Time-harmonic electromagnetic waves in anisotropic media

Patrick Ciarlet[†]

with D. Chicaud[†], A. Modave[†]

[†]POEMS, ENSTA Paris, Institut Polytechnique de Paris, France

Outline

- **Introduction/Motivation**
- The model and its well-posedness
- A priori regularity of the fields
- Discretization and error estimates
- Numerical illustrations
- Conclusion and perspectives

Introduction/Motivation

- A starting point:
  - Study of *electromagnetic wave propagation in plasmas*, a popular model in plasma physics [Stix'92].
  - Time-harmonic model, rigorously derived and studied mathematically in [PhD-Hattori'14], [Back-Hattori-Labrunie-Roche-Bertrand'15].
- The model:
  - Find $\boldsymbol{E} \in \boldsymbol{H}(\mathbf{curl};\Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \mid \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\}$ governed by:
    - $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E} - \dfrac{\omega^2}{c^2}\underline{\underline{\boldsymbol{K}}}\boldsymbol{E} = 0$ in Ω, where

$$\underline{\underline{\boldsymbol{K}}}(\boldsymbol{x}) = \begin{pmatrix} S(\boldsymbol{x}) & -i\,D(\boldsymbol{x}) & 0 \\ i\,D(\boldsymbol{x}) & S(\boldsymbol{x}) & 0 \\ 0 & 0 & P(\boldsymbol{x}) \end{pmatrix}$$

      is the anisotropic *plasma response tensor* (S, D, P $\mathbb{C}$-valued coefficients) ;
    - a boundary condition on $\partial\Omega$ (with non-zero data): value of $\boldsymbol{E} \times \boldsymbol{n}_{|\partial\Omega}$, of $\mathbf{curl}\,\boldsymbol{E} \times \boldsymbol{n}_{|\partial\Omega}$, or of $\boldsymbol{E} \times \boldsymbol{n}_{|\Gamma}$ and $\mathbf{curl}\,\boldsymbol{E} \times \boldsymbol{n}_{|\partial\Omega\setminus\Gamma}$, is given.

Introduction/Motivation

- Key properties of the anisotropic plasma response tensor:
  - $\underline{\underline{\boldsymbol{K}}}$ is a normal, non-hermitian, matrix field of $\underline{\underline{\boldsymbol{L}}}^{\infty}(\Omega)$;
  - $\underline{\underline{\boldsymbol{K}}}$ fulfills an ellipticity condition:

$$\exists \eta > 0,\ \forall \boldsymbol{z} \in \mathbb{C}^3,\ \eta |\boldsymbol{z}|^2 \leq \Im[\boldsymbol{z}^* \underline{\underline{\boldsymbol{K}}} \boldsymbol{z}] \quad \text{ae in } \Omega.$$

- Main results from [PhD-Hattori'14], [Back-Hattori-Labrunie-Roche-Bertrand'15]:
  - the model, expressed variationally, involves a sesquilinear form that is automatically coercive: hence it is well-posed.
  - *Plain, mixed* and *augmented variational formulations* are analyzed.
  - *Discretization* is achieved with the help of the piecewise $\boldsymbol{H}^1$-conforming Finite Element Method (vector-valued Lagrange FE), together with a Fourier expansion (Ω is a torus).
  - A *Domain Decomposition Method* is proposed.
  - No numerical analysis is provided.

Introduction/Motivation

- Our goals and assumptions:
  - Study a time-harmonic electromagnetic wave propagation model with $\underline{\underline{\boldsymbol{L}}}^{\infty}$, anisotropic magnetic permeability $\underline{\underline{\boldsymbol{\mu}}}$ and electric permittivity $\underline{\underline{\boldsymbol{\varepsilon}}}$.
  - Assume a "generalized" ellipticity condition for $\underline{\underline{\boldsymbol{\xi}}} \in \{\underline{\underline{\boldsymbol{\varepsilon}}}, \underline{\underline{\boldsymbol{\mu}}}\}$:

$$(Ell) \qquad \exists \theta_{\xi} \in \mathbb{R},\ \exists \xi_{-} > 0,\ \forall \boldsymbol{z} \in \mathbb{C}^3,\ \xi_{-}|\boldsymbol{z}|^2 \leq \Re[e^{i\theta_{\xi}} \cdot \boldsymbol{z}^* \underline{\underline{\boldsymbol{\xi}}} \boldsymbol{z}] \quad \text{ae in } \Omega.$$

- Main results [Chicaud-PC-Modave'21], [PhD-Chicaud'2x]:
  - the model, expressed variationally, enters Fredholm alternative (coerciveness does not always hold) ;
  - derivation of the *a priori* regularity of the field $\boldsymbol{E}$ and of its curl ;
  - discretization with the help of the $\boldsymbol{H}(\mathbf{curl})$-conforming Finite Element Method ;
  - numerical analysis.

Introduction/Motivation

- Our goals and assumptions:
  - Study a time-harmonic electromagnetic wave propagation model with $\underline{\underline{\boldsymbol{L}}}^{\infty}$, anisotropic magnetic permeability $\underline{\underline{\boldsymbol{\mu}}}$ and electric permittivity $\underline{\underline{\boldsymbol{\varepsilon}}}$.
  - Assume a "generalized" ellipticity condition for $\underline{\underline{\boldsymbol{\xi}}} \in \{\underline{\underline{\boldsymbol{\varepsilon}}}, \underline{\underline{\boldsymbol{\mu}}}\}$:

$$(Ell) \qquad \exists \theta_\xi \in \mathbb{R},\ \exists \xi_- > 0,\ \forall \boldsymbol{z} \in \mathbb{C}^3,\ \xi_-|\boldsymbol{z}|^2 \leq \Re[e^{i\theta_\xi} \cdot \boldsymbol{z}^* \underline{\underline{\boldsymbol{\xi}}} \boldsymbol{z}] \quad \text{ae in } \Omega.$$

- Some references on these issues:
  - "Useful" monographs [Monk'03], [Costabel-Dauge-Nicaise'10], [Roach-Stratis-Yannacopoulos'13], [Assous-PC-Labrunie'18].

Introduction/Motivation

- Our goals and assumptions:
  - Study a time-harmonic electromagnetic wave propagation model with $\underline{\underline{\boldsymbol{L}}}^\infty$, anisotropic magnetic permeability $\underline{\underline{\boldsymbol{\mu}}}$ and electric permittivity $\underline{\underline{\boldsymbol{\varepsilon}}}$.
  - Assume a "generalized" ellipticity condition for $\underline{\underline{\boldsymbol{\xi}}} \in \{\underline{\underline{\boldsymbol{\varepsilon}}}, \underline{\underline{\boldsymbol{\mu}}}\}$:

$$(Ell) \qquad \exists \theta_\xi \in \mathbb{R},\ \exists \xi_- > 0,\ \forall \boldsymbol{z} \in \mathbb{C}^3,\ \xi_- |\boldsymbol{z}|^2 \leq \Re[e^{i\theta_\xi} \cdot \boldsymbol{z}^* \underline{\underline{\boldsymbol{\xi}}} \boldsymbol{z}] \quad \text{ae in } \Omega.$$

- Some references on these issues:
  - On the regularity results:
    - in the $(\boldsymbol{PH}^s(\Omega))_{s>0}$ scale: assuming piecewise smooth, elliptic, scalar fields/hermitian tensors [Costabel-Dauge-Nicaise'99], [Jochmann'99], [Bonito-Guermond-Luddens'13], [PC'20] ;
    - in the $(\boldsymbol{L}^r(\Omega))_{r>1}$ scale: assuming $\underline{\underline{\boldsymbol{L}}}^\infty$, elliptic, perturbation of hermitian tensors [Xiang'20] ;
    - in the $(\boldsymbol{\mathcal{C}}^{0,\alpha}(\overline{\Omega}))_{\alpha>0}$ scale: assuming (Hölder-)continuous, elliptic tensors [Alberti-Capdeboscq'14], [Alberti'18], [Tsering-xiao-Wang'20].

ENSTA IP PARIS

Outline

- Introduction/Motivation
- **The model and its well-posedness**
- A priori regularity of the fields
- Discretization and error estimates
- Numerical illustrations
- Conclusion and perspectives

Time-harmonic Maxwell equations

- Ω is a *domain* ; $\omega > 0$ is the pulsation.

- Given volume data $\boldsymbol{f}$ and surface data $\boldsymbol{g}$, solve:

$$\left\{ \begin{array}{ll} \textit{Find } \boldsymbol{E} \in \boldsymbol{H}(\mathbf{curl};\Omega) \textit{ s.t.} & \\ \mathbf{curl}\left(\underline{\underline{\boldsymbol{\mu}}}^{-1}\,\mathbf{curl}\,\boldsymbol{E}\right) - \omega^2 \underline{\underline{\boldsymbol{\varepsilon}}}\boldsymbol{E} = \boldsymbol{f} & \text{in } \Omega\,; \\ \boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{g} & \text{on } \partial\Omega. \end{array} \right.$$

[Dirichlet boundary condition.]

- The tensors $\underline{\underline{\boldsymbol{\xi}}} \in \{\underline{\underline{\boldsymbol{\varepsilon}}}, \underline{\underline{\boldsymbol{\mu}}}\}$ are elliptic (Ell), and they belong to $\underline{\underline{\boldsymbol{L}}}^{\infty}(\Omega)$.

- Volume data: $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$.
- Surface data: $\boldsymbol{g} = \boldsymbol{E}_d \times \boldsymbol{n}_{|\partial\Omega}$, with $\boldsymbol{E}_d \in \boldsymbol{H}(\mathbf{curl};\Omega)$.

Helmholtz decompositions-1

- Define the function spaces

$$\boldsymbol{H}_0(\mathbf{curl};\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega) \,|\, \boldsymbol{v}\times\boldsymbol{n}_{|\partial\Omega} = 0\},$$
$$\boldsymbol{H}(\operatorname{div}\underline{\underline{\boldsymbol{\xi}}};\Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \,|\, \underline{\underline{\boldsymbol{\xi}}}\boldsymbol{v} \in \boldsymbol{H}(\operatorname{div};\Omega)\},$$
$$\boldsymbol{H}(\operatorname{div}\underline{\underline{\boldsymbol{\xi}}}0;\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\operatorname{div}\underline{\underline{\boldsymbol{\xi}}};\Omega) \,|\, \operatorname{div}\underline{\underline{\boldsymbol{\xi}}}\boldsymbol{v} = 0\},$$
$$\boldsymbol{K}_N(\underline{\underline{\boldsymbol{\xi}}};\Omega) := \boldsymbol{H}_0(\mathbf{curl};\Omega) \cap \boldsymbol{H}(\operatorname{div}\underline{\underline{\boldsymbol{\xi}}}0;\Omega).$$

- Helmholtz decompositions:

$$\boldsymbol{L}^2(\Omega) = \nabla[H^1_0(\Omega)] \oplus \boldsymbol{H}(\operatorname{div}\underline{\underline{\boldsymbol{\xi}}}0;\Omega) \quad \text{and} \quad \boldsymbol{H}_0(\mathbf{curl};\Omega) = \nabla[H^1_0(\Omega)] \oplus \boldsymbol{K}_N(\underline{\underline{\boldsymbol{\xi}}};\Omega).$$

NB. Notion of orthogonality does not apply when $\underline{\underline{\boldsymbol{\xi}}}$ is non-hermitian.

- To solve our model, we rely on the second Helmholtz decompostion.

Helmholtz decompositions-2

- First Helmholtz decomposition $\boldsymbol{L}^2(\Omega) = \nabla[H_0^1(\Omega)] \oplus \boldsymbol{H}(\mathrm{div}\,\underline{\underline{\boldsymbol{\xi}}}0;\Omega)$.
  Idea of proof: Let $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$.
  - The Dirichlet problem

$$(P_{Dir}) \quad \begin{cases} \text{Find } p \in H_0^1(\Omega) \text{ such that} \\ (\underline{\underline{\boldsymbol{\xi}}}\nabla p|\nabla q) = (\underline{\underline{\boldsymbol{\xi}}}\boldsymbol{v}|\nabla q), \quad \forall q \in H_0^1(\Omega), \end{cases}$$

    is well-posed, thanks to (Ell): $\exists! p$, with $\|p\|_{H^1(\Omega)} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}$.
  - Let $\boldsymbol{v}_T = \boldsymbol{v} - \nabla p \in \boldsymbol{L}^2(\Omega)$: $\|\boldsymbol{v}_T\|_{\boldsymbol{L}^2(\Omega)} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}$ and

$$(\underline{\underline{\boldsymbol{\xi}}}\boldsymbol{v}_T|\nabla q) = 0, \quad \forall q \in H_0^1(\Omega),$$

    ie. $\boldsymbol{v}_T \in \boldsymbol{H}(\mathrm{div}\,\underline{\underline{\boldsymbol{\xi}}}0;\Omega)$.
  - The sum is direct because (P_{Dir}) is well-posed.
- Second Helmholtz decomposition $\boldsymbol{H}_0(\mathbf{curl};\Omega) = \nabla[H_0^1(\Omega)] \oplus \boldsymbol{K}_N(\underline{\underline{\boldsymbol{\xi}}};\Omega)$ follows as a straightforward corollary.

Study of $\boldsymbol{K}_N(\underline{\underline{\boldsymbol{\xi}}};\Omega)$

- The embedding of $\boldsymbol{K}_N(\underline{\underline{\boldsymbol{\xi}}};\Omega)$ into $\boldsymbol{L}^2(\Omega)$ is compact.

  NB. One has a similar property for the function space $\boldsymbol{H}_0(\mathbf{curl};\Omega) \cap \boldsymbol{H}(\operatorname{div}\underline{\underline{\boldsymbol{\xi}}};\Omega)$.

- Property relies on a key ingredient: a decomposition of $\boldsymbol{H}(\operatorname{div}\underline{\underline{\boldsymbol{\xi}}}0;\Omega)$.
  - Let $(\Gamma_k)_{k=0,K}$ be the (maximal) connected components of $\partial\Omega$:

    $$Q_N(\underline{\underline{\boldsymbol{\xi}}};\Omega) := \{q \in H^1(\Omega) \,|\, \operatorname{div}\underline{\underline{\boldsymbol{\xi}}}\nabla q = 0 \text{ in } \Omega, q_{|\Gamma_0} = 0,\ q_{|\Gamma_k} = cst_k,\ 1 \leq k \leq K\}.$$

  - $\boldsymbol{H}(\operatorname{div}\underline{\underline{\boldsymbol{\xi}}}0;\Omega) = \nabla[Q_N(\underline{\underline{\boldsymbol{\xi}}};\Omega)] \oplus \underline{\underline{\boldsymbol{\xi}}}^{-1}\,\mathbf{curl}[\boldsymbol{H}^1(\Omega)]$. Moreover,

    $$\forall \boldsymbol{z} \in \boldsymbol{H}(\operatorname{div}\underline{\underline{\boldsymbol{\xi}}}0;\Omega),\ \exists (q^\Gamma, \boldsymbol{w}) \in Q_N(\underline{\underline{\boldsymbol{\xi}}};\Omega) \times \boldsymbol{H}^1(\Omega),$$
    $$\boldsymbol{z} = \nabla q^\Gamma + \underline{\underline{\boldsymbol{\xi}}}^{-1}\,\mathbf{curl}\,\boldsymbol{w} \text{ with } \|\nabla q^\Gamma\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)} \lesssim \|\boldsymbol{z}\|_{\boldsymbol{L}^2(\Omega)}.$$

    Idea of proof: extraction of vector potential Thm 3.4.1 [Assous-PC-Labrunie'18].

Well-posedness (1)

- Recall our model:

$$\begin{cases} \textit{Find } \boldsymbol{E} \in \boldsymbol{H}(\mathbf{curl};\Omega) \textit{ s.t.} \\ \mathbf{curl}\left(\underline{\underline{\boldsymbol{\mu}}}^{-1}\,\mathbf{curl}\,\boldsymbol{E}\right) - \omega^2 \underline{\underline{\boldsymbol{\varepsilon}}}\boldsymbol{E} = \boldsymbol{f} & \text{in } \Omega\,; \\ \boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{g} & \text{on } \partial\Omega. \end{cases}$$

- An equivalent variational formulation reads

$$(FVE) \quad \begin{cases} \text{Find } \boldsymbol{E} \in \boldsymbol{H}(\mathbf{curl};\Omega) \text{ such that} \\ (\underline{\underline{\boldsymbol{\mu}}}^{-1}\,\mathbf{curl}\,\boldsymbol{E}|\,\mathbf{curl}\,\boldsymbol{F}) - \omega^2(\underline{\underline{\boldsymbol{\varepsilon}}}\boldsymbol{E}|\boldsymbol{F}) = \ell_{\mathrm{D}}(\boldsymbol{F}),\ \forall \boldsymbol{F} \in \boldsymbol{H}_0(\mathbf{curl};\Omega), \\ \boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{g} \text{ on } \partial\Omega, \end{cases}$$

where $\ell_{\mathrm{D}}\ :\ \boldsymbol{F} \mapsto (\boldsymbol{f}|\boldsymbol{F})$ belongs to $(\boldsymbol{H}_0(\mathbf{curl};\Omega))'$.

Well-posedness (1)

- Recall our model:

$$\left\{\begin{array}{ll} \textit{Find } \boldsymbol{E} \in \boldsymbol{H}(\mathbf{curl};\Omega) \textit{ s.t.} & \\ \mathbf{curl}\left(\underline{\underline{\boldsymbol{\mu}}}^{-1}\,\mathbf{curl}\,\boldsymbol{E}\right) - \omega^2 \underline{\underline{\boldsymbol{\varepsilon}}}\boldsymbol{E} = \boldsymbol{f} & \text{in } \Omega\ ; \\ \boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{g} & \text{on } \partial\Omega. \end{array}\right.$$

- Introduce the new unknown $\boldsymbol{E}_0 := \boldsymbol{E} - \boldsymbol{E}_d \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$, which is governed by

$$(FVE_0) \quad \left\{\begin{array}{l} \text{Find } \boldsymbol{E}_0 \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \text{ such that} \\ (\underline{\underline{\boldsymbol{\mu}}}^{-1}\,\mathbf{curl}\,\boldsymbol{E}_0|\,\mathbf{curl}\,\boldsymbol{F}) - \omega^2(\underline{\underline{\boldsymbol{\varepsilon}}}\boldsymbol{E}_0|\boldsymbol{F}) = \ell_{\mathrm{D},0}(\boldsymbol{F}),\ \forall \boldsymbol{F} \in \boldsymbol{H}_0(\mathbf{curl};\Omega), \end{array}\right.$$

where $\ell_{\mathrm{D},0} : \boldsymbol{F} \mapsto (\boldsymbol{f} + \omega^2\underline{\underline{\boldsymbol{\varepsilon}}}\boldsymbol{E}_d|\boldsymbol{F}) - (\underline{\underline{\boldsymbol{\mu}}}^{-1}\,\mathbf{curl}\,\boldsymbol{E}_d|\,\mathbf{curl}\,\boldsymbol{F})$ belongs to $(\boldsymbol{H}_0(\mathbf{curl};\Omega))'$.

ENSTA IP PARIS

Well-posedness (1)

- Recall our model:

$$\left\{\begin{array}{ll} \textit{Find } \boldsymbol{E} \in \boldsymbol{H}(\mathbf{curl};\Omega) \textit{ s.t.} & \\ \mathbf{curl}\left(\underline{\underline{\boldsymbol{\mu}}}^{-1}\,\mathbf{curl}\,\boldsymbol{E}\right) - \omega^2\underline{\underline{\boldsymbol{\varepsilon}}}\boldsymbol{E} = \boldsymbol{f} & \text{in } \Omega\,; \\ \boldsymbol{E}\times\boldsymbol{n} = \boldsymbol{g} & \text{on } \partial\Omega. \end{array}\right.$$

- Using the second Helmholtz decomposition, split $\boldsymbol{E}_0$ as $\boldsymbol{E}_0 = \nabla p_0 + \boldsymbol{k}_0$, where $p_0 \in H_0^1(\Omega)$ and $\boldsymbol{k}_0 \in \boldsymbol{K}_N(\underline{\underline{\varepsilon}};\Omega)$ are respectively governed by

$$(P_{Dir}^E)\quad \left\{\begin{array}{l} \text{Find } p_0 \in H_0^1(\Omega) \text{ such that} \\ -\omega^2(\underline{\underline{\varepsilon}}\nabla p_0|\nabla q) = \ell_{\mathrm{D},0}(\nabla q),\ \forall q \in H_0^1(\Omega), \end{array}\right.$$

$$(P_K^E)\quad \left\{\begin{array}{l} \text{Find } \boldsymbol{k}_0 \in \boldsymbol{K}_N(\underline{\underline{\varepsilon}};\Omega) \text{ such that} \\ (\underline{\underline{\boldsymbol{\mu}}}^{-1}\,\mathbf{curl}\,\boldsymbol{k}_0|\,\mathbf{curl}\,\boldsymbol{k}) - \omega^2(\underline{\underline{\varepsilon}}\boldsymbol{k}_0|\boldsymbol{k}) = \omega^2(\underline{\underline{\varepsilon}}\nabla p_0|\boldsymbol{k}) + \ell_{\mathrm{D},0}(\boldsymbol{k}),\ \forall \boldsymbol{k} \in \boldsymbol{K}_N(\underline{\underline{\varepsilon}};\Omega), \end{array}\right.$$

with $\ell_{\mathrm{D},0} : \boldsymbol{F} \mapsto (\boldsymbol{f} + \omega^2\underline{\underline{\varepsilon}}\boldsymbol{E}_d|\boldsymbol{F}) - (\underline{\underline{\boldsymbol{\mu}}}^{-1}\,\mathbf{curl}\,\boldsymbol{E}_d|\,\mathbf{curl}\,\boldsymbol{F})$.

Well-posedness (2)

- (P^E_{Dir}) is well-posed, thanks to the ellipticity condition (Ell).

- (P^E_K) enters Fredholm alternative.
  Idea of proof: Let $\alpha > 0$.
  The form $(\boldsymbol{v}, \boldsymbol{w}) \mapsto (\underline{\underline{\boldsymbol{\mu}}}^{-1} \operatorname{\mathbf{curl}} \boldsymbol{v} | \operatorname{\mathbf{curl}} \boldsymbol{w}) - \omega^2 (\underline{\underline{\boldsymbol{\varepsilon}}} \boldsymbol{v} | \boldsymbol{w})$ is split as
  - $(\boldsymbol{v}, \boldsymbol{w}) \mapsto (\underline{\underline{\boldsymbol{\mu}}}^{-1} \operatorname{\mathbf{curl}} \boldsymbol{v} | \operatorname{\mathbf{curl}} \boldsymbol{w}) + \alpha e^{i\theta_\mu} (\boldsymbol{v} | \boldsymbol{w})$, which is coercive on $\boldsymbol{K}_N(\underline{\underline{\boldsymbol{\varepsilon}}}; \Omega)$.
  - $(\boldsymbol{v}, \boldsymbol{w}) \mapsto -((\omega^2 \underline{\underline{\boldsymbol{\varepsilon}}} + \alpha e^{i\theta_\mu}) \boldsymbol{v} | \boldsymbol{w})$, which is a compact perturbation on $\boldsymbol{K}_N(\underline{\underline{\boldsymbol{\varepsilon}}}; \Omega)$, thanks to the compact embedding of $\boldsymbol{K}_N(\underline{\underline{\boldsymbol{\varepsilon}}}; \Omega)$ into $\boldsymbol{L}^2(\Omega)$.

Well-posedness (2)

- (P^E_{Dir}) is well-posed, thanks to the ellipticity condition (Ell).
- (P^E_K) enters Fredholm alternative.
- So (FVE_0) and (FVE) also enter Fredholm alternative:
  - either (FVE_0) admits a unique solution $\boldsymbol{E}_0$ in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, which depends continuously on the data $\boldsymbol{f}$ and $\boldsymbol{E}_d$:

$$\|\boldsymbol{E}_0\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{E}_d\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}\,;$$

  - or, (FVE_0) has solutions if, and only if, $\boldsymbol{f}$ and $\boldsymbol{E}_d$ satisfy a finite number of compatibility conditions.

  Moreover, each alternative occurs simultaneously for (FVE_0) and (FVE).
- For the rest of the talk we *assume that the problems are well-posed*, ie. existence, uniqueness and continuous dependence of the solution wrt the data.

Outline

- Introduction/Motivation
- The model and its well-posedness
- **A priori regularity of the fields**
- Discretization and error estimates
- Numerical illustrations
- Conclusion and perspectives

Extra-regularity of $\boldsymbol{E}$

- For $\boldsymbol{E}_0 \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$, two key ingredients...
- Regular-gradient splitting, see Lemma 2.4 [Hiptmair'02]:
  In a domain Ω, for all $\boldsymbol{u}$ in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, there exist $\boldsymbol{u}^{\mathrm{reg}}$ in $\boldsymbol{H}^1(\Omega)$ and ϕ in $H^1_0(\Omega)$, such that $\boldsymbol{u} = \boldsymbol{u}^{\mathrm{reg}} + \nabla\phi$ in Ω, with $\|\boldsymbol{u}^{\mathrm{reg}}\|_{\boldsymbol{H}^1(\Omega)} + \|\phi\|_{H^1(\Omega)} \lesssim \|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$.
- Shift theorem, see Thm 3.4.5 [Costabel-Dauge-Nicaise'10]:
  Assume that $\underline{\underline{\varepsilon}} \in \underline{\underline{\boldsymbol{C}}}^1(\overline{\Omega})$ and that $\partial\Omega$ is of **class** $\mathcal{C}^2$.
  Let ℓ in $\left(H^1_0(\Omega)\right)'$, and p governed by the Dirichlet problem

$$\left\{\begin{array}{l}\text{Find } p \in H^1_0(\Omega) \text{ such that} \\ (\underline{\underline{\varepsilon}}\nabla p|\nabla q) = \ell(q),\ \forall q \in H^1_0(\Omega).\end{array}\right.$$

Then, for all $\sigma \in [0,1] \setminus \{\frac{1}{2}\}$:

$$\ell \in \left(H^{1-\sigma}_0(\Omega)\right)' \Longrightarrow p \in H^{\sigma+1}(\Omega);$$

$$\exists C_\sigma > 0,\ \forall \ell \in \left(H^{1-\sigma}_0(\Omega)\right)',\ \|p\|_{H^{\sigma+1}(\Omega)} \leq C_\sigma \|\ell\|_{\left(H^{1-\sigma}_0(\Omega)\right)'}.$$

Extra-regularity of E

- For $\boldsymbol{E}_0 \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$, two key ingredients...
- Regular-gradient splitting, see Lemma 2.4 [Hiptmair'02]:
  In a domain Ω, for all $\boldsymbol{u}$ in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, there exist $\boldsymbol{u}^{\text{reg}}$ in $\boldsymbol{H}^1(\Omega)$ and ϕ in $H_0^1(\Omega)$, such that $\boldsymbol{u} = \boldsymbol{u}^{\text{reg}} + \nabla\phi$ in Ω, with $\|\boldsymbol{u}^{\text{reg}}\|_{\boldsymbol{H}^1(\Omega)} + \|\phi\|_{H^1(\Omega)} \lesssim \|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$.
- Shift theorem, see Thm 3.4.5 [Costabel-Dauge-Nicaise'10]:
  Assume that $\underline{\underline{\varepsilon}} \in \underline{\underline{C}}^1(\overline{\Omega})$ and that $\partial\Omega$ is of **class** $\mathcal{C}^2$.

- Applying these results to $\boldsymbol{E}_0$, one concludes that:
  if $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ is such that $\operatorname{div}\boldsymbol{f} \in H^{\mathtt{s}-1}(\Omega)$ with $\mathtt{s} \in [0,1] \setminus \{\frac{1}{2}\}$,
  if $\boldsymbol{E}_d \in \boldsymbol{H}^{\mathtt{r}}(\Omega)$ with $\mathtt{r} \in [0,1] \setminus \{\frac{1}{2}\}$,
  then

$$\begin{cases} \boldsymbol{E} \in \boldsymbol{H}^{\min(\mathtt{s},\mathtt{r})}(\Omega) \text{ and} \\ \|\boldsymbol{E}\|_{\boldsymbol{H}^{\min(\mathtt{s},\mathtt{r})}(\Omega)} \lesssim \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\operatorname{div}\boldsymbol{f}\|_{H^{\mathtt{s}-1}(\Omega)} + \|\boldsymbol{E}_d\|_{\boldsymbol{H}^{\mathtt{r}}(\Omega)} + \|\mathbf{curl}\,\boldsymbol{E}_d\|_{\boldsymbol{L}^2(\Omega)}. \end{cases}$$

Extra-regularity of curl $\boldsymbol{E}$

- For $\underline{\underline{\boldsymbol{\mu}}}^{-1}\,\mathbf{curl}\,\boldsymbol{E} \in \boldsymbol{H}(\mathbf{curl};\Omega)$, again two key ingredients...
- Regular-gradient splitting of elements of $\boldsymbol{H}(\mathbf{curl};\Omega)$ in a domain Ω of the $\mathfrak{A}$-type, see Thm 3.6.7 [Assous-PC-Labrunie'18].
- Shift theorem, see Thm 3.4.5 [Costabel-Dauge-Nicaise'10], for the Neumann problem. One assumes that $\underline{\underline{\boldsymbol{\mu}}} \in \underline{\underline{\mathcal{C}}}^1(\overline{\Omega})$ and again that $\partial\Omega$ is of **class** $\mathcal{C}^2$.

- Applying these results to $\underline{\underline{\boldsymbol{\mu}}}^{-1}\,\mathbf{curl}\,\boldsymbol{E}$, one concludes that:
  if $\mathbf{curl}\,\boldsymbol{E}_d \in \boldsymbol{H}^{\mathrm{r}'}(\Omega)$ with $\mathrm{r}' \in [0,1] \setminus \{\frac{1}{2}\}$,
  then

$$\begin{cases} \mathbf{curl}\,\boldsymbol{E} \in \boldsymbol{H}^{\mathrm{r}'}(\Omega) \text{ and} \\ \|\mathbf{curl}\,\boldsymbol{E}\|_{\boldsymbol{H}^{\mathrm{r}'}(\Omega)} \lesssim \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{E}_d\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{E}_d\|_{\boldsymbol{H}^{\mathrm{r}'}(\Omega)}. \end{cases}$$

NB. If Ω is a domain with boundary of class $\mathcal{C}^2$, it is automatically of the $\mathfrak{A}$-type.

Outline

- Introduction/Motivation
- The model and its well-posedness
- A priori regularity of the fields
- **Discretization and error estimates**
- Numerical illustrations
- Conclusion and perspectives

ENSTA IP PARIS

$\boldsymbol{H}(\mathrm{curl};\Omega)$-conforming discretization

- For the sake of simplicity, assume that Ω is a (Lipschitz) polyhedron.

- Let $(\mathcal{T}_h)_{h>0}$ be a shape regular family of tetrahedral meshes of Ω.
- We choose the first family of edge finite elements for the discretization [Nédélec'80]: for $h>0$, and $K\in\mathcal{T}_h$, let $\mathcal{R}_1(K) := \{\boldsymbol{v}\in\boldsymbol{P}_1(K) \,|\, \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x},\ \boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^3\}$.
- For $h>0$, introduce the discrete spaces $\boldsymbol{V}_h := \{\boldsymbol{v}_h \in \boldsymbol{H}(\mathbf{curl};\Omega) \,|\, \boldsymbol{v}_{h|K}\in\mathcal{R}_1(K),\ \forall K\in\mathcal{T}_h\}$, $\boldsymbol{V}_h^0 := \boldsymbol{H}_0(\mathbf{curl};\Omega)\cap\boldsymbol{V}_h$.

- The discrete variational formulation reads

$$(FVE_h)\qquad \begin{cases} \text{Find } \boldsymbol{E}_h\in\boldsymbol{V}_h \text{ such that} \\ a_\omega(\boldsymbol{E}_h,\boldsymbol{F}_h) = \ell_{\mathrm{D}}(\boldsymbol{F}_h),\ \forall \boldsymbol{F}_h\in\boldsymbol{V}_h^0, \\ \boldsymbol{E}_h\times\boldsymbol{n} = \boldsymbol{g}_h \text{ on } \partial\Omega, \end{cases}$$

with the sesquilinear form $a_\omega : (\boldsymbol{u},\boldsymbol{v}) \mapsto (\underline{\underline{\boldsymbol{\mu}}}^{-1}\,\mathbf{curl}\,\boldsymbol{u}|\,\mathbf{curl}\,\boldsymbol{v}) - \omega^2(\underline{\underline{\boldsymbol{\varepsilon}}}\boldsymbol{u}|\boldsymbol{v})$.

Error estimates-1

- If the form a_ω is coercive in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, one uses Céa's lemma.

- If the form a_ω is not coercive, one must prove a uniform discrete inf-sup condition:

$$\exists C_\omega, h_\omega > 0,\ \forall h \le h_\omega,\ \forall \boldsymbol{u}_h \in \boldsymbol{V}_h^0,\ \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h^0 \setminus \{0\}} \frac{|a_\omega(\boldsymbol{u}_h, \boldsymbol{v}_h)|}{\|\boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}} \ge C_\omega \, \|\boldsymbol{u}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}.$$

– Tedious proof ! –

Error estimates-1

- If the form a_ω is coercive in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, one uses Céa's lemma.

- If the form a_ω is not coercive, one must prove a uniform discrete inf-sup condition.

- In both instances, one concludes that

$$(Cea) \quad \exists C > 0,\ \forall h(\leq h_\omega),\ \|\boldsymbol{E}-\boldsymbol{E}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \leq C \inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \|\boldsymbol{E}-\boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}\,.$$

  - Without any regularity assumption on $\boldsymbol{E}$: $\lim_{h\to 0+} \|\boldsymbol{E}-\boldsymbol{E}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} = 0$.

  - When $\boldsymbol{E} \in \boldsymbol{PH}^{\mathtt{t}}(\Omega)$ and $\mathbf{curl}\,\boldsymbol{E} \in \boldsymbol{PH}^{\mathtt{t}'}(\Omega)$ for some $\mathtt{t}, \mathtt{t}' > 0$, one can bound the right-hand side of (Cea) with respect to $h^{\min(\mathtt{t},\mathtt{t}',1)}$...

Error estimates-2

- When $\boldsymbol{E} \in \boldsymbol{PH}^{\mathtt{t}}(\Omega)$ and $\mathbf{curl}\,\boldsymbol{E} \in \boldsymbol{PH}^{\mathtt{t}'}(\Omega)$ for $\mathtt{t}, \mathtt{t}' > 0$, one uses interpolation...
  - If $\mathtt{t} > 1/2$, $\mathtt{t}' > 0$, one uses Nédélec's classical interpolation [Bermudez-Rodriguez-Salgado'05] to find the estimate

$$\|\boldsymbol{E} - \Pi_h^{curl}\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim h^{\min(\mathtt{t},\mathtt{t}',1)} \left(\|\boldsymbol{E}\|_{\boldsymbol{PH}^{\mathtt{t}}(\Omega)} + \|\,\mathbf{curl}\,\boldsymbol{E}\|_{\boldsymbol{PH}^{\mathtt{t}'}(\Omega)}\right).$$

  - If $\mathtt{t} \in (0, 1/2]$, $\mathtt{t}' > 0$, one uses quasi-interpolation [Ern-Guermond'18], or combined interpolation [PC'16]-[PC-preprint], to derive a similar estimate.

Error estimates-2

- When $\boldsymbol{E} \in \boldsymbol{PH}^{\mathtt{t}}(\Omega)$ and $\mathbf{curl}\, \boldsymbol{E} \in \boldsymbol{PH}^{\mathtt{t}'}(\Omega)$ for $\mathtt{t}, \mathtt{t}' > 0$, one uses interpolation...
  - If $\mathtt{t} > 1/2$, $\mathtt{t}' > 0$, one uses Nédélec's classical interpolation [Bermudez-Rodriguez-Salgado'05] to find the estimate

$$\|\boldsymbol{E} - \Pi_h^{curl} \boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim h^{\min(\mathtt{t},\mathtt{t}',1)} \left(\|\boldsymbol{E}\|_{\boldsymbol{PH}^{\mathtt{t}}(\Omega)} + \|\,\mathbf{curl}\, \boldsymbol{E}\|_{\boldsymbol{PH}^{\mathtt{t}'}(\Omega)} \right).$$

  - If $\mathtt{t} \in (0, 1/2]$, $\mathtt{t}' > 0$, one uses quasi-interpolation [Ern-Guermond'18], or combined interpolation [PC'16]-[PC-preprint], to derive a similar estimate.

- When Ω has a boundary of class $\mathcal{C}^2$, one follows [§8,Dello Russo-Alonso'09] to take into account the approximation of the domain by the meshes. One concludes that

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim h^{\min(\mathtt{s},\mathtt{r},\mathtt{r}')} \Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\, \boldsymbol{f}\|_{H^{\mathtt{s}-1}(\Omega)} + \|\boldsymbol{E}_d\|_{\boldsymbol{H}^{\mathtt{r}}(\Omega)} + \|\,\mathbf{curl}\, \boldsymbol{E}_d\|_{\boldsymbol{H}^{\mathtt{r}'}(\Omega)} \Big)$$

where the exponents $\mathtt{s}, \mathtt{r}, \mathtt{r}' \in [0, 1] \setminus \{\frac{1}{2}\}$ are related to the regularity of the data.

Outline

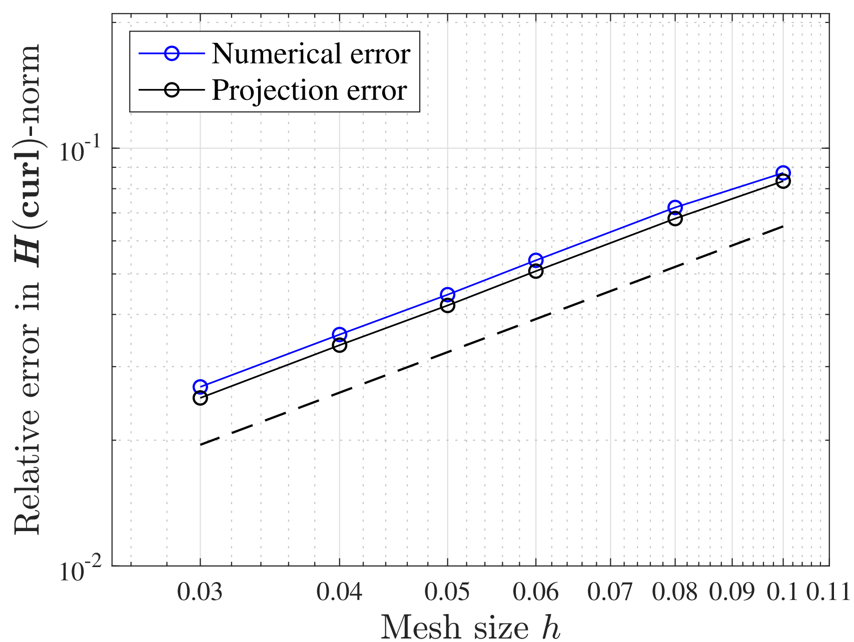
- Introduction/Motivation
- The model and its well-posedness
- A priori regularity of the fields
- Discretization and error estimates
- **Numerical illustrations**
- Conclusion and perspectives

Numerical results

- Software: `Freefem++`.
- Example 1: $\Omega := \{\boldsymbol{x} \,|\, |\boldsymbol{x}| < 1\}\,;\, \omega = 1\,;$
  material tensors: $\underline{\underline{\boldsymbol{\mu}}} = \text{diag}\big(1, 1, 1\big)$, $\underline{\underline{\boldsymbol{\varepsilon}}} = \text{diag}\big(1 + 10^{-1}i, 1 + 10^{-1}i, -2 + 10^{-1}i\big)$.
  The sesquilinear form is *coercive*.
  Manufactured solution: $\boldsymbol{E}_{\text{ref}}(\boldsymbol{x}) = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \exp(i\pi \boldsymbol{k} \cdot \boldsymbol{x})$, with $\boldsymbol{k} = \dfrac{1}{\sqrt{14}} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.
  The volume data $\boldsymbol{f}$ and surface data $\boldsymbol{g}$ are chosen accordingly.

Numerical results

- Software: `Freefem++`.
- Example 1: $\Omega := \{\boldsymbol{x} \,|\, |\boldsymbol{x}| < 1\}$; $\omega = 1$;
  material tensors: $\underline{\underline{\boldsymbol{\mu}}} = \mathrm{diag}\big(1, 1, 1\big)$, $\underline{\underline{\boldsymbol{\varepsilon}}} = \mathrm{diag}\big(1 + 10^{-1}i, 1 + 10^{-1}i, -2 + 10^{-1}i\big)$.
  The sesquilinear form is *coercive*.

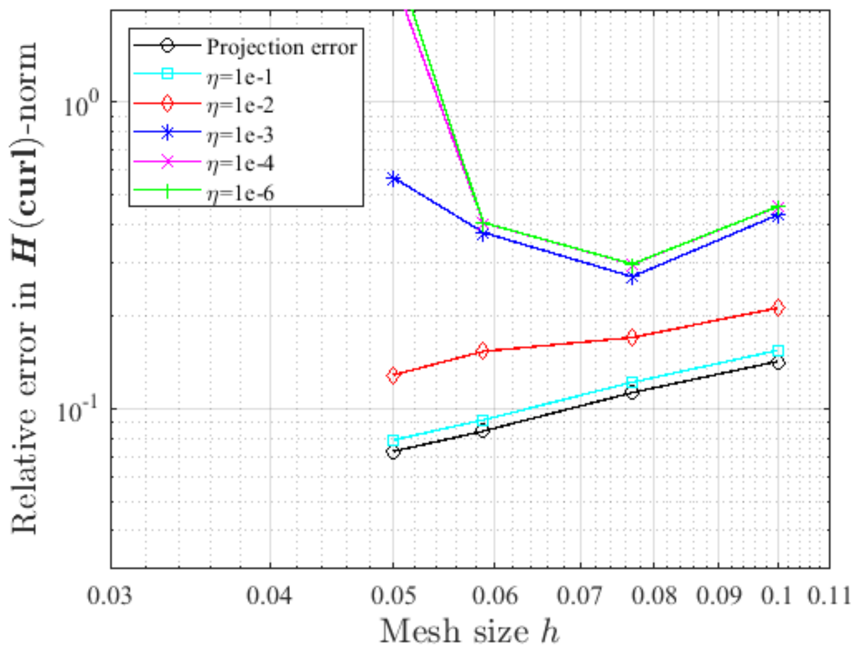


Numerical results

- Software: `Freefem++`.
- <u>Example 2:</u> $\Omega := (0,1)^3$; $\omega = 1$;
  material tensors: $\underline{\underline{\boldsymbol{\mu}}} = \text{diag}\big(1,1,1\big)$, $\underline{\underline{\boldsymbol{\varepsilon}}}^\eta = \text{diag}\big(1,1,-2+i\eta\big)$, for $\eta > 0$.
  Manufactured solution: $\boldsymbol{E}_{\text{ref}}(\boldsymbol{x}) = \begin{pmatrix} 2\cos(\pi x_1)\sin(\pi x_2)\sin(\pi x_3) \\ -\sin(\pi x_1)\cos(\pi x_2)\sin(\pi x_3) \\ -\sin(\pi x_1)\sin(\pi x_2)\cos(\pi x_3) \end{pmatrix}$.
  The volume data $\boldsymbol{f}$ and surface data $\boldsymbol{g}$ are chosen accordingly.

Numerical results

- Software: `Freefem++`.
- Example 2: $\Omega := (0,1)^3$; $\omega = 1$;
  material tensors: $\underline{\underline{\boldsymbol{\mu}}} = \mathrm{diag}\big(1,1,1\big)$, $\underline{\underline{\boldsymbol{\varepsilon}}}^\eta = \mathrm{diag}\big(1,1,-2+i\eta\big)$, for $\eta > 0$.



Outline

- Introduction/Motivation
- The model and its well-posedness
- A priori regularity of the fields
- Discretization and error estimates
- Numerical illustrations
- **Conclusion and perspectives**

Conclusion

- We solved theoretically and numerically the time-harmonic Maxwell equations with a Dirichlet boundary condition.
- For the problem with a Neumann boundary condition, ie. $\underline{\underline{\boldsymbol{\mu}}}^{-1}\,\mathbf{curl}\,\boldsymbol{E}\times\boldsymbol{n}=\boldsymbol{j}$ on $\partial\Omega$, see [Chicaud-PC-Modave'21].
- For the problem with a mixed boundary condition:
  - variational formulation, see [Back-Hattori-Labrunie-Roche-Bertrand'15] ;
  - compact embedding, see [Fernandes-Gilardi'97] ;
  - a priori regularity, see [Jochmann'99] for partial results.
- "Limit case" of a hyperbolic metamaterial, ie. $\underline{\underline{\boldsymbol{\varepsilon}}}^0=\text{diag}(1,1,-2)$, see [PC-Kachanovska-preprint] for some preliminary results.
- Adding a Domain Decomposition Method "layer" is possible.
- Other anisotropic models, see [PhD-Chicaud'2x].

Thank you for your attention.

Domain of the $\mathfrak{A}$-type

A domain Ω is said of the $\mathfrak{A}$-type if, for any $\boldsymbol{x} \in \partial\Omega$, there exists a neighbourhood $\mathcal{V}$ of $\boldsymbol{x}$ in $\mathbb{R}^3$, and a $\mathcal{C}^2$ diffeomorphism that transforms $\Omega \cap \mathcal{V}$ into one of the following types, where (x_1, x_2, x_3) denote the cartesian coordinates and $(\rho, \tilde{\omega}) \in \mathbb{R} \times \mathbb{S}^2$ the spherical coordinates:

1. $[x_1 > 0]$, *i.e.* $\boldsymbol{x}$ is a regular point;
2. $[x_1 > 0, x_2 > 0]$, *i.e.* $\boldsymbol{x}$ is a point on a salient (outward) edge;
3. $\mathbb{R}^3 \setminus [x_1 \geq 0, x_2 \geq 0]$, *i.e.* $\boldsymbol{x}$ is a point on a reentrant (inward) edge;
4. $[\rho > 0, \tilde{\omega} \in \tilde{\Omega}]$, where $\tilde{\Omega} \subset \mathbb{S}^2$ is a topologically trivial domain. In particular, if $\partial\tilde{\Omega}$ is smooth, $\boldsymbol{x}$ is a conical vertex; if $\partial\tilde{\Omega}$ is a made of arcs of great circles, $\boldsymbol{x}$ is a polyhedral vertex.