $\sigma = 0$, i.e. dielectric inhomogeneities

1. No real transmission eigenvalues exist in the case of $\sigma > 0$ (if the background is dielectric).
2. Nevertheless, spectral analysis of the transmission eigenvalue problem for conducting inhomogeneities is largely open, even in the scalar case (except for a discreteness result under restrictive assumptions on the coefficients).

The transmission eigenvalue problem for spherically symmetric media.

Spherical Symmetric Case

$D := B_1$ the ball centered at 0 and electromagnetic parameter $\epsilon(r)$, $\mu(r)$ transmission eigenvalues are:

$$\det \mathcal{A}_\ell(k) \det \mathcal{B}_\ell(k) = 0 \qquad \text{for at least one } \ell \in \mathbb{N}, \text{ where}$$

$$\mathcal{A}_\ell(k) := \begin{pmatrix} j_\ell(k) & j_\ell^E(1) \\ (rj_\ell(kr))'\big|_{r=1} & \epsilon_0/\epsilon(r)\,\left(rj_\ell^E(r)\right)'\Big|_{r=1} \end{pmatrix}$$

$$\mathcal{B}_\ell(k) := \begin{pmatrix} j_\ell(k) & j_\ell^H(1) \\ (rj_\ell(kr))'\big|_{r=1} & \mu_0/\mu(r)\,\left(rj_\ell^H(r)\right)'\Big|_{r=1} \end{pmatrix}.$$

where $j_\ell^E(r)$ and $j_\ell^E(r)$ solve ODEs (for constants ϵ, μ they become the spherical Bessel functions).

The part E_0, H_0 of eigenfunctions involve $j_\ell(kr)$ and expressions in terms of $Y_\ell^m(\hat{x})$, and are entire solutions to Maxwell's equations.

The transmission eigenvalues for spherically symmetric media are all non-scattering frequencies

Spherical Symmetric Case

The electromagnetic transmission eigenvalues are transmission eigenvalues for the scalar case:

$$\begin{cases} \nabla \cdot c_0 \nabla u_0 + k^2 n_0 u_0 = 0, \quad \nabla \cdot c \nabla u + k^2 n u = 0, & \text{for } r < 1, \\ u = u_0, \qquad c \dfrac{\partial u}{\partial r} = c_0 \dfrac{\partial u_0}{\partial r}, & \text{for } r = 1. \end{cases}$$

For $\det \mathcal{A}_\ell(k) = 0$: $c(r) = \frac{\epsilon_0}{\epsilon(r) r^2}$, $n(r) = \frac{\mu(r)}{\mu_0 r^2}$, $c_0(r) = \frac{1}{r^2}$, $n_0(r) = \frac{1}{r^2}$

For $\det \mathcal{B}_\ell(k) = 0$: $c(r) = \frac{\mu_0}{\mu(r) r^2}$, $n(r) = \frac{\epsilon(r)}{\epsilon_0 r^2}$, $c_0(r) = \frac{1}{r^2}$, $n_0(r) = \frac{1}{r^2}$

CAKONI-MENG-XIAO, *Inverse Problems and Imaging (2021)*

Spherical Symmetric Case

- There exist an infinite set of real and complex electromagnetic transmission eigenvalues.
- There are infinitely many complex electromagnetic transmission eigenvalues in a neighborhood of real and imaginary axis.

The scalar case of this form

$$\begin{cases} \Delta u_0 + k^2 u_0 = 0, \quad \Delta u + k^2 n u = 0, & \text{for } r < 1, \\ u = u_0, \quad \dfrac{\partial u}{\partial r} = \dfrac{\partial u_0}{\partial r}, & \text{for } r = 1. \end{cases}$$

is better studied, including the solution of inverse spectral problem.

Work by D. Colton, YJ Leung, G. Vodev etc.

D. COLTON, R. KRESS, Inverse Acoustic and Electromagnetic Scattering Theory, Springer, 4th Edition (2019).

A duality between transmission eigenvalues and the scattering poles is established in the scalar case in

F. CAKONI, D. COLTON AND H. HADDAR, A duality between scattering poles and transmission eigenvalues in scattering theory, *Proc. A. 476 (2020).*

Existence of real transmission eigenvalues and monotonicity properties

Transmission Eigenvalues

The transmission eigenvalue problem is non-selfadjoint.

$$
\begin{aligned}
\text{curl curl } E_0 - k^2 E_0 = 0 \qquad & \text{in} \quad D \\
\text{curl } A \,\text{curl } E - k^2 N E = 0 \qquad & \text{in} \quad D \\
\nu \times E = \nu \times E_0 \qquad & \text{on} \quad \partial D \\
\nu \times A \,\text{curl } E = \nu \times \text{curl } E_0 \qquad & \text{on} \quad \partial D
\end{aligned}
$$

In a "natural" variational form this problem reads

$$
\int_D (\text{curl } AE) \cdot (\text{curl } \overline{E}')\, dx - \int_D (\text{curl } E_0) \cdot (\text{curl } \overline{E}_0')\, dx
$$
$$
-k^2 \int_D NE \cdot \overline{E}'\, dx + k^2 \int_D E_0 \cdot \overline{E}_0'\, dx = 0
$$

for all $E', E_0' \in X(D)$, where

$$
X(D) := \{(w, v) \in H(curl, D) \times H(curl, D) \,|\, \nu \times w = \nu \times v \text{ on } \Gamma\}.
$$

Transmission Eigenvalues

Consider $A = I$, letting $k^2 := \tau$ and assume that $N - I > 0$.
It is possible to write

$$\begin{aligned} \text{curl curl } E - \tau N E = 0 \quad & \text{in} \quad D \\ \text{curl curl } E_0 - \tau E_0 = 0 \quad & \text{in} \quad D \\ \nu \times E = \nu \times E_0 \quad & \text{on} \quad \Gamma \\ \nu \times \text{curl } E = \nu \times \text{curl } E_0 \quad & \text{on} \quad \Gamma \end{aligned}$$

$E, E_0 \in L^2(D)$, for the difference $W = E - E_0 \in H_0(\text{curl}^{\,2}, D)$ as

$$(\nabla \times \nabla \times -\tau)(N - I)^{-1}(\nabla \times \nabla \times -\tau N)W = 0$$

and in the variational form, for all $W' \in H_0(cur^2, D)$

$$\int_D (N - I)^{-1}(\nabla \times \nabla \times W - \tau N W)(\nabla \times \nabla \times \overline{W}' - \tau \overline{W}')\, dx = 0$$

$$H_0(\text{curl}^{\,2}, D) = \{u \in H(\text{curl}\,, D), \text{curl } u \in H(\text{curl}\,, D), \nu \times u = 0, \nu \times \text{curl } u = 0 \text{ on } \partial D\}$$

Existence of Real Transmission Eigenvalues

$$(\mathbb{A}_\tau - \tau\mathbb{B})u = 0 \quad \text{in} \;\; H_0(\text{curl}^{\,2}, D)$$

$$\begin{aligned}
(\mathbb{A}_\tau W, W') &= \int_D (N - I)^{-1}(\text{curl curl } W - \tau W) \cdot (\text{curl curl } \overline{W'} - \tau \overline{W'})\, dx \\
&+ \tau^2 \int_D W \cdot \overline{W'}\, dx \\
(\mathbb{B} W, W') &= \int_D \text{curl } W \cdot \text{curl } \overline{W'}\, dx
\end{aligned}$$

The mapping $\tau \to \mathbb{A}_\tau$ is continuous from $(0, +\infty)$ to the set of self-adjoint coercive operators from $H_0(\text{curl}^{\,2}, D) \to H_0(\text{curl}^{\,2}, D)$.

$\mathbb{B} : H_0(\text{curl}^{\,2}, D) \to H_0(\text{curl}^{\,2}, D)$ is self-adjoint, compact and non-negative.

Existence of Real Transmission Eigenvalues

Now we consider the generalized eigenvalue problem

$$(\mathbb{A}_\tau - \lambda(\tau)\mathbb{B})W = 0 \quad \text{in} \;\; H_0^2(\text{curl}^{\,2}, D)$$

For a fixed $\tau > 0$ there exists an increasing sequence of eigenvalues $\lambda_j(\tau)_{j\geq 1}$ going to $+\infty$ and satisfy Courant-Fisher max-min principle.

$$\lambda_j(\tau) = \min_{\mathcal{U}\subset\mathcal{H}_j} \left(\max_{W\in\mathcal{U}\subset\{0\}} \frac{(\mathbb{A}_\tau W, W)}{(\mathbb{B}W, W)} \right)$$

where $\mathcal{H}_j$ denotes all j-dimensional subspaces $\mathcal{H}_j$ of $H_0^2(\text{curl}^{\,2}, D)$ that don't intersect

$$\text{Kern}(\mathbb{B}) = \left\{ U \in H_0^2(\text{curl}^{\,2}, D),\; u := \nabla\varphi,\; \varphi \in H^1(D) \right\}.$$

τ is a transmission eigenvalue if and only $\lambda_j(\tau) = \tau$

Existence of Real Transmission Eigenvalues

- For $0 < \tau_0 < \dfrac{\lambda_1(D)}{N_{max}}$,

$$\mathbb{A}_{\tau_0} - \tau_0 \mathbb{B} > 0 \quad \text{on} \quad H_0^2(\text{curl}^{\,2}, D),$$

where $\lambda_1(D)$ is the first Dirichlet eigenvalue for $-\Delta$ in D.

- There exists τ_1 such that

$$\mathbb{A}_{\tau_1} - \tau_1 \mathbb{B} \leq 0$$

on some m dimensional subspace of $H_0^2(\text{curl}^{\,2}, D)$ that daesn't intersect Kern($\mathbb{B}$). This can be done for m arbitrarily large

Max-min principle for $\lambda_j(\tau)$ implies each $\lambda_j(\tau) = \tau$ for $j = 1, \dots, m$, has at least one solution in $[\tau_0, \tau_1]$ meaning that there exists m transmission eigenvalues counting multiplicity within $[\tau_0,\ \tau_1]$.

Existence of Real Transmission Eigenvalues

Theorem

Assume that $N_{min} > 1$. Then, there exists an infinite discrete set of real transmission eigenvalues τ_j accumulating at $+\infty$ and satisfying

$$\tau_j(N_{max}, B_1) \le \tau_j(N_{max}, D) \le \tau_j(N(x), D) \le \tau_j(N_{min}, D) \le \tau_j(N_{min}, B_2)$$

where $B_2 \subset D \subset B_1$.

Similar results can be obtained for the case when
$0 < N_{min} \le N(x) \le N_{max} < 1$.

CAKONI-GINTIDES-HADDAR, *SIAM J. Math. Anal. (2010)*

This idea is extended to the case of $\mu \neq \mu_0$ (i.e. $A \neq I$) only if μ is constant.

CAKONI-KIRSCH, *Int. Jour. Comp. Sci. Math. (2010)*

Discreteness and location of electromagnetic transmission eigenvalues.

Discreteness of Transmission Eigenvalues

The discreteness is proven using integral equations approach for the case $A = I$, and $N := nI$ where $n > 1$ of $n < 1$ constant in a neighborhood of ∂D

F. CAKONI, H. HADDAR, S. MENG, *J. Int. Eqns. Appl. (2015).*

The most up-to-date result on the discreteness and location of transmission eigenvalues is in

F. CAKONI, H.M. NGUYEN, On the Discreteness of Transmission Eigenvalues for the Maxwell Equations *SIAM J. Math Analysis (2021).*

where we improve substantially the result in

L. CHESNEL, *Inverse Problems (2012).*

Discreteness of Transmission Eigenvalues

$$\begin{cases} \text{curl } E = i\omega\mu H \\ \text{curl } H = -i\omega\epsilon E \end{cases} \text{in } D, \qquad \begin{cases} \text{curl } E_0 = i\omega\mu_0 H_0 \\ \text{curl } H_0 = -i\omega\epsilon_0 E_0 \end{cases} \text{in } D,$$

$$(E_0 - E) \times \nu = 0 \ \text{ on } \partial D, \quad \text{and} \quad (H_0 - H) \times \nu = 0 \ \text{ on } \partial D$$

Theorem

Assume that

i) $\epsilon,\ \mu,\ \epsilon_0,\ \mu_0$ are of class C^1 in some neighborhood of ∂D,

ii) $\epsilon,\ \mu,\ \epsilon_0,\ \mu_0$ are isotropic on ∂D,

iii) $\epsilon \neq \epsilon_0, \quad \mu \neq \mu_0, \quad \epsilon/\mu \neq \epsilon_0/\mu_0 \quad$ on ∂D.

The set of the transmission eigenvalues is discrete with ∞ as the only possible accumulation point.

Discreteness of Transmission Eigenvalues

Our analysis is inspired by the concept of complementary conditions due to Agmon, Douglis, and Nirenberg in their celebrated papers

AGMON-DOUGLIS-NIRENBERG, *Comm. Pure Appl. Math (1959), (1964)*

adapted to Maxwell's equations for media with sign changing coefficients in

NGUYEN-SIL, *Comm. Math Physics (2020).*

Discreteness of Transmission Eigenvalues

We study the resolvent

$$\begin{cases} \text{curl } E = i\omega\mu H + J_e \\ \text{curl } H = -i\omega\epsilon E + J_m \end{cases} \text{in } D, \qquad \begin{cases} \text{curl } E_0 = i\omega\mu_0 H_0 + J_e^0 \\ \text{curl } H_0 = -i\omega\epsilon_0 E_0 + J_m^0 \end{cases} \text{in } D,$$

$$(E_0 - E) \times \nu = 0 = 0 \text{ on } \partial D, \quad \text{and} \quad (H_0 - H) \times \nu = 0 \text{ on } \partial D$$

Step 1. We first analyze the problem in the half space $\mathbb{R}^3_+ := \mathbb{R}^3_{x_3>0}$ using 2d Fourier transform in $\mathbb{R}^3_0 := \mathbb{R}^3_{x_3=0}$. We prove explicit a priori estimates for ω in a region in $\mathbb{C}$.

$$C\left(\|(E,H,E_0,H_0)\|_{H^1(\mathbb{R}^3_+)} + |\omega|\,\|(E,H,E_0,H_0)\|_{L^2(\mathbb{R}^3_+)}\right) \le \|(J_e,J_m,J_e^0,J_m^0)\|_{L^2(\mathbb{R}^3_+)}$$

$$+\frac{1}{|\omega|}\|(\text{div}\, J_e, \text{div}\, J_m, \text{div}\, J_e^0, \text{div}\, J_m^0)\|_{L^2(\mathbb{R}^3_+)} + \frac{1}{|\omega|}\|(J_{e,3}-J_{e,3}^0, J_{m,3}-J_{m,3}^0)\|_{H^{1/2}(\mathbb{R}^3_0)}$$

Discreteness of Transmission Eigenvalues

Step 2. Applying the standard technique of straitening locally the boundary, together with decaying behavior

$$\|(E,H)\|_{L^2(D\setminus D_\delta)} \le c_1 e^{-c_2|\omega|}\|(E,H)\|_{L^2(D_\delta)} + c_1\|(J_e,J_m)\|_{L^2(D)}$$

of the solution of

$$\left\{\begin{array}{c} \text{curl}\, E = i\omega\mu H + J_e \\ \text{curl}\, H = -i\omega\epsilon E + J_m \end{array}\right. \quad \text{in } D$$

and assuming *i*), *ii*), *iii*) we then prove in this region of ω similar a priori estimates as above.

Here D_δ the region D without a neighborhood of ∂D.

Discreteness of Transmission Eigenvalues

Step 3 Define the Hilbert space

$$\mathbf{H}(D) = \Big\{(u, v, u_0, v_0) \in L^2; \operatorname{div}(\epsilon u) = \operatorname{div}(\mu v) = \operatorname{div}(\epsilon_0 u_0) = \operatorname{div}(\mu_0 v_0) = 0 \text{ in } D$$

$$\text{and } \epsilon u \cdot \nu - \epsilon_0 u_0 \cdot \nu = \mu v \cdot \nu - \mu_0 v_0 \cdot \nu = 0 \text{ on } \partial D\Big\}$$

equipped with $\quad \|(u, v, u_0, v_0)\|_{\mathbf{H}(D)} = \|(u, v, u_0, v_0)\|_{L^2(D)}.$

Then we show that TE are the eigenvalues of the compact operator

$$\begin{aligned} T: \quad & \mathbf{H}(D) && \to && \mathbf{H}(D) \\ & (\tilde{J}_e, \tilde{J}_m, \tilde{J}_e^0, \tilde{J}_m^0) && \mapsto && (E, H, E_0, H_0), \end{aligned}$$

where (E, H, E_0, H_0) is unique solution discussed in Step 2 with

$$(J_e, J_m, J_e^0, J_m^0) = (\mu \tilde{J}_m, -\epsilon \tilde{J}_e, \mu_0 \tilde{J}_m^0, -\epsilon_0 \tilde{J}_e^0).$$

(Note: Since $\operatorname{div} J_e = \operatorname{div} J_m = \operatorname{div} J_e^0 = \operatorname{div} J_m^0 = 0$, it follows that

$$\operatorname{div}(\epsilon E) = \operatorname{div}(\mu H) = \operatorname{div}(\epsilon_0 E_0) = \operatorname{div}(\mu_0 H_0) = 0.$$

thus $(E, H, E_0, H_0) \in \mathbf{H}(D)$)

Location of Transmission Eigenvalues

Theorem

Assume that

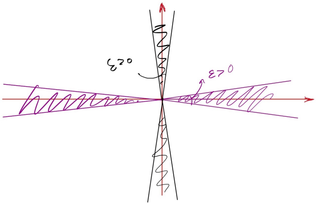
i) $\epsilon,\ \mu,\ \epsilon_0,\ \mu_0$ are of class C^1 in some neighborhood of ∂D,

ii) $\epsilon,\ \mu,\ \epsilon_0,\ \mu_0$ are isotropic on ∂D,

iii) $\epsilon \neq \epsilon_0, \quad \mu \neq \mu_0, \quad \epsilon/\mu \neq \epsilon_0/\mu_0 \quad$ on ∂D.

For $\gamma > 0$, there exists $\omega_0 > 0$ such that if $\omega \in \mathbb{C}$ with $|\Im(\omega^2)| \geq \gamma|\omega|^2$ and $|\omega| \geq \omega_0$, then ω is not a transmission eigenvalue.

Location of Transmission Eigenvalues



The above theorem states that TE frequencies ω are located in wedges of arbitrary small angle around both real and imaginary axis. Spherically symmetric media case shows the location is optimal.

1. In particular for the case of Maxwell's equation there is no half complex plane free of the transmission eigenvalues, even if there is no contrast in magnetic permeability.
2. This can complicate the solvability of the interior transmission problem for the time dependent Maxwell's equation.
3. This situation is contrary to the scalar case modeled by

$$\Delta u + \omega^2 n(x) u = 0$$

all transmission eigenvalues lie in a strip around real axis.

CAKONI-MONK-SELGAS, *Analysi & PDEs (2021).*

Spectral Analysis

Finally there is a lot of work to be done on the spectral analysis of the transmission eigenvalue problem for Maxwell's equations, including completeness of the eigenfunctions and Weyl's asymptotic laws for the counting function of eigenvalues.

Some preliminary work on completeness is done for the case of $\mu = \mu_0$ (i.e. $A = I$) and C^∞ regularity for the domain and coefficients

HADDAR-MENG, *Journal de Mathématiques Pures et Appliquées (2018).*

Applications of transmission eigenvalue in electromagnetic inverse scattering theory

Applications in Inverse Scattering

Transmission eigenvalues can be determine from scattering data

To fix the ideas we take a plane wave incident field

$$E^i(x, d, p) := ik(d \times p) \times d e^{ikx\cdot d}$$

propagating in the direction $d \in \mathbb{S}^2$ with polarization $p \in \mathbb{R}^3$

The scattered field E^s has the asymptotic behavior

$$E^s(x; d, p, k) = \frac{e^{ik|x|}}{|x|} \left\{ E_\infty(\hat{x}; d, p, k) + O\left(\frac{1}{|x|}\right) \right\}$$

as $|x| \to \infty$ uniformly with respect $\hat{x} = x/|x|$.

$E_\infty(\hat{x}, d, p, k)$ is the far field pattern of the scattered field E^s.

Scattering Data

$E_\infty(\hat{x}; d, p, k)$, for $d \in \mathbb{S}^2_i \subset \mathbb{S}^2$, $\hat{x} \in \mathbb{S}^2_m \subset \mathbb{S}^2$ and (possibly) $k \in [k_1,\ k_2]$.

Far Field Operator

The far field operator $F : L^2_t(\mathbb{S}^2) \to L^2_t(\mathbb{S}^2)$ is defined by

$$(Fg)(\hat{x}) := \int_{\mathbb{S}^2} E_\infty(\hat{x}; d, g(d), k) ds_d.$$

- Fg is the far field pattern of the scattered due to

$$E_g(x) := \int_{\mathbb{S}^2} e^{ikx\cdot d} g(d) ds_d \qquad g \in L^2_t(\mathbb{S}^2) \quad (g(\hat{x}) \cdot \hat{x} = 0)$$

known as a electric Herglotz wave function

Theorem

F is injective and has dense range if and only if k is not a transmission eigenvalue

$$\begin{aligned} \text{curl curl } E_0 - k^2 E_0 = 0, \quad \text{curl } A \text{ curl } E - k^2 N E = 0 \qquad &\text{in} \quad D \\ \nu \times E = \nu \times E_0, \quad \nu \times A \text{ curl } E = \nu \times \text{curl } E_0 \qquad &\text{on} \quad \partial D \end{aligned}$$

such that $E_0 := E_g$ is an electric Herglotz wave function.

Linear Sampling Method

CAKONI-COLTON-HADDAR (2010) *C. R. Math. Acad. Sci. Paris*

We use the far field equation for $z \in D$

$$(Fg)(\hat{x}) = E_\infty(\hat{x}, z, q, k) \qquad \text{for } g \in L^2_t(\mathbb{S}^2) \text{ and } k \in [k_0,\, k_1]$$

Assume that A and N are such that the interior transmission problem is Fredholm, and let g_α be the regularized solution of the far field equation with E_{g_α} corresponding electric Herglotz function.

Let $B \subset D$ be a ball inside D. Then

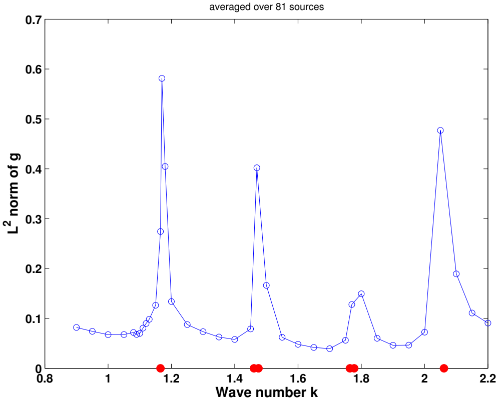
$\|E_{g_\alpha}\|_{L^2(D)}$ is bounded for all $z \in B$ as $\alpha \to 0$

if and only if

k is not a transmission eigenvalue

Computation of Real Transmission Eigenvalues

Results for an isotropic sphere of unit radius. DUE TO P. MONK.



$N = 16I$

Solving the far-field equation for several source points z inside the sphere gives obvious peaks at the first transmission eigenvalue. Red dots indicate exact transmission eigenvalues.

Non-destructive Testing and TE

Given the measured $k_1(D, N(x))$, we now compute a constant n such that $k_1(D, N(x)) = k_1(D, n)$. Then the monotonicity result implies that

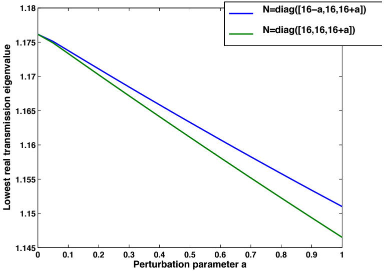
$$N_{min} \leq n \leq N_{max}$$

In the isotropic case $N(x) := n(x)I$, the above constant n gives

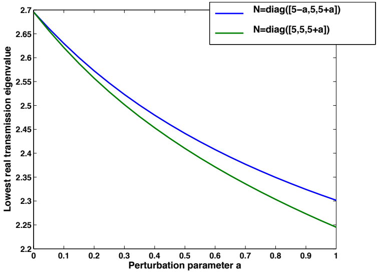
$$n \approx \frac{1}{|D|} \int_D n(x)\, dx$$

Numerical Examples

due to MONK-SUN.



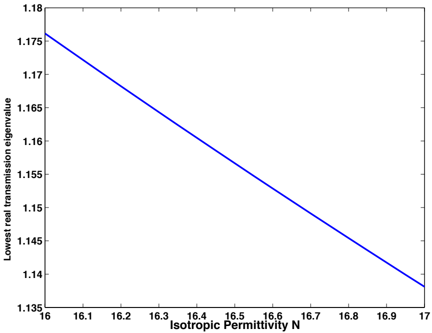
Perturbation of $N = 16I$



Perturbation of $N = 5I$

Numerical Examples (cont)

For any measured transmission eigenvalue for anisotropic N we compute the isotropic n discussed previously



N	$\lambda_{1,D,N(x)}$	n
$\mathrm{diag}([15.5, 16, 16.5])$	1.163	16.33
$\mathrm{diag}([15, 16, 17])$	1.151	16.65
$\mathrm{diag}([16, 16, 16.5])$	1.161	16.38
$\mathrm{diag}([16, 16, 17])$	1.146	16.77

Numerical Examples (cont)

Similar experiment for lower N

N	$\lambda_{1,D,N(x)}$	n
$\mathrm{diag}([4.5, 5, 5.5])$	2.442	5.339
$\mathrm{diag}([4, 5, 6])$	2.302	5.631
$\mathrm{diag}([5, 5, 5.5])$	2.410	5.397
$\mathrm{diag}([5, 5, 6])$	2.245	5.778