RADIATION CONDITION AND INSTABILITY PHENOMENON AT A CORNER INTERFACE BETWEEN A DIELECTRIC AND A NEGATIVE MATERIAL

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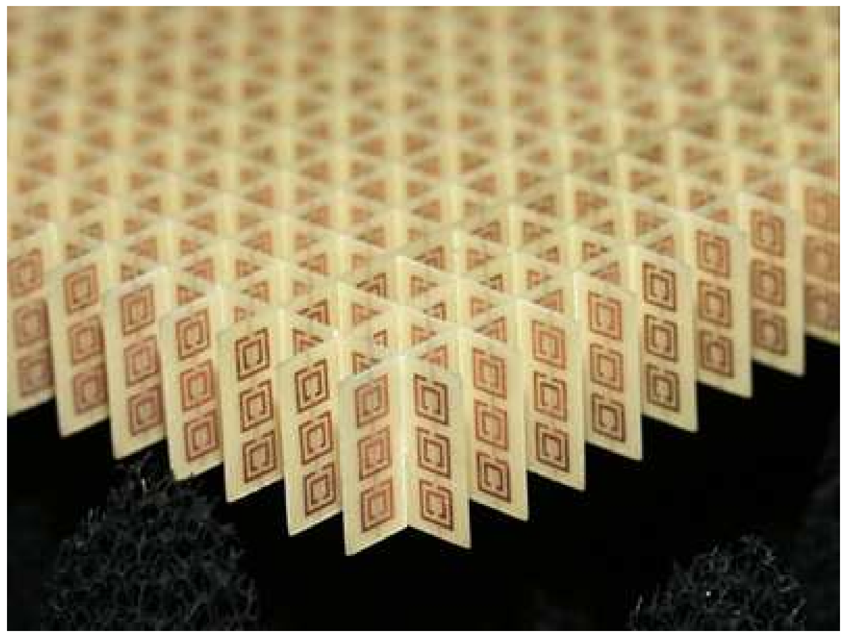
Metamaterials

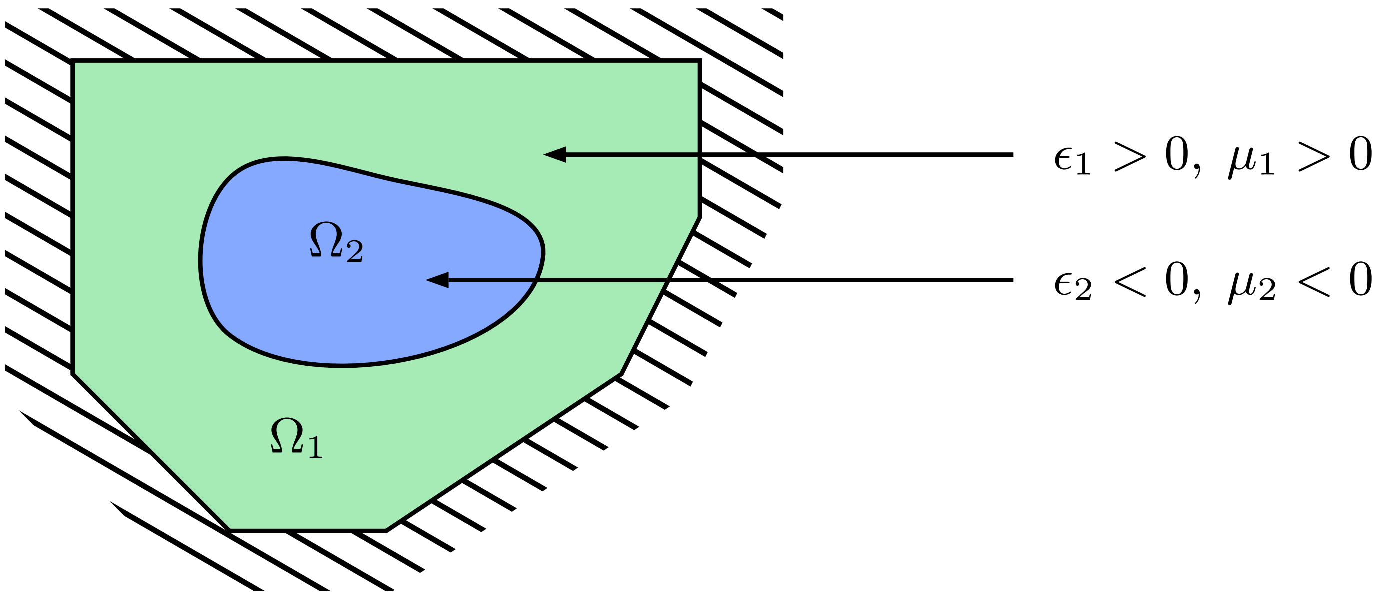
In the context of electromagnetic wave propagation, metamaterials are periodic assemblies of small resonators whose caracteristic size is much smaller than the average wavelength.



The periodicity cell (at the "micro scale") can be chosen so as to provide particular effective caracteristics at the macroscopic level.

Modelling: negative caracteristics

For particular choices of the periodicity cells, metamaterials can be modelled by homogeneous materials admitting negative effective permittivity/permeabilitty at some frequency: $\epsilon(\omega) < 0, \mu(\omega) < 0$.



Interesting applications rely on interfaces metamaterial/standard materials. The mathematical modelling is necessarily non-standard due to the sign shift of ϵ, μ through the interface.

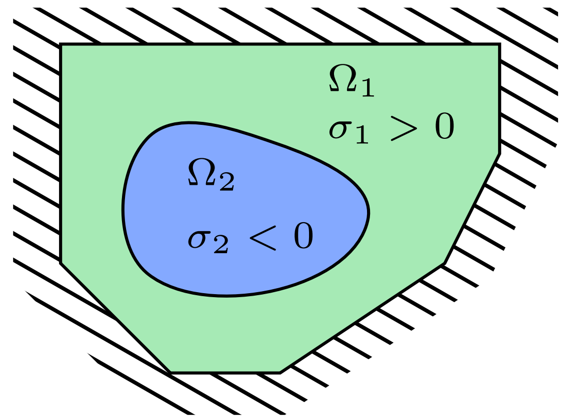
Refs: [Bouchitté-Bourel-Felbacq, 2009], [Bouchitté-Schweizer, 2010]

Model problem

Interesting mathematical difficulties are already contained in a 2-D "diffusion-like" model problem. Let $\mathrm{H}^1_0(\Omega) = \{ v \in \mathrm{L}^2(\Omega) \mid \nabla v \in \mathrm{L}^2(\Omega),\ v|_\Omega = 0 \}$. Given some $f \in \mathrm{H}^{-1}(\Omega) = \mathrm{H}^1_0(\Omega)^*$,

Find $u \in \mathrm{H}^1_0(\Omega)$ such that

$-\mathrm{div}(\sigma \nabla u) = f$ in Ω.



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$\sigma = \sigma_j$ in Ω_j

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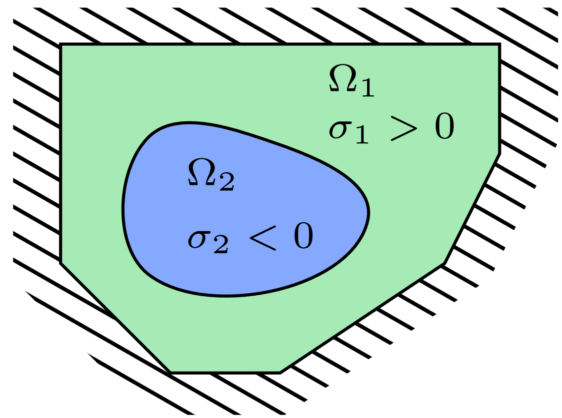

Find $u \in \mathrm{H}^1_0(\Omega)$ such that

$$\int_\Omega \sigma \nabla u \nabla v \, d\mathbf{x} = \langle f, v \rangle \quad \forall v \in \mathrm{H}^1_0(\Omega)\ .$$

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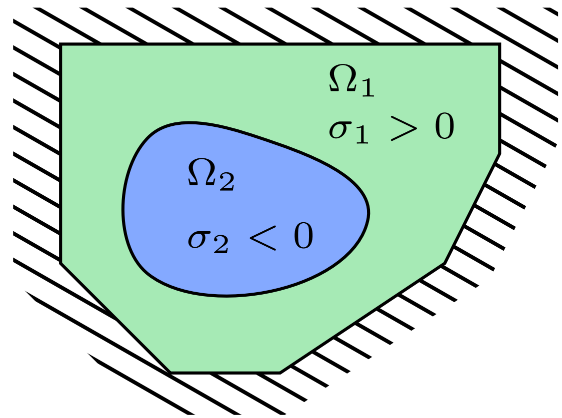

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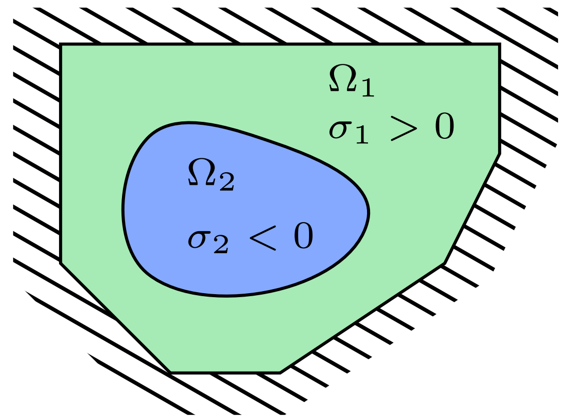

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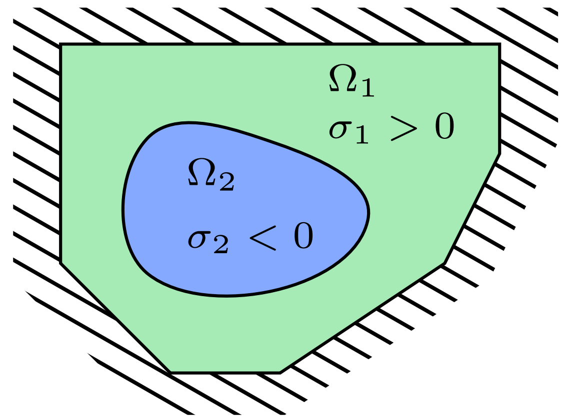

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no coercivity as σ changes sign
$\Rightarrow$ Lax-Milgram not available...

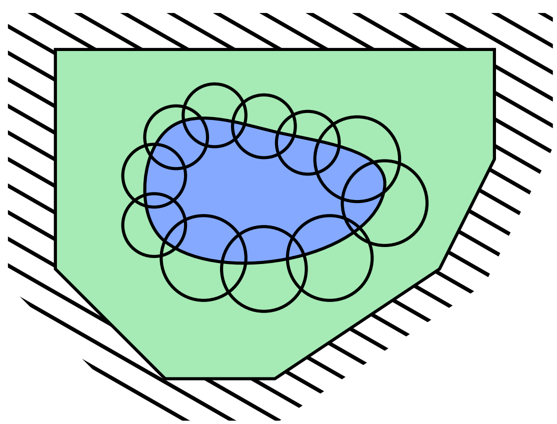
Alternative approach: T-coercivity

Sufficient condition of well-posedness = existence of an isomorphism $\mathrm{T} : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ and $\alpha > 0$ such that

$$\Re e\left\{ \int_\Omega \sigma \nabla v \nabla (\overline{\mathrm{T} v})\, d\mathbf{x} \right\} \;\geq\; \alpha \|v\|^2_{\mathrm{H}^1_0(\Omega)} \quad \forall v \in \mathrm{H}^1_0(\Omega)\ .$$

Question: Does such an isomorphism T exist?

Except for particular geometries, the operator T is not available. It can be constructed locally though (using partition of unity), leading to Fredholmness property. Several cases must be considered. . .



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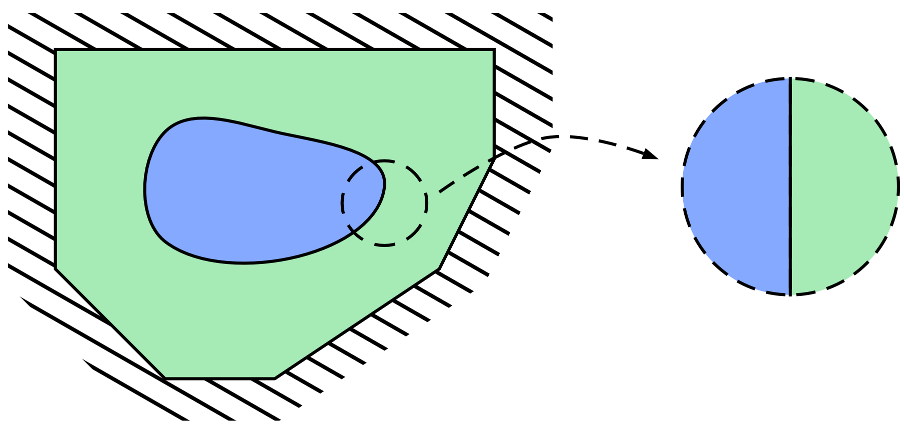
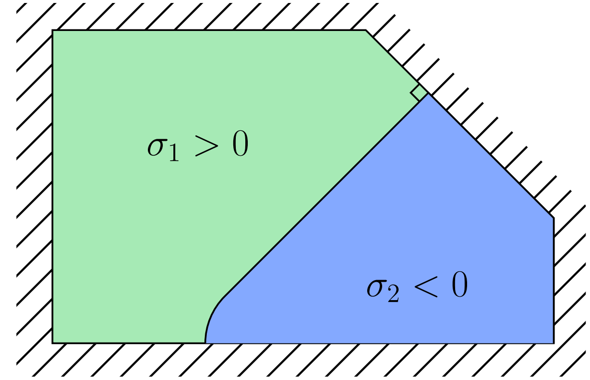


Case 1: smooth interface



Case 1 corresponds to:

- smooth interface $\Sigma := \partial\Omega_1 \cap \partial\Omega_2$,
- if Σ meets $\partial\Omega$, it does with perpendicular angle.

Then T-coercivity techniques show the following.

Theorem
If the geometry belongs to case 1, and $\kappa_\sigma := \sigma_2/\sigma_1 \neq -1$, then the operator $\mathrm{A} : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ defined by

$$\langle \mathrm{A}u, v\rangle \;:=\; \int_\Omega \sigma \nabla u \, \nabla v \; d\mathbf{x} \qquad \forall u, v \in \mathrm{H}^1_0(\Omega)$$

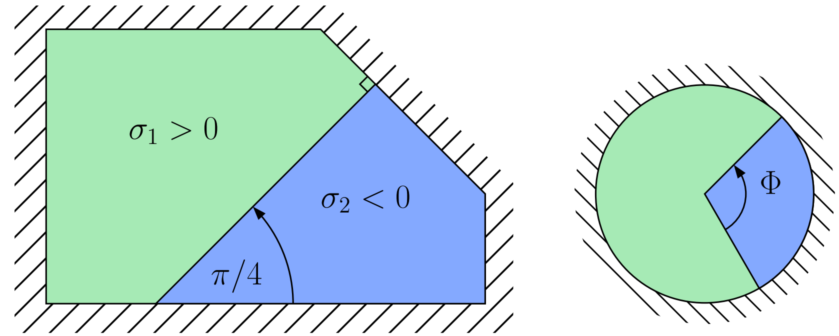
is of Fredholm type with index 0.

Refs: [Bonnet-Ben Dhia, Ciarlet Jr., Zwölf, 2010],
[Bonnet-Ben Dhia, Chesnel, Ciarlet Jr., 2012],
[Chesnel, 2012].

Case 2: corner interface

Case 2 is the same as case 1 except that:

- the interface Σ may admit corners,
- Σ may meet $\partial\Omega$ with an angle $\neq \pi/2$.



$\mathscr{I} = [-1, -\frac{1}{3}]$ $\qquad$ $\mathscr{I} = [\frac{\Phi-2\pi}{\Phi}, \frac{\Phi}{\Phi-2\pi}]$

Again $\langle \mathrm{A}u, v\rangle := \int_\Omega \sigma \nabla u \nabla v \, d\mathbf{x}$ and $\kappa_\sigma = \sigma_2/\sigma_1$.

Theorem
In case 2, there exists a closed interval $\mathscr{I} \subset \mathbb{R}_-$ depending on the corner angles of Σ, with $-1 \in \mathscr{I}$ and such that:

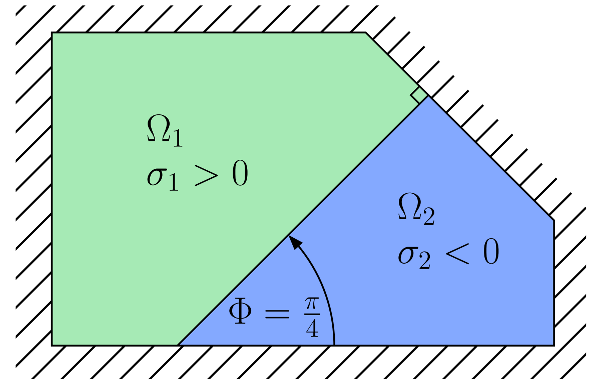
- if $\kappa_\sigma \in \mathbb{C} \setminus \mathscr{I}$, the operator $\mathrm{A} : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ is (index 0) - Fredholm,
- if $\kappa_\sigma \in \mathscr{I}$, the operator $\mathrm{A} : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ is **NOT** of Fredholm type.

Questions: What exactly happens for $\kappa_\sigma \in \mathscr{I}$? Is it possible to recover Fredholmness by changing the functional setting?

Case study

More complicated geometrical cases may be considered (ongoing work. . .).

The remaining of this talk will focus on case 2 for $\kappa_\sigma = \sigma_2/\sigma_1 \in \mathscr{I} = [-1, -1/3]$ in the following geometry (for simplicity $\Phi = \pi/4$).



Objective: Find a functional space $\mathrm{V}^{\text{out}}(\Omega) \subset \mathrm{L}^2_{\text{loc}}(\Omega)$ such that the following problem is of Fredholm type,

$$\left\{ \begin{array}{l} \text{Find } u \in \mathrm{V}^{\text{out}}(\Omega) \text{ such that} \\ -\mathrm{div}(\sigma\nabla u) = f \quad \text{in } \Omega\,. \end{array} \right.$$

To study this "corner problem", we make intensive use of Kondratiev's theory, cf. [Texier & Dauge, 97], [Nazarov&Plamenevsky, 94], [Nazarov&Taskinen, 94]. . .

Reduction of the problem

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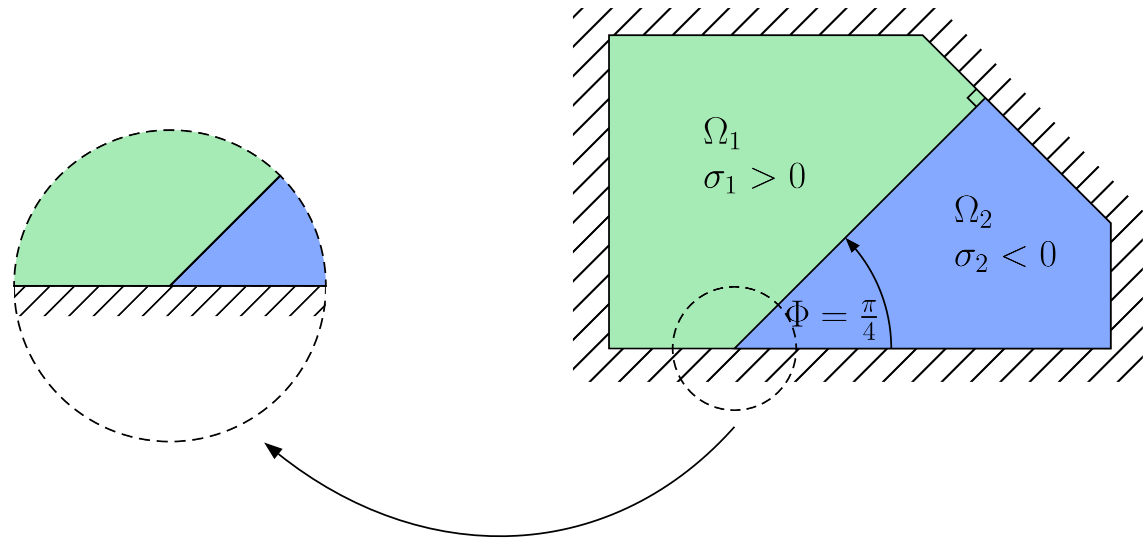
Relevant features of our problem are inherited from the metamaterial corner at the boundary.



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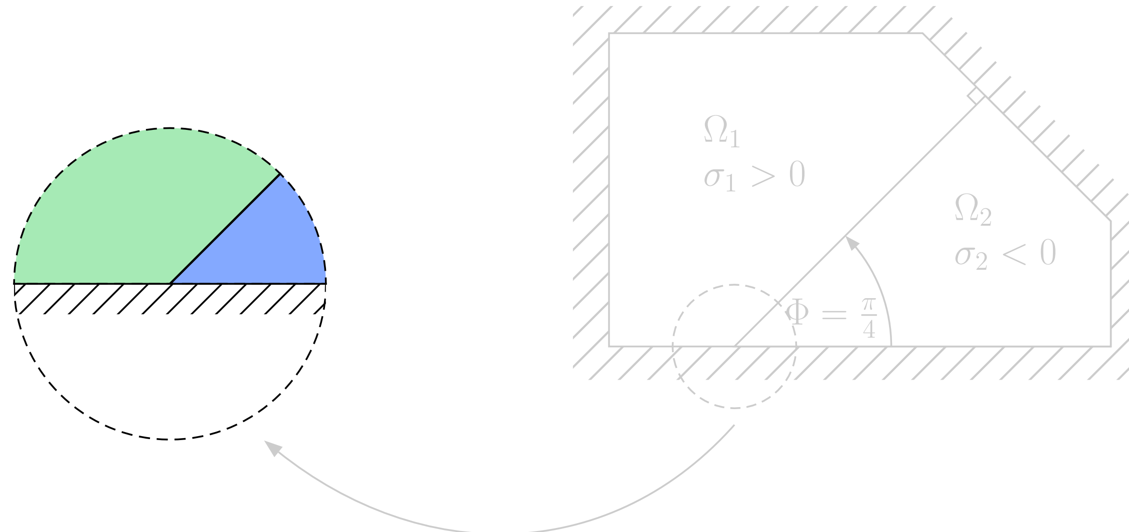

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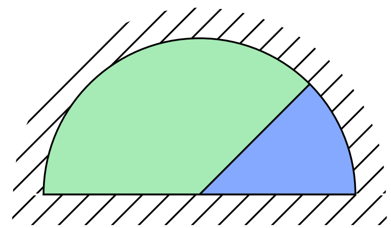

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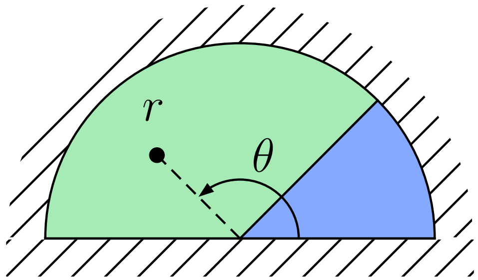

$\omega = \mathrm{D}(0,1) \cap \mathbb{R}^2_+$

$$\begin{cases} -\mathrm{div}(\sigma \nabla u) = f \text{ in } \omega \\ u = 0 \text{ in } \partial\omega \end{cases} \qquad (1)$$

Modal analysis:

For $f = 0$, how do solutions to (1) behave for $r \to 0$? We look for expansions of the form $u(r,\theta) = r^\lambda \varphi(\theta) \; + \; \dots,$

$$0 = \mathrm{div}(\sigma \nabla u) = \mathrm{div}(\sigma \nabla r^\lambda \varphi(\theta)) + \dots$$

$$\Rightarrow 0 = \mathrm{div}(\sigma \nabla r^\lambda \varphi(\theta)) = r^{\lambda-2}\big(\, \partial_\theta \sigma \partial_\theta \varphi + \sigma \lambda^2 \varphi \, \big)$$

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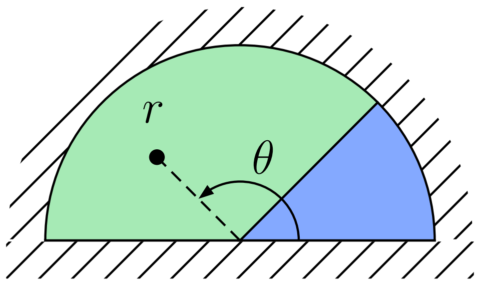

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$$\frac{\partial}{\partial\theta}\left(\sigma(\theta)\frac{\partial\varphi}{\partial\theta}\right) + \lambda^2\sigma(\theta)\varphi(\theta) = 0 \qquad \text{on } (0,\pi) \tag{2}$$

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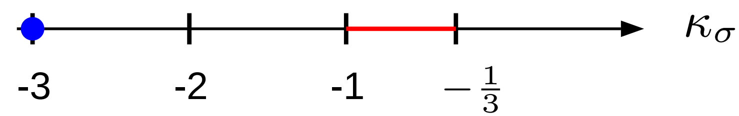
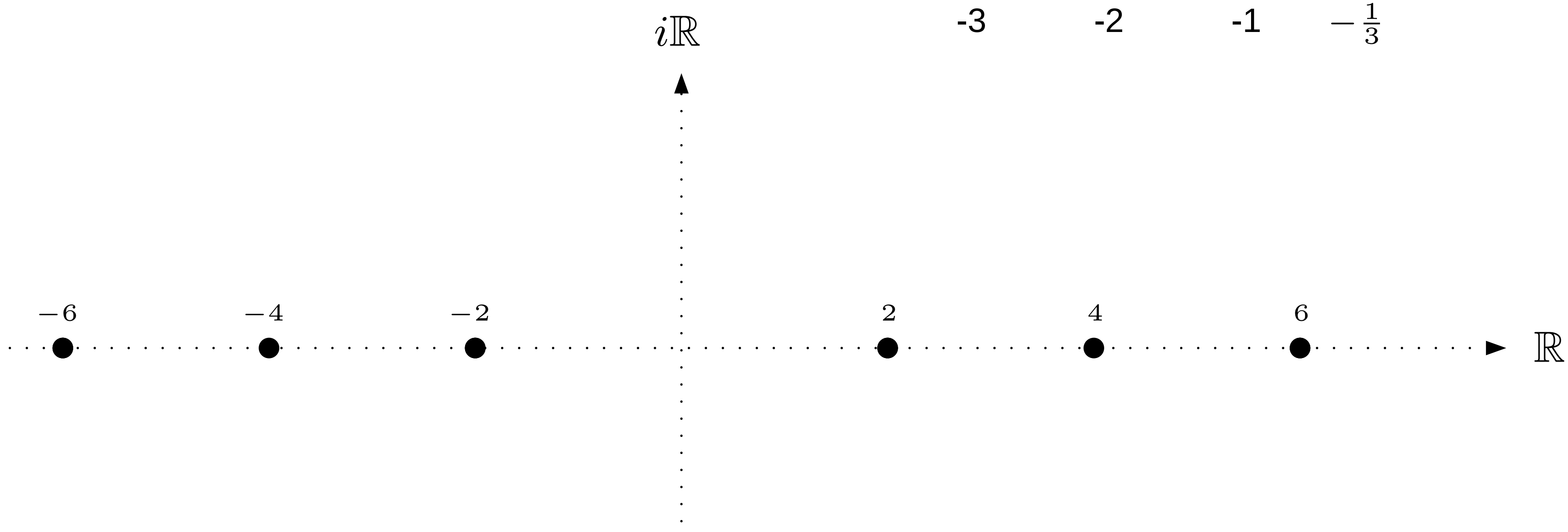
Not a standard problem due to the sign shift of σ at $\theta = \pi/4 \pm 0$. The roots of this problem strongly depend on $\kappa_\sigma = \sigma_2/\sigma_1$.





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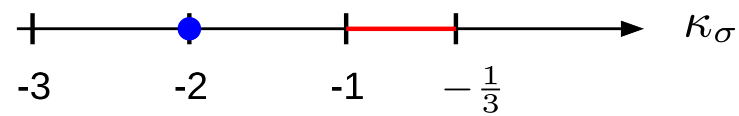

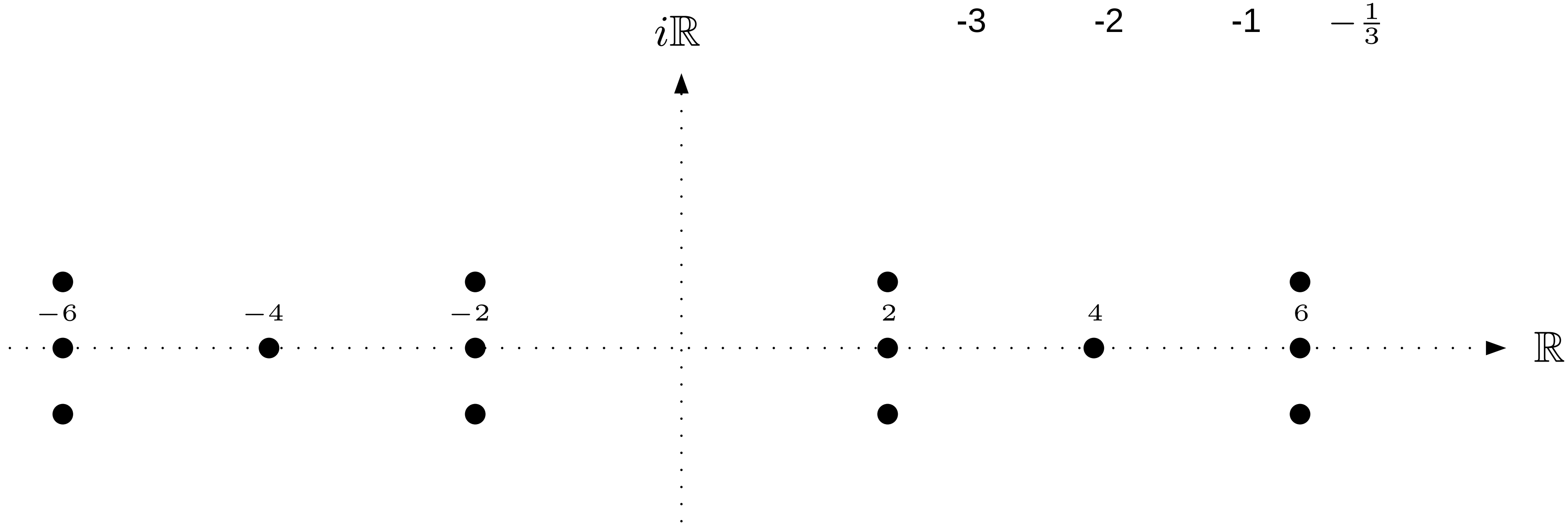

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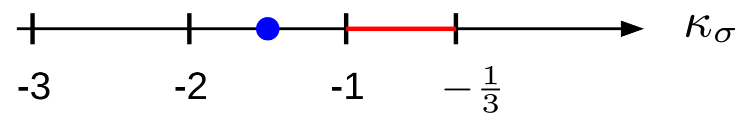

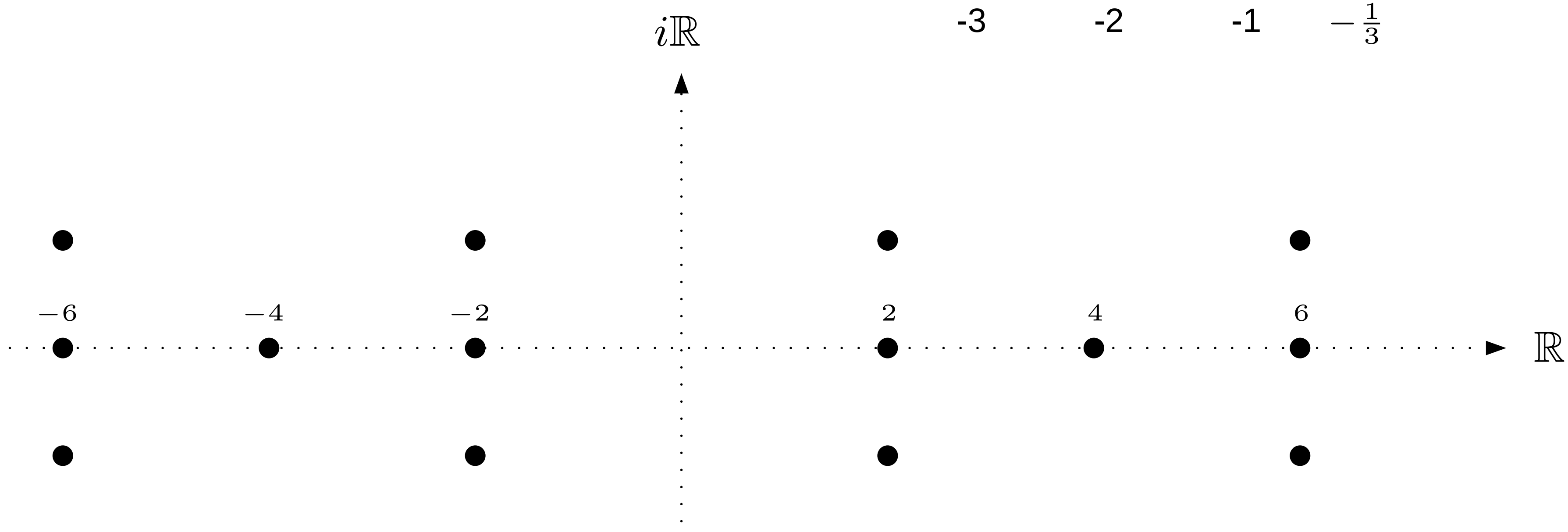

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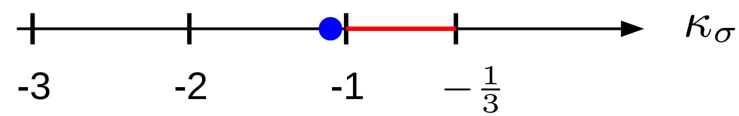

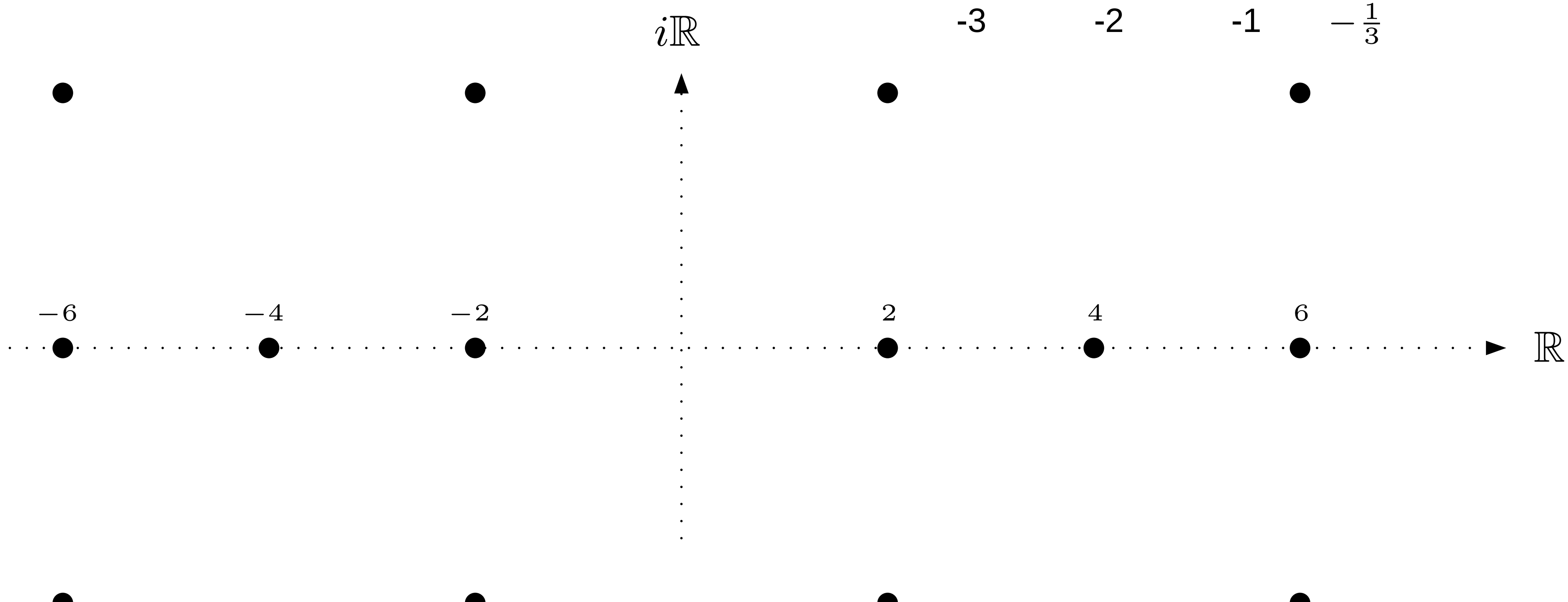

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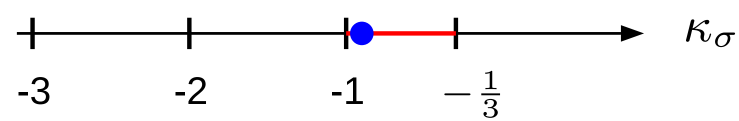

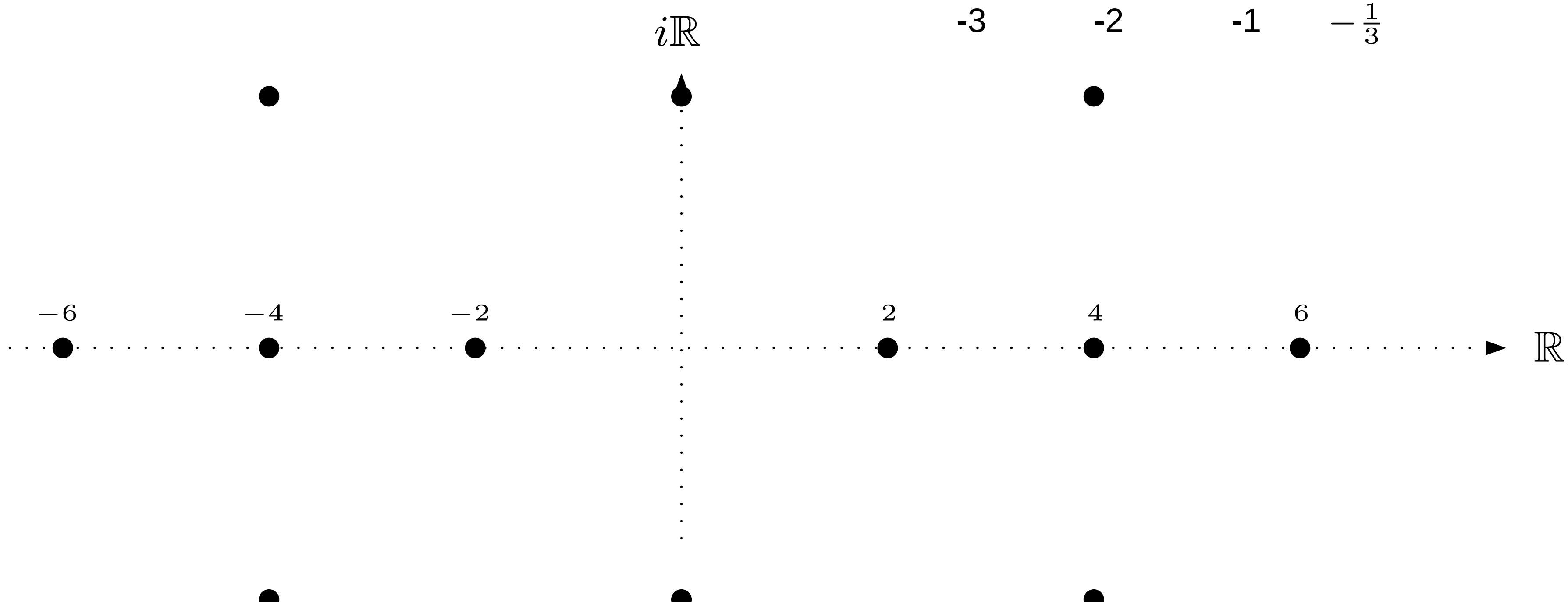

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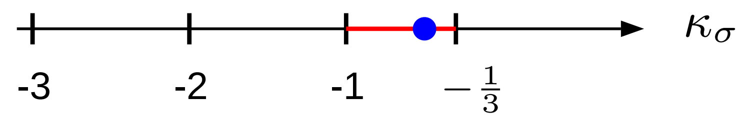

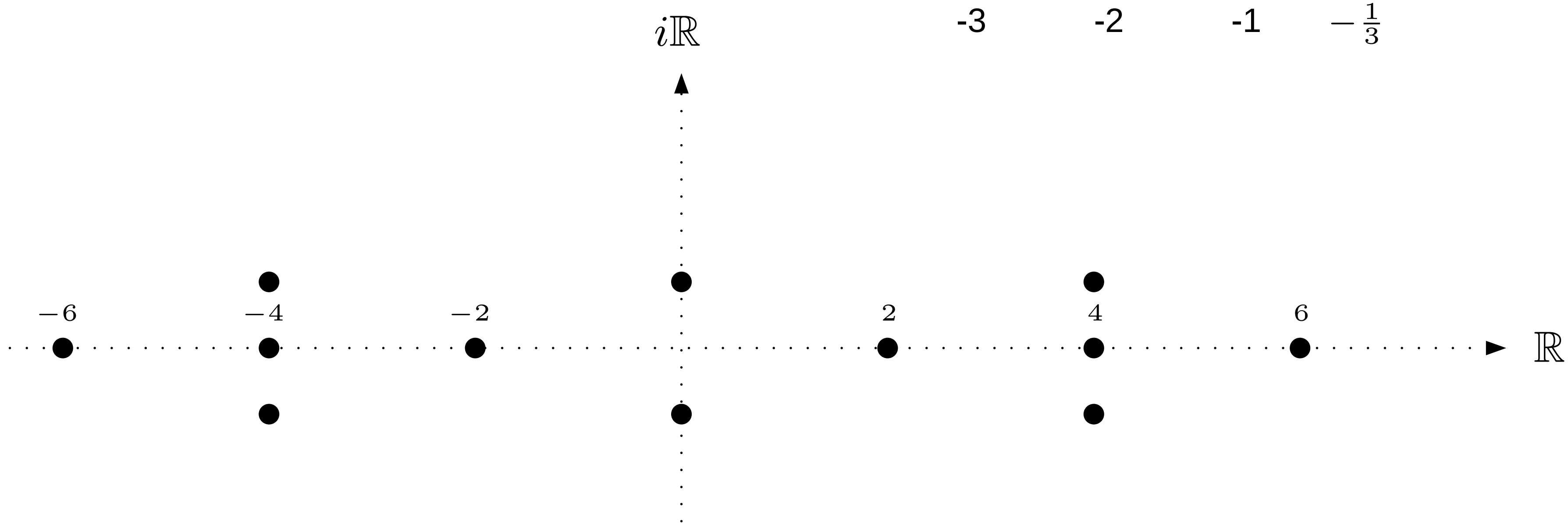

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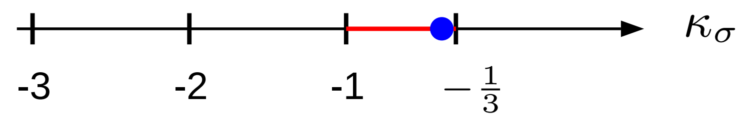

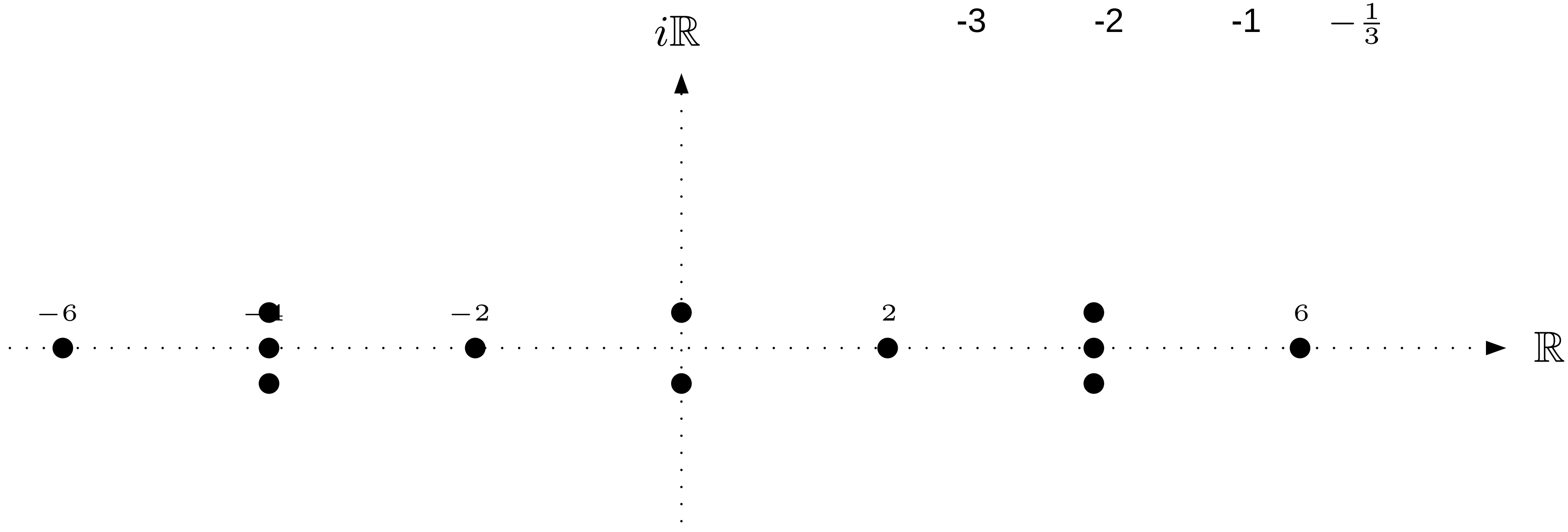

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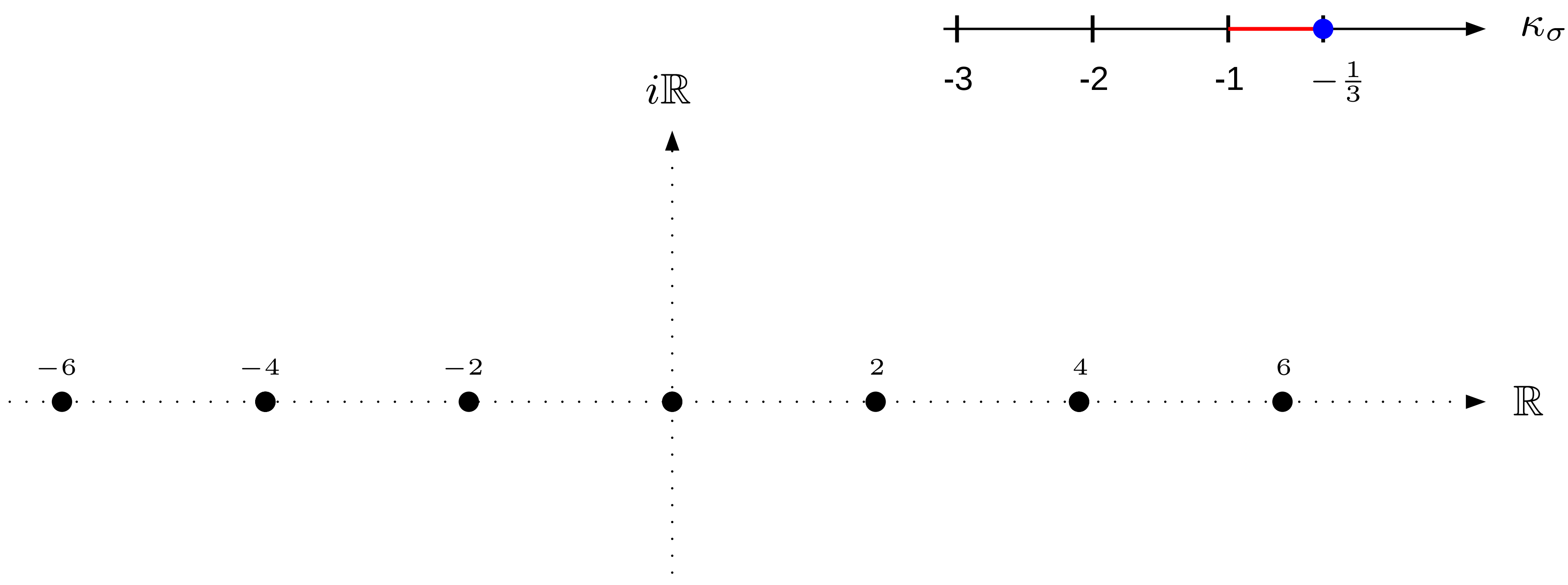

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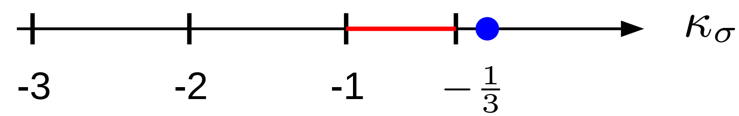
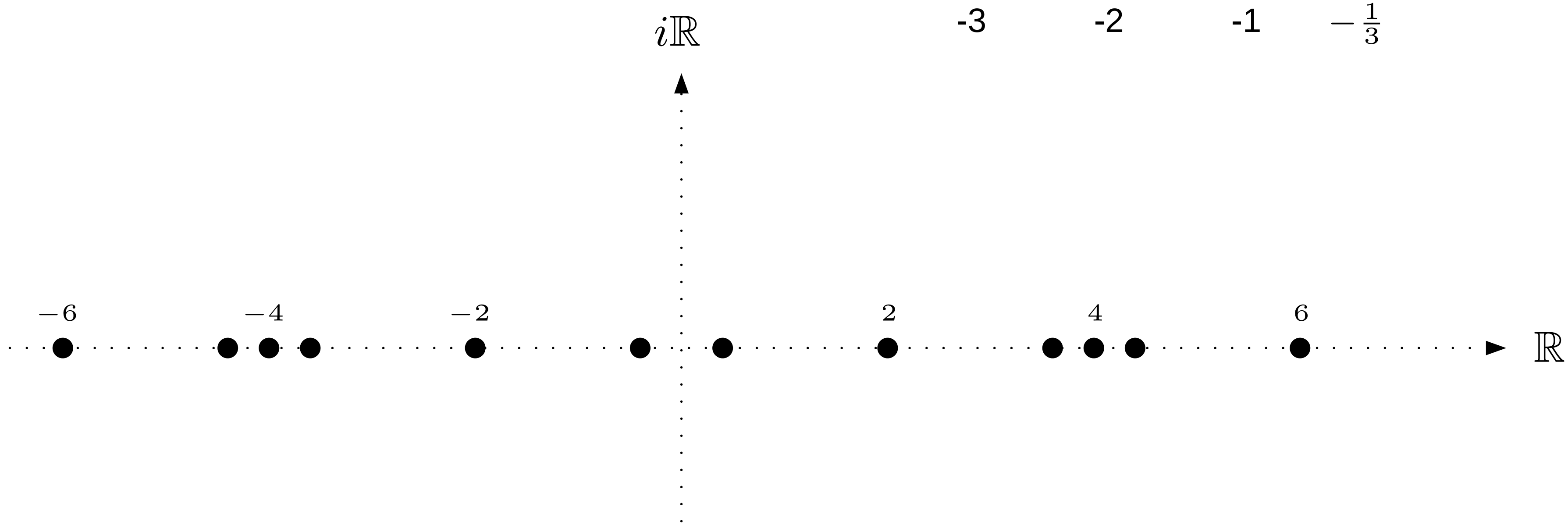


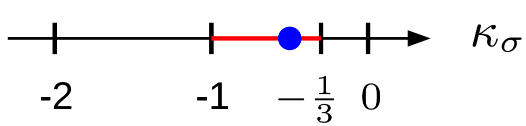


Modal analysis

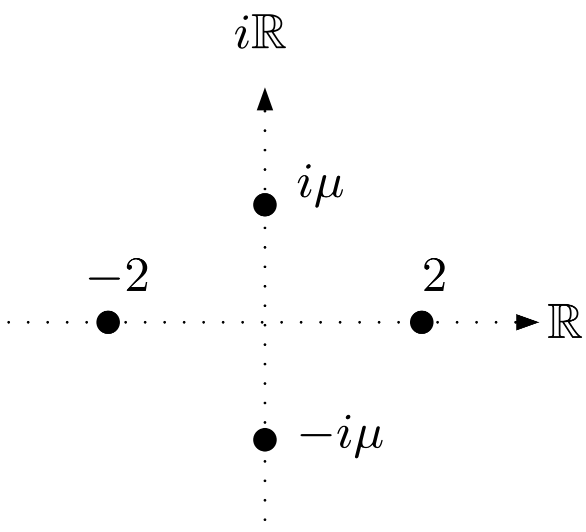
Eigenvalue problem: $(\lambda, \varphi) \in \mathbb{C} \times \mathrm{H}^1_0(0,\pi)$ must satisfy

$$\frac{\partial}{\partial\theta}\left(\sigma(\theta)\frac{\partial\varphi}{\partial\theta}\right) + \lambda^2\sigma(\theta)\varphi(\theta) = 0 \qquad \text{on } (0,\pi) \qquad (2)$$

Not a standard problem due to the sign shift of σ at $\theta = \pi/4 \pm 0$. The roots of this problem strongly depend on $\kappa_\sigma = \sigma_2/\sigma_1$.



For $\beta \in (0,2)$, there exists $\mu \in \mathbb{R}_+$ such that the strip $|\Re e\{\lambda\}| < 2$ only contains the roots: $\pm i\mu$.



The corresponding solution to (2) oscillates as $r \to 0$ (propagative behaviour) and takes the form:

$$r^{i\mu}\varphi_{\mathrm{p}}(\theta) \notin \mathrm{H}^1_0(\omega)$$

Kondratiev's analysis

Weighted Sobolev space: for $\beta \in \mathbb{R}$, define

$$\mathrm{V}^1_\beta(\omega) := \{\ r^{-\beta} v(r,\theta)\ ,\ v \in \mathrm{H}^1_0(\omega)\ \}$$

and for $\beta \in (0,2)$ the operators

$$\begin{array}{l} \mathrm{A}_{+\beta} : \mathrm{V}^1_{+\beta}(\omega) \to \mathrm{V}^1_{-\beta}(\omega)^* \\ \qquad\qquad\ \ \bigcup \qquad\qquad\quad \bigcup \\ \mathrm{A}_{-\beta} : \mathrm{V}^1_{-\beta}(\omega) \to \mathrm{V}^1_{+\beta}(\omega)^* \end{array} \quad \text{with} \quad \langle \mathrm{A}_{\pm\beta} u, v\rangle := \int_\omega \sigma \nabla u\, \nabla v\; d\mathbf{x}$$

Proposition

- A_β is Fredholm if and only if the line $\Re e\{\lambda\} = \beta$ contains no root of the eigenvalue problem.
- If $u \in \mathrm{V}^1_{+\beta}(\omega)$ with $\beta \in (0,2)$ and we have $\mathrm{A}_\beta u \in \mathrm{V}^1_{+\beta}(\omega)^* \subset \mathrm{V}^1_{-\beta}(\omega)^*$ then the following expansion holds:

$$u(r,\theta)\ =\ \alpha_+\, r^{i\mu}\varphi_{\mathrm{p}}(\theta) + \alpha_-\, r^{-i\mu}\varphi_{\mathrm{p}}(\theta) + u_{-\beta}(r,\theta)$$

$$\text{with}\ \ \alpha_\pm \in \mathbb{C},\ u_{-\beta} \in \mathrm{V}^1_{-\beta}(\omega)\ .$$

Kondratiev's analysis

Weighted Sobolev space: for $\beta \in \mathbb{R}$, define

$$\mathrm{V}^1_\beta(\omega) := \{\, r^{-\beta} v(r,\theta)\ ,\ v \in \mathrm{H}^1_0(\omega)\ \}$$

functions possibly blowing up as $r \to 0$ like $r^{-\beta+1}$

and for $\beta \in (0,2)$ the operators

$$\mathrm{A}_{+\beta} : \mathrm{V}^1_{+\beta}(\omega) \to \mathrm{V}^1_{-\beta}(\omega)^*$$

$$\cup \qquad \cup$$

$$\mathrm{A}_{-\beta} : \mathrm{V}^1_{-\beta}(\omega) \to \mathrm{V}^1_{+\beta}(\omega)^*$$

with $\langle \mathrm{A}_{\pm\beta} u, v\rangle := \int_\omega \sigma \nabla u\, \nabla v\, d\mathbf{x}$

functions decaying as $r \to 0$ as fast as $r^{+\beta+1}$

Proposition

- A_β is Fredholm if and only if the line $\Re e\{\lambda\} = \beta$ contains no root of the eigenvalue problem.
- If $u \in \mathrm{V}^1_{+\beta}(\omega)$ with $\beta \in (0,2)$ and we have $\mathrm{A}_\beta u \in \mathrm{V}^1_{+\beta}(\omega)^* \subset \mathrm{V}^1_{-\beta}(\omega)^*$ then the following expansion holds:

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with $\alpha_\pm \in \mathbb{C},\ u_{-\beta} \in \mathrm{V}^1_{-\beta}(\omega)$.

Kondratiev's analysis

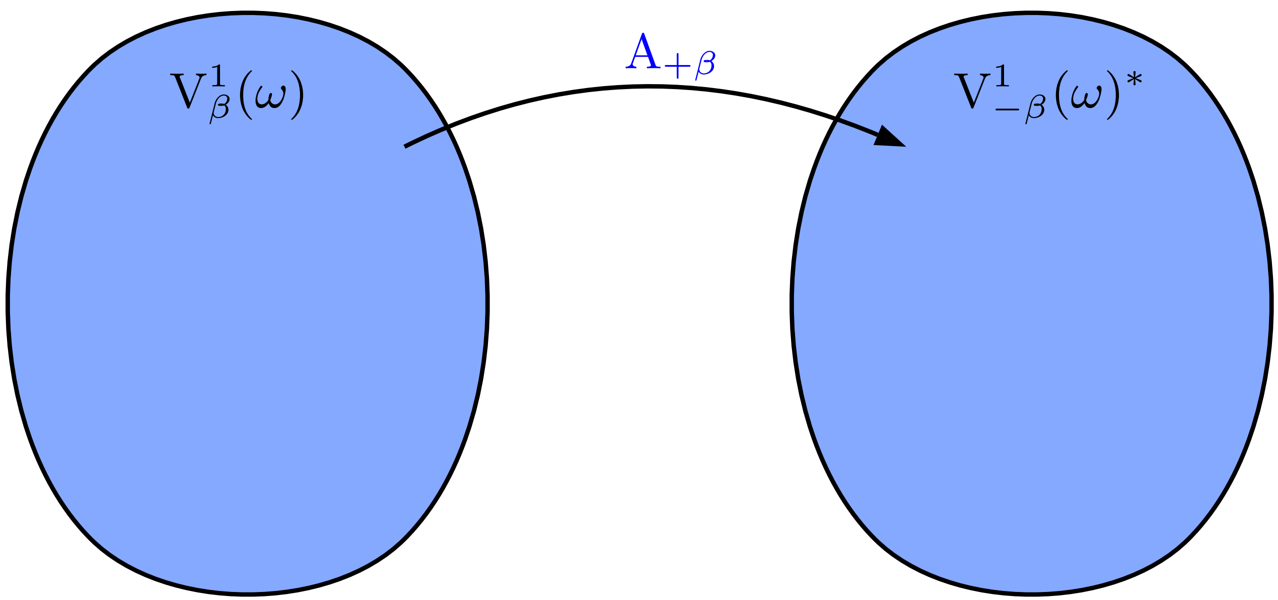
Proposition

For $\beta \in (0,2)$ we have

- $A_{+\beta} : V^1_{+\beta}(\omega) \to V^1_{-\beta}(\omega)^*$ is onto but not one-to-one,

Set $V^{\text{out}}_\beta(\omega) := \text{span}\{\, r^{+i\mu}\varphi_{\text{p}}(\theta)\chi(r)\,\} \oplus V^1_{-\beta}(\omega)$

hence $V^1_{-\beta}(\omega) \subset V^{\text{out}}_\beta(\omega) \subset V^1_{+\beta}(\omega)$



Kondratiev's analysis

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For $\beta \in (0,2)$ we have

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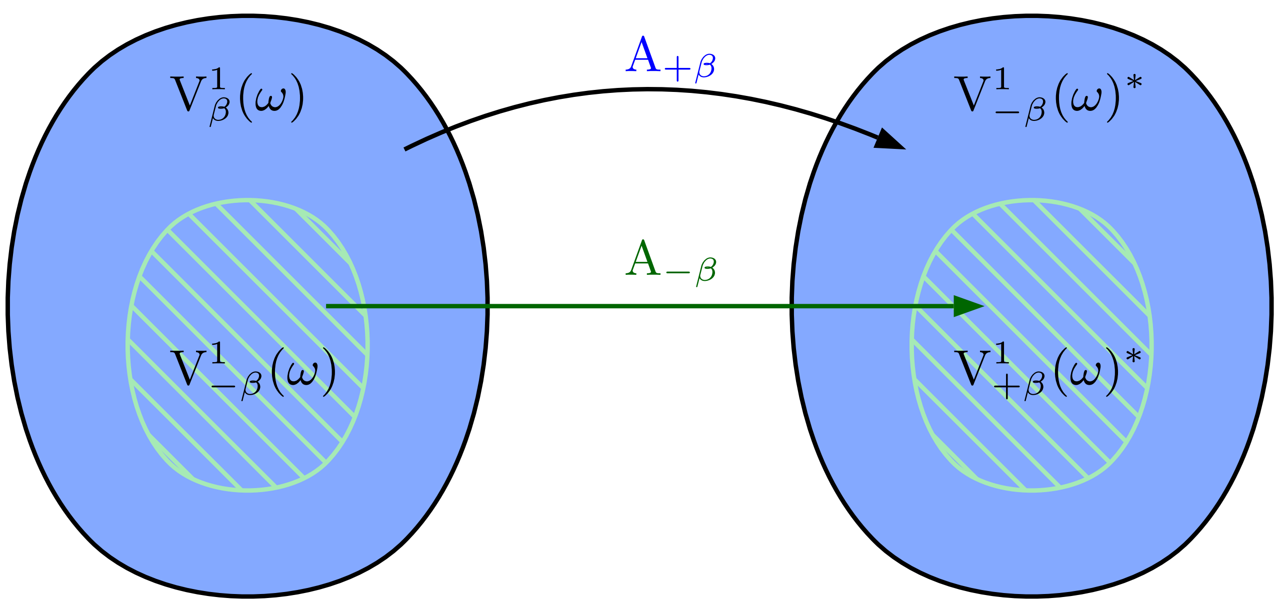

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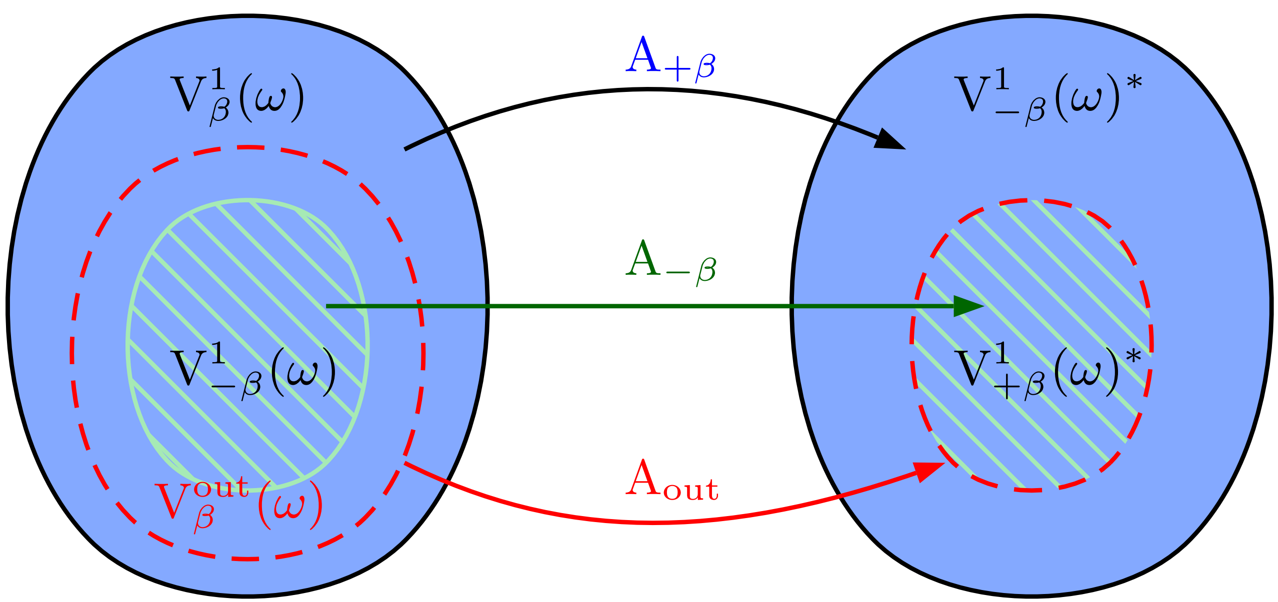

Kondratiev's analysis

Theorem

For $\beta \in (0,2)$ we have

- $\mathrm{A}_{+\beta} : \mathrm{V}^1_{+\beta}(\omega) \to \mathrm{V}^1_{-\beta}(\omega)^*$ is onto but not one-to-one,
- $\mathrm{A}_{\mathrm{out}} : \mathrm{V}^{\mathrm{out}}_{\beta}(\omega) \to \mathrm{V}^1_{+\beta}(\omega)^*$ is an isomorphism,
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Set $\mathrm{V}^{\mathrm{out}}_{\beta}(\omega) := \mathrm{span}\{\ r^{+i\mu}\varphi_{\mathrm{p}}(\theta)\chi(r)\ \} \oplus \mathrm{V}^1_{-\beta}(\omega)$

Considering $\mathrm{V}^1_{\pm\beta}(\Omega), \mathrm{V}^{\mathrm{out}}_{\beta}(\Omega)$, a similar result holds for the initial problem set in Ω up to a compact perturbation [Bonnet-BenDhia, Chesnel, Claeys, 2013].

Theorem

If $\kappa_\sigma \in \mathscr{I}$, suppose that $f \in \mathrm{V}^1_{+\beta}(\Omega)^*$ for some $\beta \in (0,2)$. Then the following problem is of Fredholm type with index 0:

Find $u \in \mathrm{V}^{\mathrm{out}}_{\beta}(\Omega)$ such that

$$\int_\Omega \sigma \nabla u \nabla v \, d\mathbf{x} \ = \langle f, v\rangle \quad \forall v \in \mathrm{V}^1_{+\beta}(\Omega).$$

Rounded corner problem

Sobolev spaces not adapted to our corner problem.
$\Rightarrow$ uncomfortable for numerical simulation.

Is it possible to regularize this problem so as to make it fit the standard Sobolev framework?

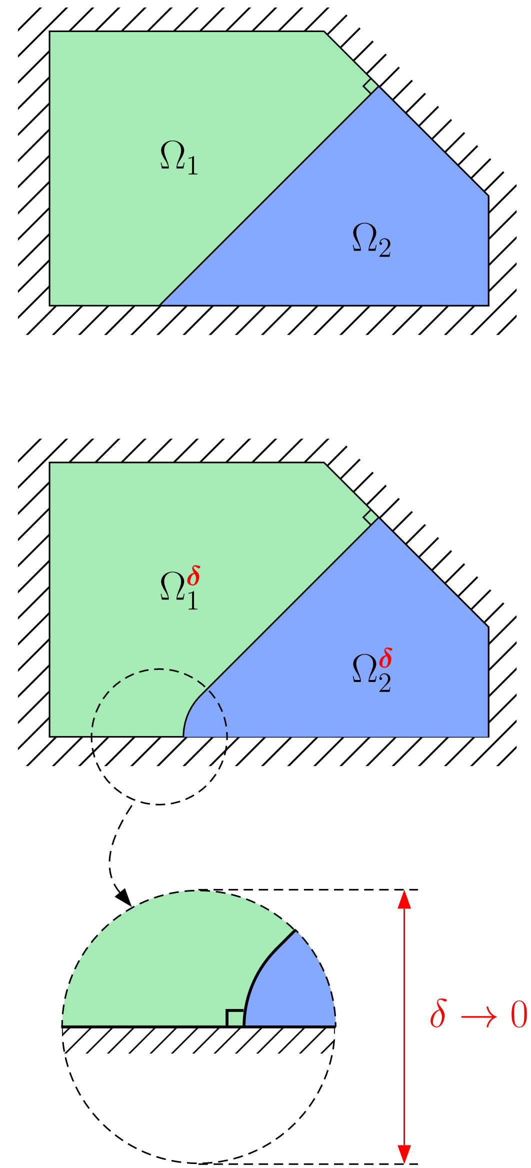
Question: What about rounding the corner?
Is this an admissible regularization process?

Set $\sigma^\delta := \sigma_j$ in Ω_j^δ, and take $f \in \mathrm{H}^{-1}(\Omega)$, with $f = 0$ next to $r = 0$ (for simplicity...).

$$\left\{ \begin{array}{l} \text{Find } u^\delta \in \mathrm{H}^1_0(\Omega) \ \text{ such that} \\ -\mathrm{div}(\sigma^\delta \nabla u^\delta) = f \text{ in } \ \Omega \,. \end{array} \right. \qquad (3)$$

Question: Assuming (3) well posed, $u^\delta \underset{\delta \to 0}{\sim} ?$

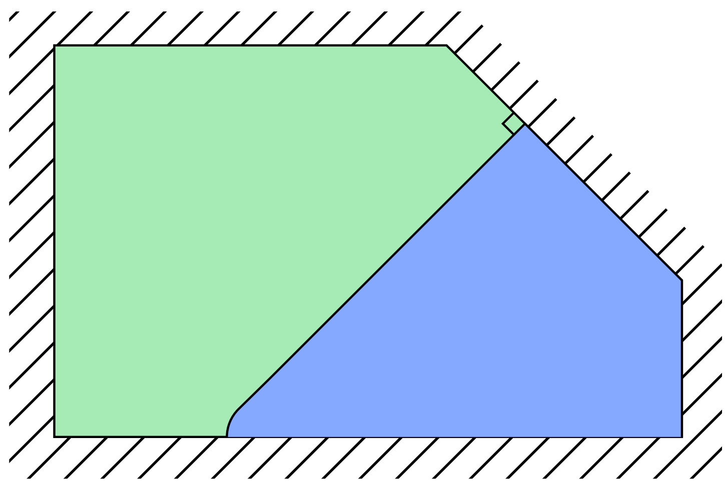
We present formal matched asymptotics...



Construction of the asymptotics

The method of matched asymptotics consist in approximating the exact solution u^δ with a "more explicit" function defined by

$$\widetilde{u}^\delta(r,\theta) :=$$



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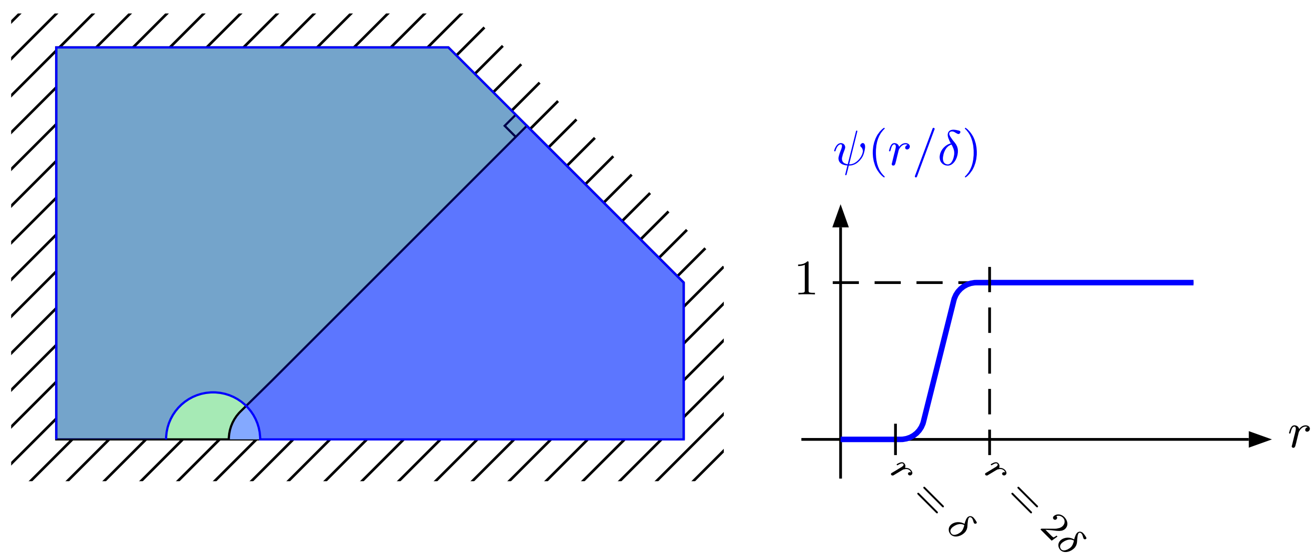
The method of matched asymptotics consist in approximating the exact solution u^δ with a "more explicit" function defined by

$$\widetilde{u}^\delta(r,\theta) := \psi(r/\delta) \times \text{far field expansion}$$



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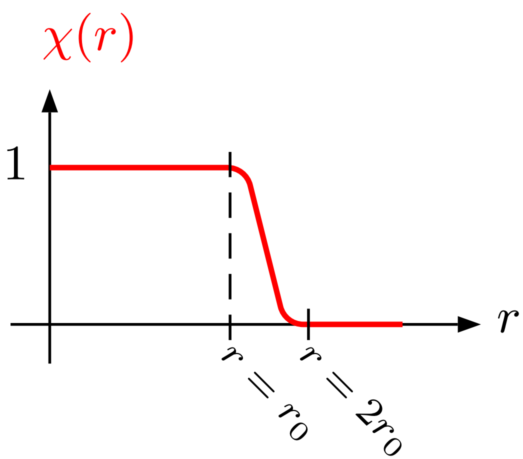
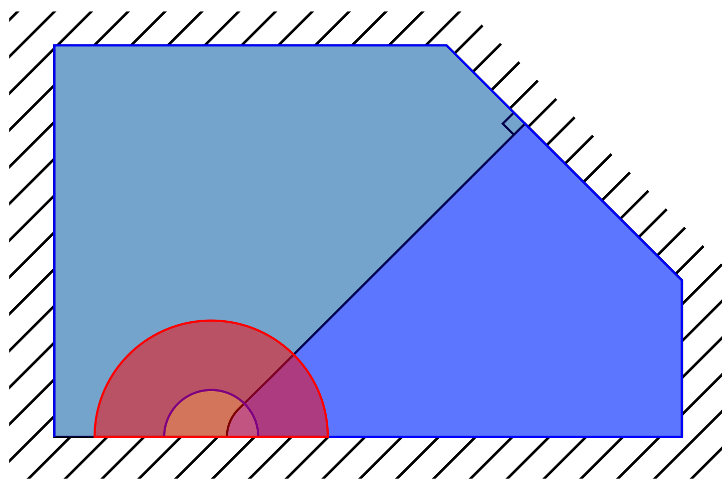
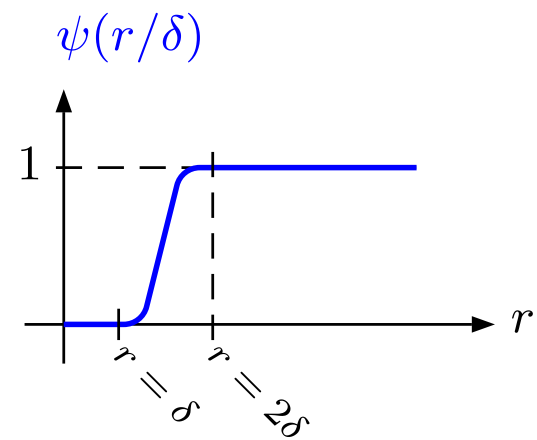
$$\widetilde{u}^\delta(r,\theta) := {\color{blue}\psi(r/\delta) \times \text{far field expansion}} + {\color{red}\chi(r) \times \text{near field expansion}}$$



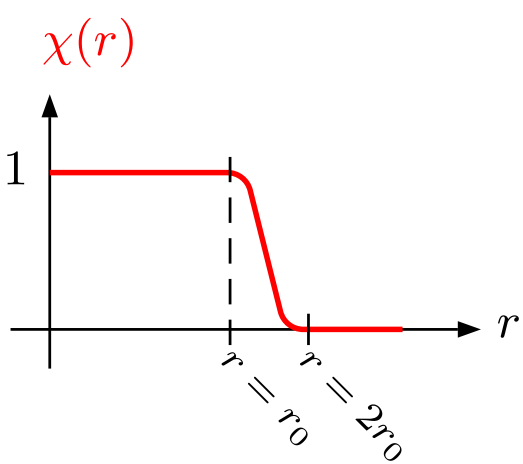


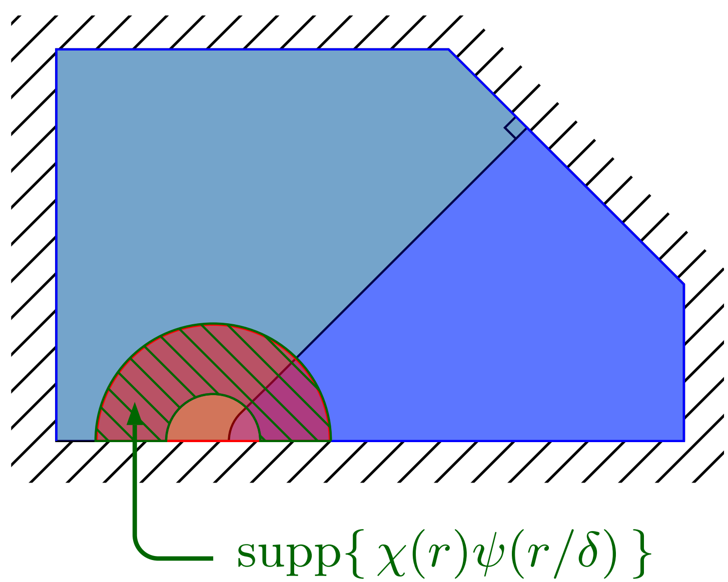


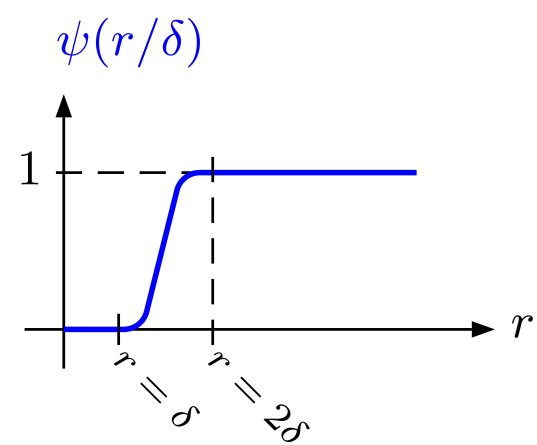
Construction of the asymptotics

The method of matched asymptotics consist in approximating the exact solution u^δ with a "more explicit" function defined by

$$\widetilde{u}^\delta(r,\theta) := \psi(r/\delta) \times \text{far field expansion} + \chi(r) \times \text{near field expansion}$$
$$- \chi(r)\psi(r/\delta) \times \text{matching contribution}$$

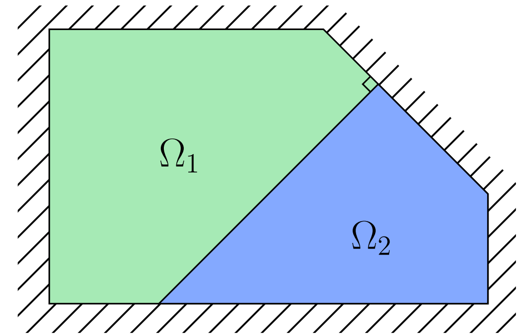






Far field expansion

Ansatz: $u^\delta(r,\theta) = u^0(r,\theta) + a(\delta)\,\zeta(r,\theta) + \dots$



Far field expansion

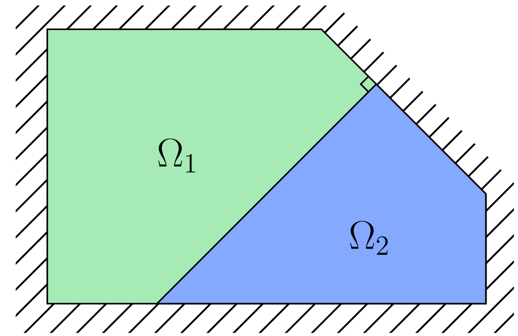
Ansatz: $u^\delta(r,\theta) = u^0(r,\theta) + a(\delta)\,\zeta(r,\theta) + \dots$

limit field

corrector (=?)

$u^0 \in \mathrm{V}^{\mathrm{out}}_\beta(\Omega)$ and

$-\mathrm{div}(\sigma^0 \nabla u^0) = f$ in Ω



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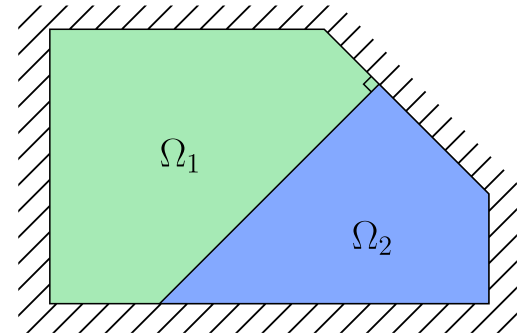

$$f = -\mathrm{div}(\sigma^\delta \nabla u^\delta) \approx -\mathrm{div}(\sigma^0 \nabla u^\delta)$$

Far field expansion

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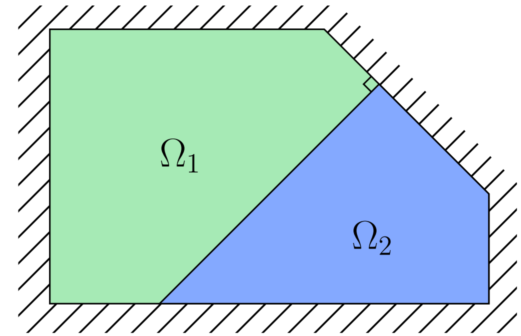

$$f \approx -\mathrm{div}\big(\,\sigma^0 \nabla (u^0 + a(\delta)\zeta + \dots)\,\big)$$

Far field expansion

Ansatz: $u^\delta(r,\theta) = u^0(r,\theta) + a(\delta)\,\zeta(r,\theta) + \dots$

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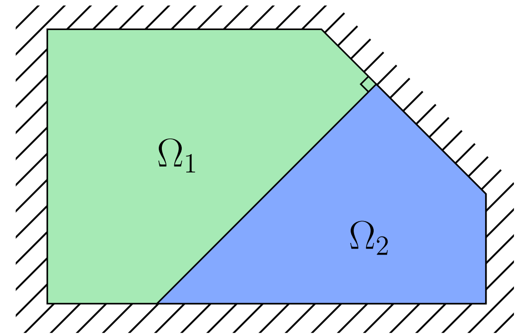


$f \approx -\mathrm{div}(\sigma^0 \nabla u^0) - a(\delta)\mathrm{div}(\sigma^0 \nabla \zeta) + \dots \quad \Longrightarrow \quad -\mathrm{div}(\sigma^0 \nabla \zeta) = 0$ in Ω

$\zeta = 0$ on $\partial\Omega$, $\zeta \neq 0$

Recall: $\mathrm{V}^1_{-\beta}(\Omega) \subset \mathrm{V}^{\mathrm{out}}_\beta(\Omega) \subset \mathrm{V}^1_{+\beta}(\Omega)$

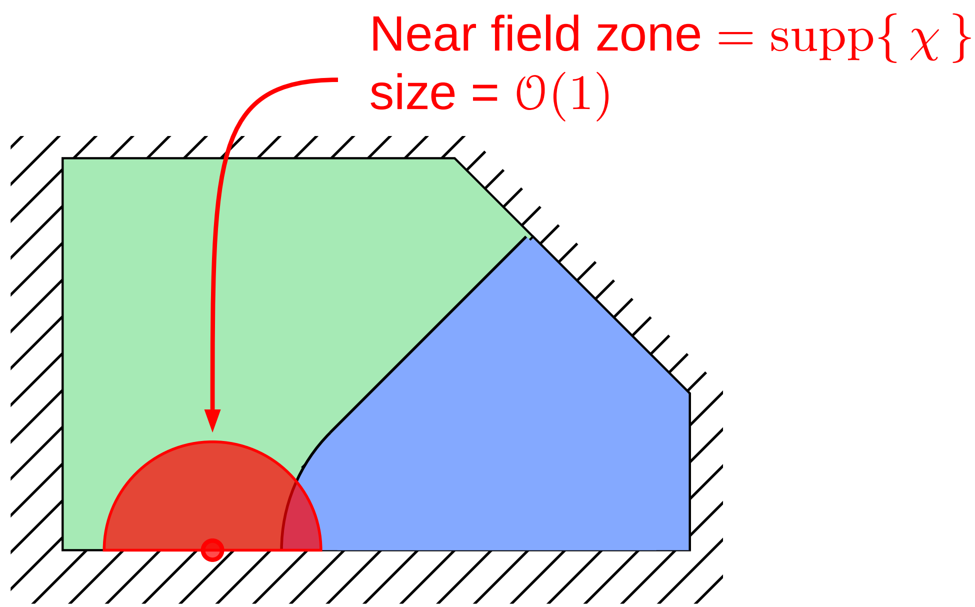
$\langle \mathrm{A}_\star u, v\rangle = \int_\Omega \sigma^0 \nabla u\, \nabla v\, d\mathbf{x}$, where $\star = \beta, \mathrm{out}$.

Proposition
Assume $\mathrm{Ker}(\mathrm{A}_{\mathrm{out}}) = \{0\}$. Then $\mathrm{Ker}(\mathrm{A}_\beta) = \mathrm{span}\{\,\zeta\,\}$ where $\zeta \in \mathrm{V}^1_{+\beta}(\Omega)$ is the unique function satisfying

$$\begin{cases} \zeta(r,\theta) - r^{-i\mu}\varphi_{\mathrm{p}}(\theta)\chi(r) \ \in \mathrm{V}^{\mathrm{out}}_\beta(\Omega), \\ -\mathrm{div}(\sigma^0 \nabla \zeta) = 0 \quad \text{in } \Omega\,. \end{cases}$$

Near field expansion

Next to the rounded corner, we use the fast variable $\rho = r/\delta$ so as to normalize the geometry of the perturbation.



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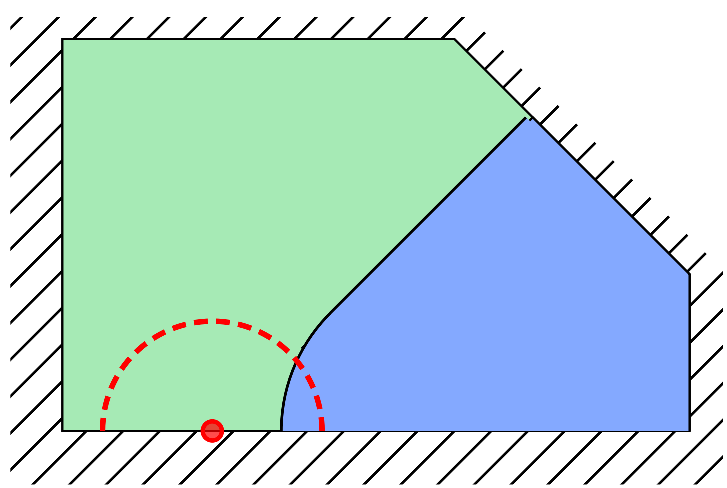

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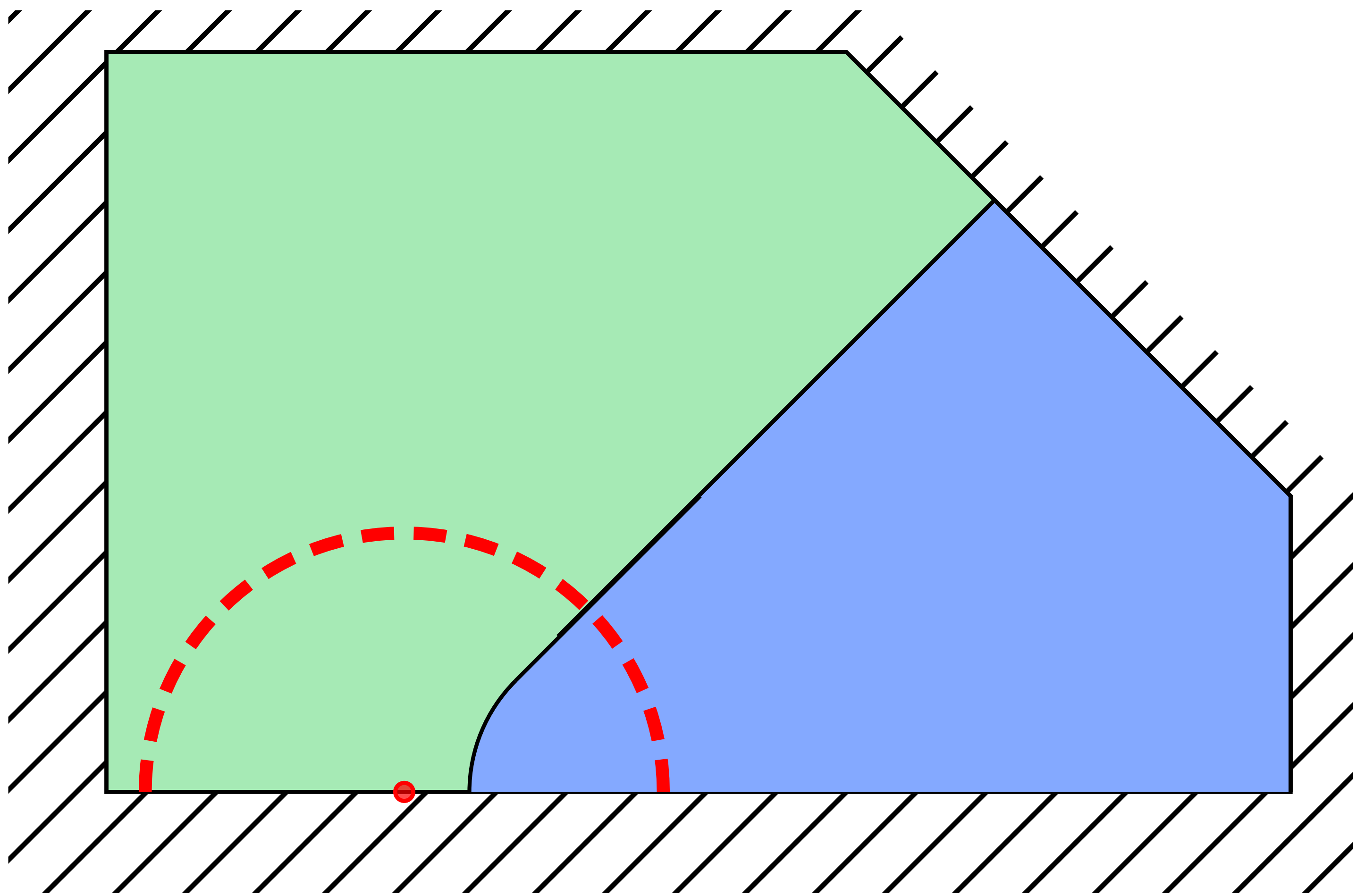
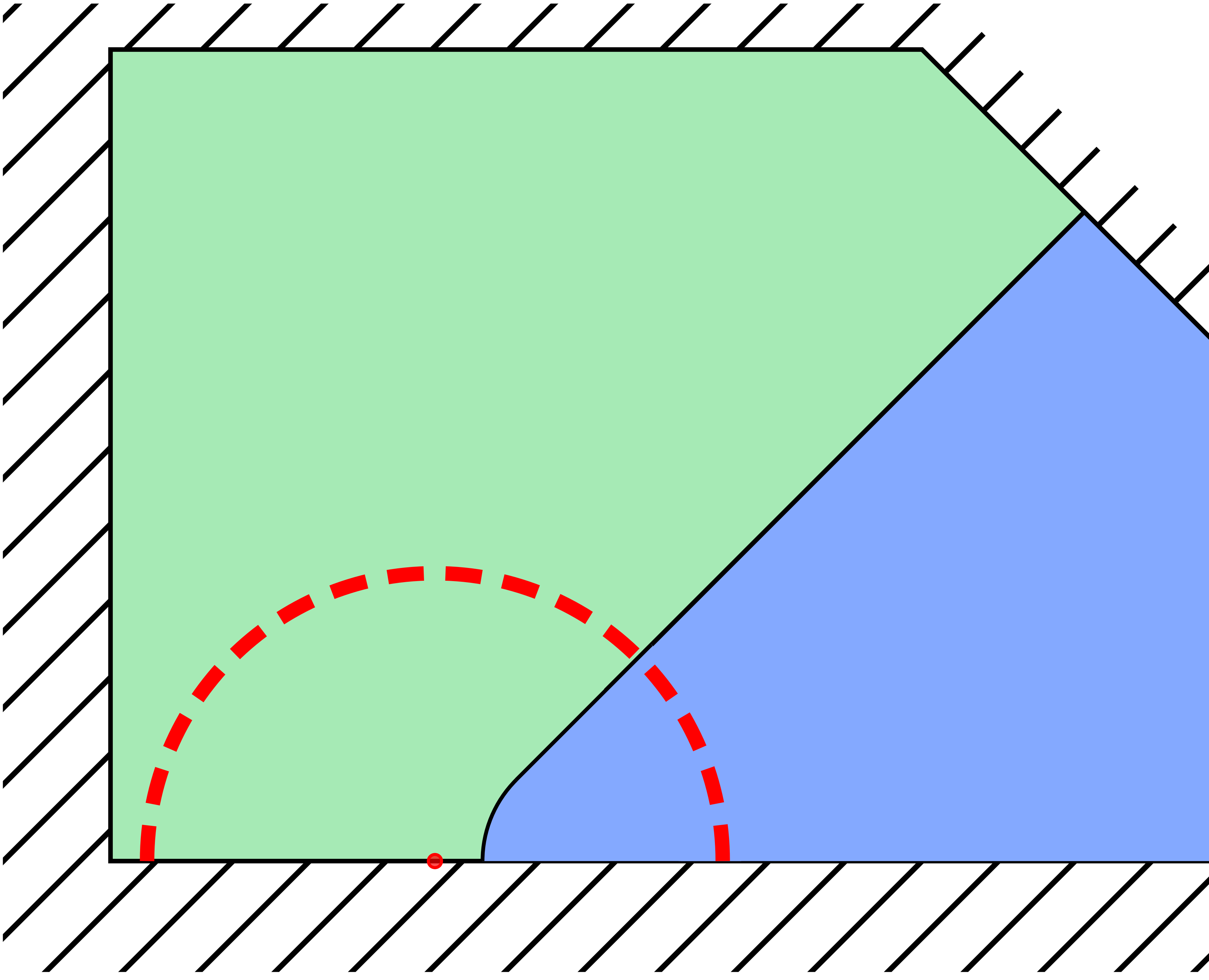
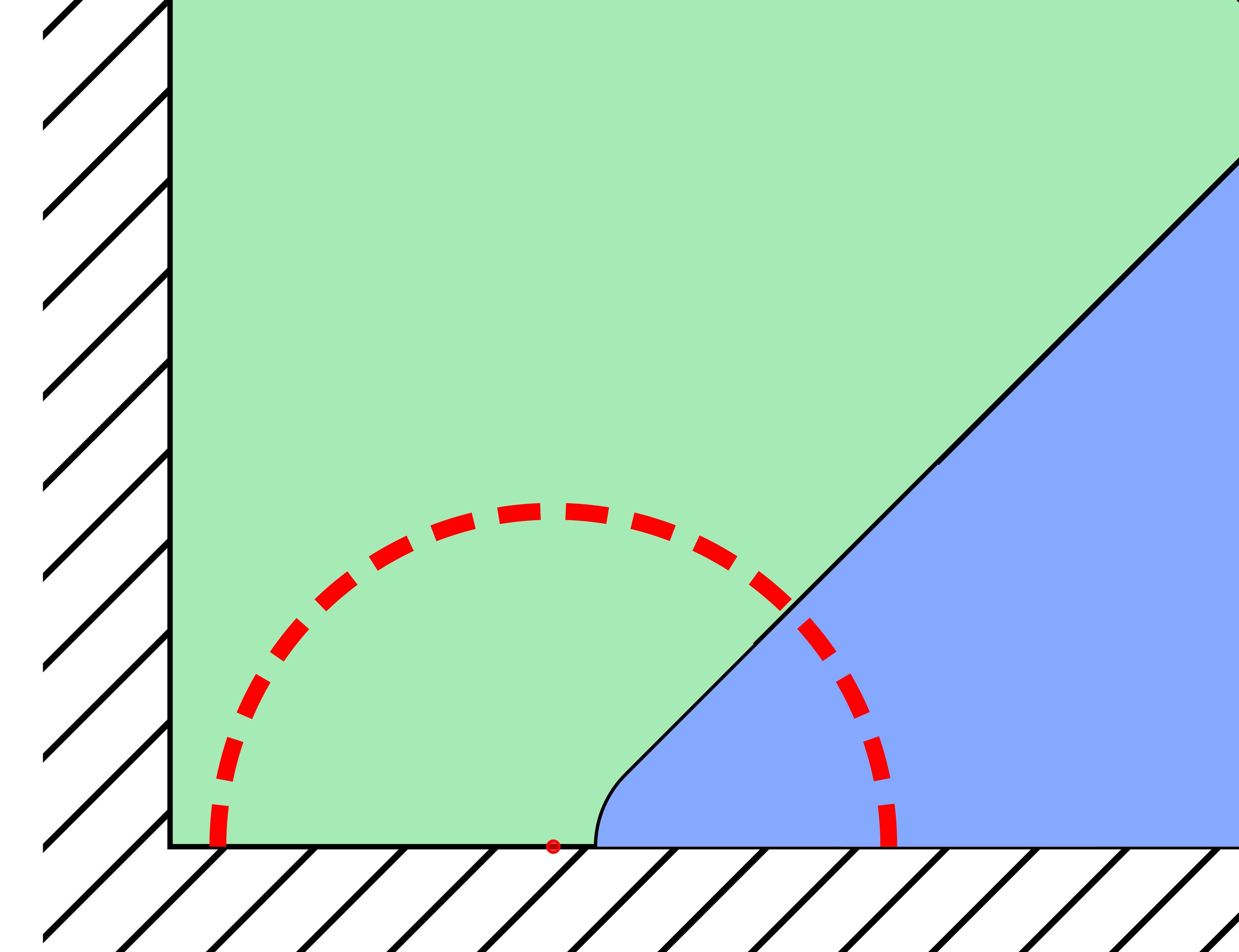
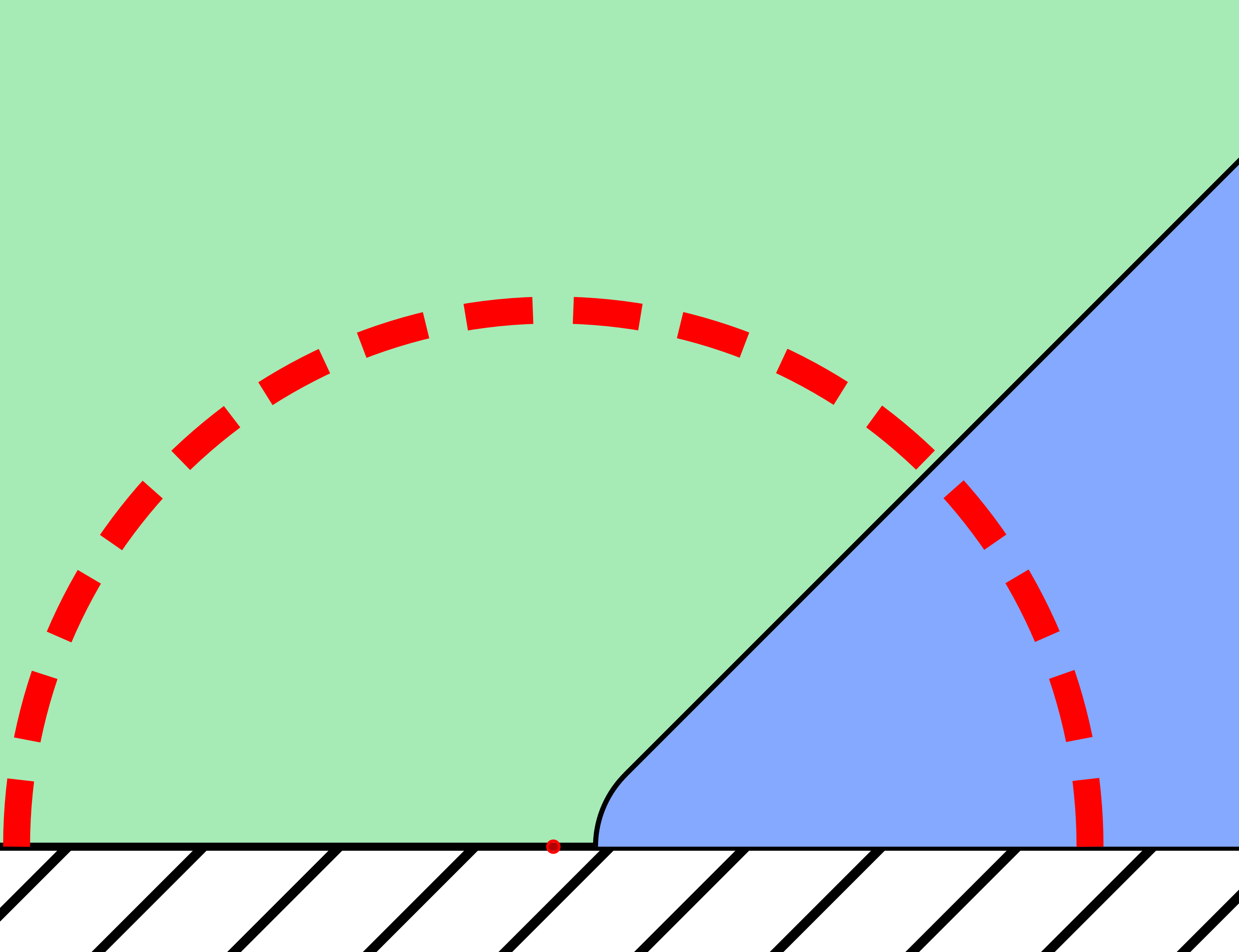
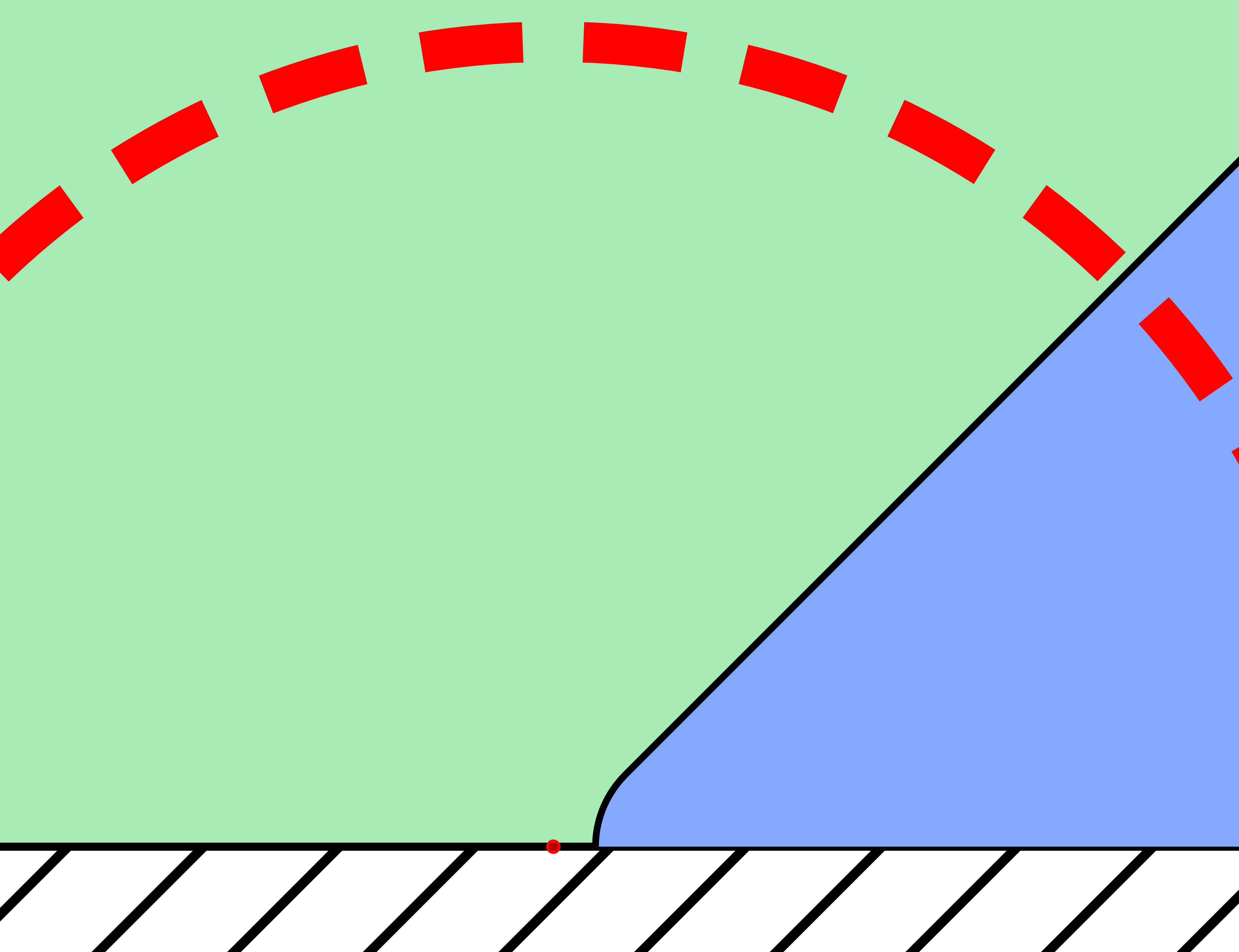


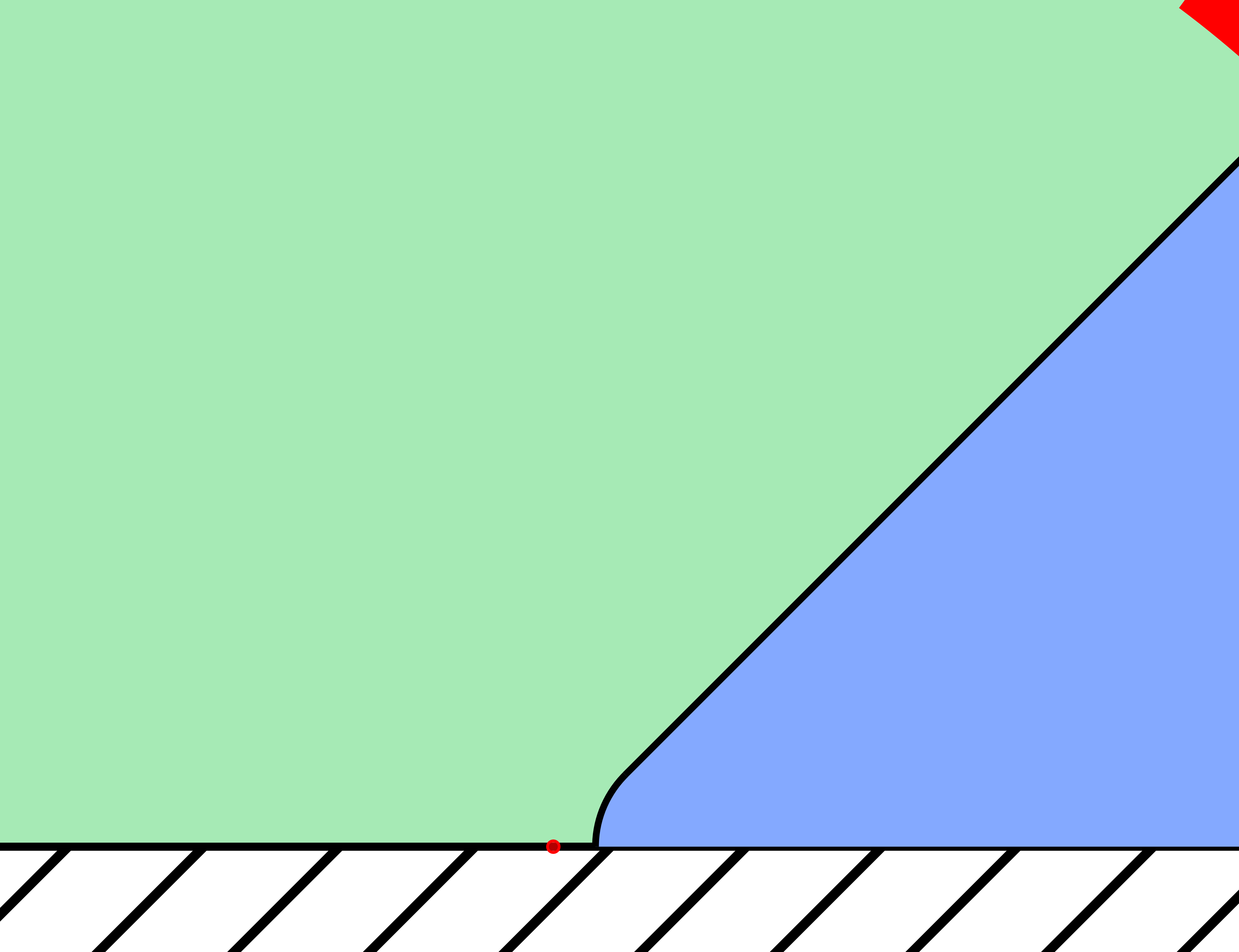
Near field expansion

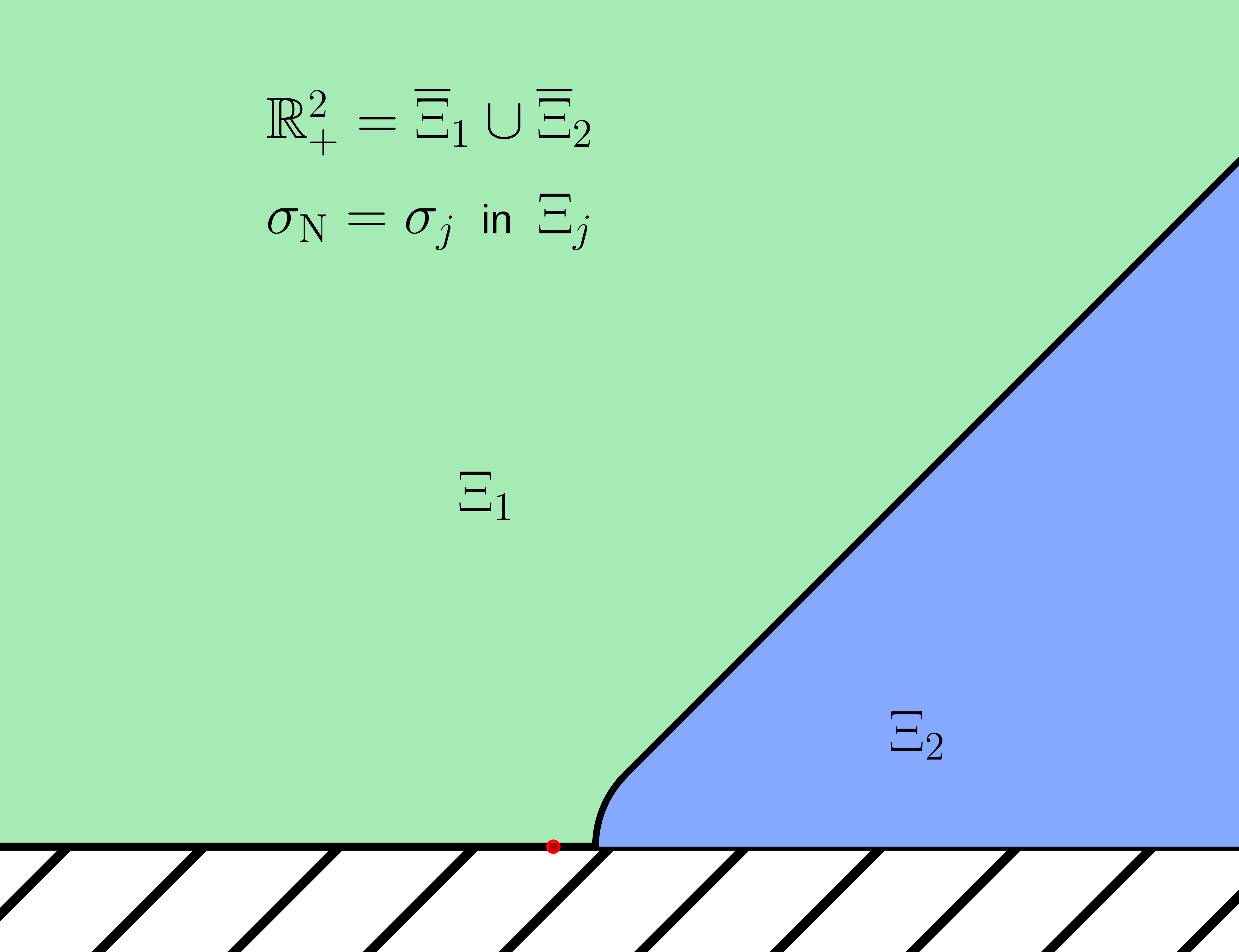








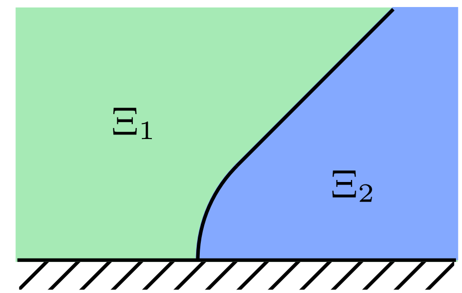



$\mathbb{R}^2_+ = \overline{\Xi}_1 \cup \overline{\Xi}_2$
$\sigma_{\mathrm{N}} = \sigma_j$ in Ξ_j
Ξ_1
Ξ_2

Near field expansion

Next to the rounded corner, we use the fast variable $\rho = r/\delta$ so as to normalize the geometry of the perturbation. The normalized field $\mathrm{U}^\delta(\rho,\theta) := u^\delta(\delta\rho,\theta)$ satisfies

$$\begin{cases} -\mathrm{div}(\sigma_{\mathrm{N}} \nabla \mathrm{U}^\delta) = 0 \quad \text{in } \mathbb{R}^2_+, \\ \mathrm{U}^\delta = 0 \quad \text{on } \partial\mathbb{R}^2_+ . \end{cases}$$



Ansatz: $\mathrm{U}^\delta(\rho,\theta) = b(\delta)\,\mathrm{Z}(\rho,\delta) + \dots$

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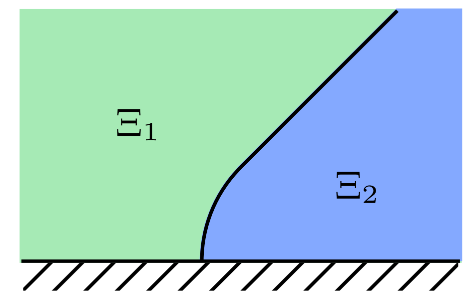

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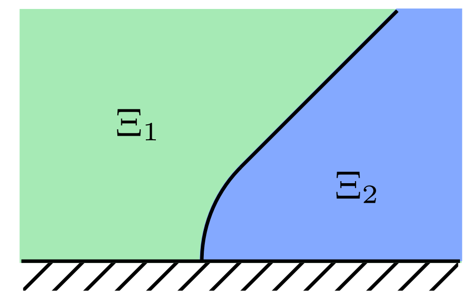

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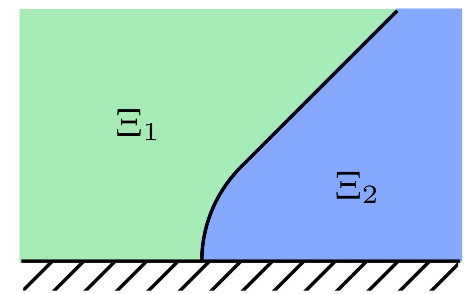
$$\begin{cases} -\mathrm{div}(\sigma_{\mathrm{N}}\nabla \mathrm{Z}) = 0 \quad \text{in } \mathbb{R}^2_+, \\ \mathrm{Z} = 0 \quad \text{on } \partial\mathbb{R}^2_+ \ . \end{cases}$$



Weighted Sobolev framework:

$$\mathrm{W}^1_\beta(\mathbb{R}^2_+) = \{\ (1+\rho)^\beta v(\rho,\theta),\ v \in \mathrm{H}^1_0(\mathbb{R}^2_+)\ \}$$

$$\mathrm{W}^{\mathrm{out}}_\beta(\mathbb{R}^2_+) = \mathrm{span}\{\ \rho^{-i\mu}\varphi_{\mathrm{p}}(\theta)\psi(\rho)\ \} \oplus \mathrm{W}^1_{-\beta}(\mathbb{R}^2_+)$$

Proposition

Suppose $\kappa_\sigma \in \mathscr{I}$, take $\beta \in (0,2)$. Then $\mathscr{A}_{\mathrm{out}} : \mathrm{W}^{\mathrm{out}}_\beta(\mathbb{R}^2_+) \to \mathrm{W}^1_\beta(\mathbb{R}^2_+)^*$ defined by $\langle \mathscr{A}_{\mathrm{out}} u, v\rangle := \int_{\mathbb{R}^2_+} \sigma_{\mathrm{N}} \nabla u \nabla v \, d\mathbf{x}$ is of Fredholm type with index 0.

Proposition

If $\mathrm{Ker}(\mathscr{A}_{\mathrm{out}}) = \{0\}$, there exists a unique $\mathrm{Z} \in \mathrm{W}^1_\beta(\mathbb{R}^2_+)$ such that

$$\mathrm{Z}(\rho,\theta) - \rho^{+i\mu}\varphi_{\mathrm{p}}(\theta)\psi(\rho) \in \mathrm{W}^{\mathrm{out}}_\beta(\mathbb{R}^2_+) \quad \text{and} \quad -\mathrm{div}(\sigma_{\mathrm{N}}\nabla \mathrm{Z}) = 0 \ \text{ in } \mathbb{R}^2_+ \ .$$

Matching

We have defined $\zeta(r,\theta), \mathrm{Z}(\rho,\theta)$, but we still have to determine the jauge functions $a(\delta), b(\delta)$. We do so by matching radial expansions.

$$
\begin{aligned}
u^{\delta}(r,\theta) &= u^{0}(r,\theta) + a(\delta)\,\zeta(r,\theta) + \dots \\
&= c_0\, r^{+i\mu}\varphi_p(\theta) + a(\delta)\big(\, r^{-i\mu}\varphi_p(\theta) + c_\zeta\, r^{+i\mu}\varphi_p(\theta)\,\big) + \dots
\end{aligned}
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$$u^0(r,\theta) = c_0\, r^{+i\mu}\varphi_p(\theta) + \dots$$

since $u^0 \in \mathrm{V}_\beta^{\mathrm{out}}(\Omega)$

$$\zeta(r,\theta) = r^{-i\mu}\varphi_p(\theta) + c_\zeta r^{+i\mu}\varphi_p(\theta) + \dots$$

since $\zeta(r,\theta) - r^{-i\mu}\varphi_p(\theta)\chi(r) \in \mathrm{V}_\beta^{\mathrm{out}}(\Omega)$

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&= (\, c_0 + a(\delta)c_\zeta\,)\, r^{+i\mu}\varphi_{\mathrm{p}}(\theta) + a(\delta) r^{-i\mu}\varphi_{\mathrm{p}}(\theta) + \dots \qquad \text{for } r \to 0.
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$$Z(\rho,\theta) = \rho^{+i\mu}\varphi_{\mathrm{p}}(\theta) + c_{\mathrm{Z}}\rho^{-i\mu}\varphi_{\mathrm{p}}(\theta) + \dots$$

since $Z(\rho,\theta) - \rho^{+i\mu}\varphi_{\mathrm{p}}(\theta) \in \mathrm{W}^{\mathrm{out}}_\beta(\mathbb{R}^2_+)$

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u^\delta(r,\theta) &= \mathrm{U}^\delta\Big(\frac{r}{\delta},\theta\Big) = b(\delta)\,\mathrm{Z}\Big(\frac{r}{\delta},\theta\Big) + \dots \\
&= b(\delta)\,\Big(\frac{r}{\delta}\Big)^{+i\mu}\varphi_{\mathrm{p}}(\theta) + b(\delta)\,c_{\mathrm{Z}}\Big(\frac{r}{\delta}\Big)^{-i\mu}\varphi_{\mathrm{p}}(\theta) + \dots \quad \text{for } \frac{r}{\delta} \to 0
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$$\begin{cases} c_0 + a(\delta)c_\zeta & = \; b(\delta)\,\delta^{-i\mu} \\ a(\delta)c_\zeta & = \; b(\delta)c_{\mathrm{Z}}\,\delta^{+i\mu} \end{cases}$$

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oscillating jauge functions!

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We have defined $\zeta(r,\theta), \mathrm{Z}(\rho,\theta)$, but we still have to determine the jauge functions $a(\delta), b(\delta)$. We do so by matching radial expansions.

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oscillating jauge functions!

Proposition
The coefficients $c_\zeta, c_{\mathrm{Z}} \in \mathbb{C}$ systematically verify $|c_\zeta| = |c_{\mathrm{Z}}| = 1$.

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We have defined $\zeta(r,\theta), \mathrm{Z}(\rho,\theta)$, but we still have to determine the jauge functions $a(\delta), b(\delta)$. We do so by matching radial expansions.

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oscillating jauge functions!

Proposition
The coefficients $c_\zeta, c_{\mathrm{Z}} \in \mathbb{C}$ systematically verify $|c_\zeta| = |c_{\mathrm{Z}}| = 1$.

Consequence: The matched asymptotic expansion is well defined only under the condition that

$$\delta \;\notin\; \mathfrak{J} = \{\; \delta \in (0,1) \;\mid\; \delta^{-2i\mu} = c_\zeta c_{\mathrm{Z}} \;\}$$

Unfortunately $\mathfrak{J}$ admits $\delta = 0$ as accumulation point.

(Simplified) convergence estimate

Theorem

Assume that $\kappa_\sigma \in \mathscr{I}$, and consider a datum $f \in \mathrm{H}^{-1}(\Omega)$ such that $f = 0$ near $r = 0$. Assume in addition:

- $\lim_{n\to\infty} \delta_n = 0$ with $\inf_{n\geq 0}|\delta_n^{-2i\mu} - c_\zeta c_{\mathrm{Z}}| > 0$,
- $\mathrm{Ker}(\mathrm{A}_\beta) = \{0\}$,
- $\mathrm{Ker}(\mathscr{A}_\beta) = \{0\}$.

Then $\forall \epsilon \in (0, 2)$, $\exists C_\epsilon > 0$ independent of δ such that

$$\|u^\delta - \widetilde{u}^\delta\|_{\mathrm{H}^1_0(\Omega)} \leq C_\epsilon\, \delta^{2-\epsilon}\, \|f\|_{\mathrm{H}^{-1}(\Omega)} \quad \forall \delta \in (0, 1)$$

where $\widetilde{u}^\delta(r, \theta)$, the matched expansion of $u^\delta(r, \theta)$, is defined by:

$$\begin{aligned} \widetilde{u}^\delta(r,\theta) = \quad & \psi(r/\delta)\,\big(\,u^0(r,\theta) + a(\delta)\,\zeta(r,\theta)\,\big) \\ & + \chi(r)\; b(\delta)\,\mathrm{Z}\big(r/\delta,\theta\big) \\ & - \chi(r)\,\psi(r/\delta)\,\big(\,b(\delta)(r/\delta)^{+i\mu} + a(\delta) r^{-i\mu}\,\big)\,\varphi_{\mathrm{P}}(\theta) \end{aligned}$$

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- $\mathrm{Ker}(\mathscr{A}_\beta) = \{0\}$.

$\widetilde{u}^\delta$ oscillates as $\delta \to 0$
and $\lim_{\delta\to 0} \|\widetilde{u}^\delta\|_{\mathrm{H}^1_0(\Omega)} = +\infty(!!!)$

Then $\forall \epsilon \in (0,2)$, $\exists C_\epsilon > 0$ independent of δ such that

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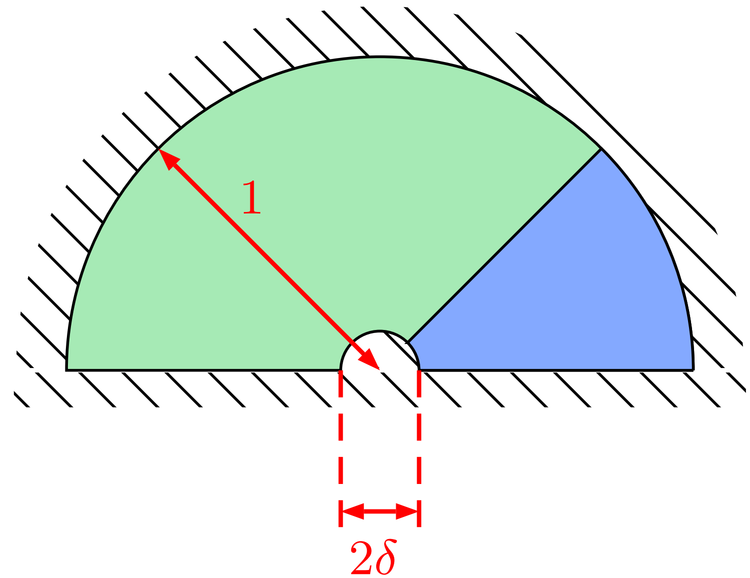
Numerical illustration

$u^\delta \in \mathrm{H}^1_0(\Omega^\delta)$ satisfies

$-\mathrm{div}(\sigma\nabla u^\delta) = f$ in Ω^δ

with $f = \begin{cases} 1 & \text{for } x < 0 \\ 0 & \text{for } x \geq 0 \end{cases}$



We represent $\Re e\{u^\delta\}$ as $\delta \to 0$ for two values of $\kappa_\sigma = \sigma_2/\sigma_1$:

a) $\kappa_\sigma = -1.0001 \notin [-1, -1/3]$

b) $\kappa_\sigma = -0.9999 \in [-1, -1/3]$

Thank you

for your attention