The Domain Derivative in Time Harmonic Electromagnetic Scattering

F. Hettlich

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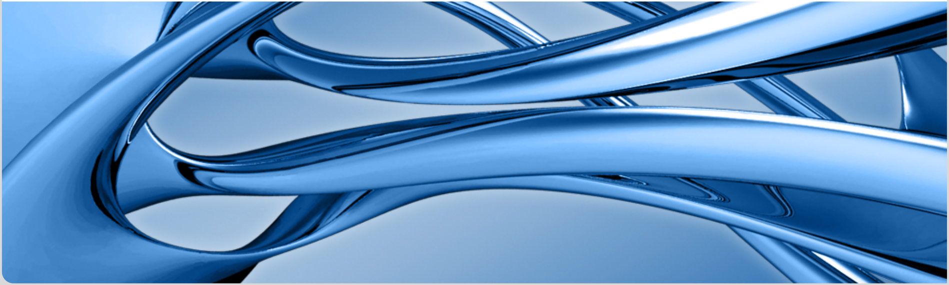
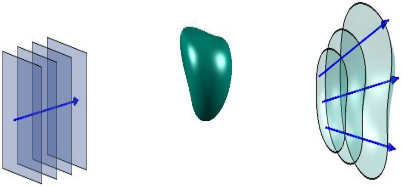


Scattering of time harmonic electromagnetic waves

E^i



$E^s = E - E^i$

S.M. rad. cond.

$$\operatorname{curl} E - \mathrm{i}kH = 0\,, \quad \operatorname{curl} H + \mathrm{i}kE = 0 \qquad \text{in } \mathbb{R}^3 \setminus \overline{D}\,.$$

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$$\operatorname{curl} E - \mathrm{i}kH = 0\,, \quad \operatorname{curl} H + \mathrm{i}kE = 0 \qquad \text{in } \mathbb{R}^3 \setminus \overline{D}\,.$$

Silver-Müller rad. cond. leads to

$$E^s(x) = \frac{\mathrm{e}^{ik|x|}}{4\pi|x|}\left(E_\infty\left(\frac{x}{|x|}\right) + \mathcal{O}\left(\frac{1}{|x|}\right)\right)\,, \quad |x| \to \infty\,.$$

Boundary conditions

- Perfectly conducting:

$$\nu \times E = 0\,, \quad \text{on } \partial D\,.$$

- Penetrable Scatterer:

$$\left[\varepsilon^{-\frac{1}{2}} \nu \times E\right]_{\pm} = 0\,, \quad \left[\mu^{-\frac{1}{2}} \nu \times H\right]_{\pm} = 0\,, \quad \text{on } \partial D\,.$$

- Impedance condition:

$$\nu \times H + \lambda\, \nu \times (E \times \nu) = 0\,, \quad \text{on } \partial D\,.$$

- ...

Inverse Scattering Theory

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Theorem

$E_\infty = 0$ on $\mathbb{S}^2$ implies $E^s = 0$ in $\mathbb{R}^3 \setminus \overline{D}$.
(see D.Colton, R.Kress, 2013)

Inverse Scattering Problems:

- Given: E_∞ for one, several, or all E^i
- Determine: D, $k|_D$, and/or λ, etc.

Inverse obstacle problem

$$F(\partial D) = E_\infty \,,$$

with $E = E^s + E^i$ solves MWEq in $\mathbb{R}^3 \setminus \overline{D}$,

Silver-Müller rad. cond. for E^s and $\nu \times E = 0$ on ∂D.

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Theorem (Uniqueness)

If $E_\infty(.; D_1, k, E^i) = E_\infty(.; D_2, k, E^i)$ *for all* $E^i(x) = p\, \mathrm{e}^{ikd\cdot x}$, *then*

$$D_1 = D_2 \,.$$

(see D.Colton, R.Kress, 2013)

Domain Derivative

Perturbation of $D \subseteq \mathbb{R}^3$ (bounded domain, sufficiently smooth)

$$D_h = \{\varphi(x) = x + h(x) : x \in D\}$$

with $h \in C_0^1(\mathbb{R}^3)$.
Note: $\|h\|_{C^1} \leq 1/2 \quad \rightsquigarrow \quad \varphi$ diffeomorphism.

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Derivative: $F'[\partial D] \in \mathcal{L}(C_0^1(\mathbb{R}^3), L^2(\mathbb{S}^2))$ with

$$\frac{1}{\|h\|_{C^1}} \|F(\partial D_h) - F(\partial D) - F'[\partial D]h\| \to 0\,, \quad \|h\|_{C^1} \to 0\,.$$

Weak formulation

$E \in H_0(\mathrm{curl}, \Omega \setminus \overline{D})$, with $\overline{D} \subseteq \Omega \subseteq \mathbb{R}^3$

$$\underbrace{(\mathrm{curl}\, E, \mathrm{curl}\, V)_{L^2(\Omega\setminus\overline{D})} - k^2 (E, V)_{L^2(\Omega\setminus\overline{D})} + \mathrm{i}k(\Lambda(\nu \times E), V)_{L^2(\partial\Omega)}}_{=\mathcal{A}(E,V)}$$

$$= \; (\mathrm{i}k\Lambda(\nu \times E^i) - \nu \times \mathrm{curl}\, E^i, V)_{L^2(\partial\Omega)}$$

for all $V \in H_0(\mathrm{curl}, \Omega \setminus \overline{D})$, with $\Lambda : \nu \times W \mapsto \nu \times \mathcal{H}^s$ Calderon operator.

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for all $V \in H_0(\mathrm{curl}, \Omega \setminus \overline{D})$, with $\Lambda : \nu \times W \mapsto \nu \times \mathcal{H}^s$ Calderon operator.

$$\rightsquigarrow \quad \mathcal{A}(E, V) = \ell(V)\,, \quad \text{for all } V \in H_0(\mathrm{curl}, \Omega \setminus \overline{D})$$

(see P. Monk (2006))

Continuous dependence

E and E_h denote the solutions w.r.t. D and D_h, respectively.

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$$\widehat{E_h} = J_\varphi^\top \widetilde{E_h}$$

Then, $\quad \widehat{E_h} \in H_0(\mathrm{curl}, \Omega \setminus \overline{D}) \quad \Longleftrightarrow \quad E_h \in H_0(\mathrm{curl}, \Omega \setminus \overline{D_h})$.

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Theorem (continuity)

It holds

$$\lim_{\|h\|_{C^1} \to 0} \left\| \widehat{E_h} - E \right\|_{H(\mathrm{curl}, \Omega \setminus \overline{D})} = 0\,.$$

Scetch of the proof

$$\mathcal{A}(\widehat{E_h} - E, V) = \mathcal{A}(\widehat{E_h}, V) - \mathcal{A}_h(E_h, \check{V})$$

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$$\begin{aligned}\mathcal{A}(\widehat{E_h} - E, V) &= \mathcal{A}(\widehat{E_h}, V) - \mathcal{A}_h(E_h, \check{V}) \\ &= \int_{\Omega\setminus\overline{D}} \operatorname{curl} \widehat{E_h} \left(I - \frac{1}{\det J_\varphi} J_\varphi^\top J_\varphi \right) \operatorname{curl} \overline{V} \\ &\qquad - k^2 \widehat{E_h} \left(I - J_\varphi^{-1} J_\varphi^{-\top} \det(J_\varphi) \right) \overline{V}\, dx\end{aligned}$$

Scetch of the proof

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$$- k^2 \widehat{E_h} \underbrace{\left(I - J_\varphi^{-1} J_\varphi^{-\top} \det(J_\varphi) \right)}_{=\mathcal{O}(\|h\|_{C^1})} \overline{V} \, dx$$

A perturbation argument leads to

$$\left\| \widehat{E_h} - E \right\|_{H(\operatorname{curl}, \Omega\setminus\overline{D})} \to 0 \,, \quad \|h\|_{C^1} \to 0 \,.$$

Domain Derivative

Theorem (material derivative)

E is differentiable, i.e.

$$\lim_{|h|_{C^1}\to 0} \frac{1}{\|h\|_{C^1}} \left\| \widehat{E_h} - E - W \right\|_{H_{\mathrm{curl}}(\Omega_R)} = 0$$

with material derivative $W \in H_0(\mathrm{curl}, \Omega \setminus \overline{D})$, linearly depending on h and satisfying

$$\mathcal{A}(W, V) = \int_{\Omega\setminus\overline{D}} \mathrm{curl}\, E^\top \left(\mathrm{div}(h) I - J_h - J_h^\top\right) \mathrm{curl}\, \overline{V} + k^2 E^\top \left(\mathrm{div}(h) I - J_h - J_h^\top\right) \overline{V}\, dx$$

for all $V \in H_0(\mathrm{curl}, \Omega \setminus \overline{D})$.

Domain Derivative

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$$W = E' + J_h^\top E + J_E h,$$

Domain Derivative

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Theorem (domain derivative)

$E' \in H(\mathrm{curl}, \Omega \setminus \overline{D})$ *radiating weak solution of Maxwell's equations*

$$\mathrm{curl}\, E' - \mathrm{i}kH' = 0\,, \quad \mathrm{curl}\, H' + \mathrm{i}kE' = 0 \qquad \text{in } \mathbb{R}^3 \setminus \overline{D}\,.$$

with

$$\nu \times E' = \nu \times \nabla_\tau (h_\nu E_\nu) - \mathrm{i}k\, h_\nu\, \nu \times (H \times \nu) \quad \text{on } \partial D\,.$$

(see R. Kress (2001), M. Costabel and F. Le Louër (2012), F.H. (2012), R. Hiptmaier and J. Li (2018), F. Hagemann (2019))

2nd Domain Derivative

$$(\partial D_{h_2})_{h_1} = \{\varphi_1(\varphi_2(x)) = x + h_2(x) + h_1(x + h_2(x)) : x \in \partial D\}$$

$\rightsquigarrow$ not symmetric !

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Definition $F''[\partial D]$ bilinear, symmetric, bounded mapping with

$$\lim_{\|h_2\|\to 0} \sup_{\|h_1\|=1} \frac{1}{\|h_2\|} \left\| F'[\partial D_2](h_1 \circ \varphi_2^{-1}) - F'[\partial D]h_1 - F''[\partial D](h_1, h_2) \right\| = 0\,.$$

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From $h_1 \circ \varphi_2^{-1} = h_1 - J_{\varphi_1} \varphi_2 + \mathcal{O}(\|h_2\|^2)$ we obtain

$$F''[\partial D](h_1, h_2) = \big(F'[\partial D] h_2\big)' [\partial D] h_1 - F'[\partial D](J_{\varphi_1} h_2)\,.$$

Theorem (2nd domain derivative)

Let ∂D be of class C^3. Then E'', H'' exist as radiating solution with

$$\begin{aligned}
\nu \times E'' = & \sum_{i \neq j = 1}^{2} \nu \times \nabla_\tau \left(h_{i,\nu} E'_{j,\nu} - E_\nu h_{i,\tau}^\top \nabla_\tau h_{j,\nu} \right) \\
& - \mathrm{i}k \sum_{i \neq j = 1}^{2} \mathrm{Div}(h_{j,\nu} H_\tau) h_{i,\nu} - h_{i,\nu} H'_{j,\tau} \\
& + \mathrm{i}k \sum_{i \neq j = 1}^{2} h_{i,\tau}^\top (\nu \times H)(\nu \times \nabla_\tau (h_{j,\nu})) \\
& + \nu \times \nabla_\tau \Big((h_{2,\tau}^\top \mathcal{R} h_{1,\tau} - 2\kappa h_{1,\nu} h_{2,\nu}) E_\nu \Big) \\
& + 2\mathrm{i}k h_{1,\nu} h_{2,\nu} (\mathcal{R} - \kappa) H_\tau - \mathrm{i}k (h_{2,\tau}^\top \mathcal{R} h_{1,\tau}) H_\tau \qquad \text{on } \partial D\,.
\end{aligned}$$

(F. Hagemann, F.H., 2020)

Iterative Regularization Methods

$$F(\partial D) = E_\infty \, .$$

domain derivative $\rightsquigarrow$ Landweber iteration, regularized Newton method, Halley-method, etc.

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Iteration step:

$$\big((F'[\partial D^n])^* F'[\partial D^n] + \alpha I\big) h = (F'[\partial D^n])^* \big(E_\infty - F(\partial D^n)\big)$$

with update $\partial D^{n+1} = \partial D^n_h$,

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stop condition:

$$\|E_\infty^\delta - F(\partial D^n)\| \leq \tau\delta < \|E_\infty^\delta - F(\partial D^j)\|$$

for $0 \leq j < n$.

Tangential cone condition ?

$$\|F(\partial D_h) - F(\partial D) - F'[\partial D]h\| \leq c\|h\|\|F(\partial D_h) - F(\partial D)\|$$

(M.Hanke, A.Neubauer, O.Scherzer (1995), M.Hanke (1997), F.H. and W.Rundell (2000), B.Kaltenbacher, A.Neubauer, O.Scherzer (2008))

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Corollary

If $-k^2$ is no eigenvalue of the Laplace-Beltrami operator on ∂D and $h_\nu =$ constant on ∂D, then $F'[\partial D]h = 0$ implies $h_\nu = 0$.
(F. Hagemann, F.H. (2020))

Integral Equation Method

Ansatz: $E^s = -\mathcal{E}\lambda$ with

$$\mathcal{E}\lambda(x) = \mathrm{i}k \int_{\partial D} \lambda(y)\,\Phi(x,y)\,ds_y - \frac{1}{\mathrm{i}k}\nabla \int_{\partial D} \mathrm{Div}\lambda(y)\,\Phi(x,y)\,ds_y\,.$$

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$\rightsquigarrow$ boundary integral equation (first kind):

$$\gamma_t \mathcal{E}\lambda = \gamma_t E^i\,,$$

k^2 no interior eigenvalue of D (A.Buffa, R.Hiptmaier (2003)).

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Karlsruhe Institute of Technology

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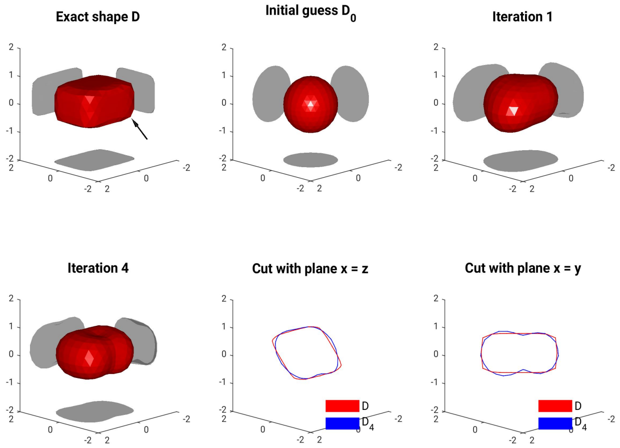
Boundary element method library: Bempp

Similiarly for E' (and E''), (e.g. E_ν or $\kappa = -\frac{1}{2}\sum_{i=1}^{3} \nu_i \Delta_{\partial D} x_i$.)

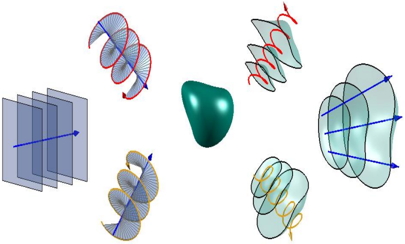
(see T.Arens, T.Betcke, F.Hagemann, F.H. (2019) and F.Hagemann, F.H. (2020))

Reconstruction (reg. Newton-Method)



(10% noise, starlike with 25 basis functions)

Chirality of Scattering Objects



⇝ shape optimization problem

Helicity of vector fields

Consider following Beltrami fields

$$W^{\pm}(B) = \{U \in H(\mathrm{curl}, B) : \mathrm{curl}\, U = \pm kU\}$$

($U \in W^{\pm}(B)$ has helicity ± 1).

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Example:
Plane waves:

$$E^i(x) = A\,\mathrm{e}^{\mathrm{i}kd\cdot x}\,, \quad H^i(x) = (d \times A)\,\mathrm{e}^{ikd\cdot x} \quad \text{with } A \cdot d = 0\,.$$

Then $\quad E^i, H^i \in W^{\pm}(B) \quad$ if and only if $\quad \mathrm{i}\, d \times A = \pm A\,.$

Herglotz wave functions

For $A \in L^2_t(\mathbb{S}^2)$ define

$$E^i[A](x) = \int_{\mathbb{S}^2} A(d)\mathrm{e}^{ikd\cdot x}\, ds_d\,, \qquad H^i[A](x) = \int_{\mathbb{S}^2} d \times A(d)\mathrm{e}^{ikd\cdot x}\, ds_d$$

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Left (or right) circularly polarized

$$E^i[A], H^i[A] \in W^{\pm}(B) \quad \Longleftrightarrow \quad \mathcal{C}A = \pm A$$

$\mathcal{C} : L^2_t(\mathbb{S}^2) \to L^2_t(\mathbb{S}^2)$ with $\mathcal{C}A(d) = \mathrm{i}\, d \times A(d)\,, \quad d \in \mathbb{S}^2$.

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It holds $\quad L^2_t(\mathbb{S}^2) = V^+ \oplus V^-$, with

$$V^{\pm} = \left\{ A \pm \mathcal{C}A : A \in L^2_t(\mathbb{S}^2) \right\}$$

Helicity of radiating solutions

Theorem

For $B \subseteq \mathbb{R}^3 \setminus \overline{D}$ holds

$$E^s, H^s \in W^{\pm}(B) \quad \Longleftrightarrow \quad E_\infty, H_\infty \in V^{\pm}\,.$$

(see T.Arens, F. Hagemann, F.H., A. Kirsch (2017))

EM-chirality

Far field operator $\mathcal{F} : L_t^2(\mathbb{S}^2) \to L_t^2(\mathbb{S}^2)$

$$\mathcal{F}[A](\widehat{x}) = \int_{\mathbb{S}^2} E_\infty(\widehat{x}; d, A(d))\, ds_d$$

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Decomposition:

$$\mathcal{F} = \mathcal{F}^{++} + \mathcal{F}^{+-} + \mathcal{F}^{-+} + \mathcal{F}^{--}, \qquad \mathcal{F}^{pq} := \mathcal{P}^p \mathcal{F} \mathcal{P}^q$$

with orth. projections $\mathcal{P}^\pm : L^2_t(\mathbb{S}^2) \to V^\pm$, $\mathcal{P}^\pm = \frac{1}{2}(I \pm \mathcal{C})$.

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Definition D is called **em-achiral** if there exist unitary transformations $\mathcal{U}^{(j)} : L^2_t(\mathbb{S}^2) \to L^2_t(\mathbb{S}^2)$ with $\mathcal{U}^{(j)}\mathcal{C} = -\mathcal{C}\mathcal{U}^{(j)}$, $j = 1, \ldots, 4$, such that

$$\mathcal{F}^{++} = \mathcal{U}^{(1)} \mathcal{F}^{--} \mathcal{U}^{(2)} \quad \text{and} \quad \mathcal{F}^{-+} = \mathcal{U}^{(3)} \mathcal{F}^{+-} \mathcal{U}^{(4)}$$

Measure of chirality

Observation: D em-achiral implies that $\mathcal{F}^{++}$ has the same singular values as $\mathcal{F}^{--}$ and analogously for $\mathcal{F}^{+-}$ and $\mathcal{F}^{-+}$.

Definition Let σ_j^{pq}, $j \in \mathbb{N}$, denote the singular values of $\mathcal{F}^{pq}$, $p, q \in \{+, -\}$.

$$\chi(\mathcal{F}) = \left(\|\sigma_j^{++} - \sigma_j^{--}\|_{\ell^2}^2 + \|\sigma_j^{+-} - \sigma_j^{-+}\|_{\ell^2}^2 \right)^{\frac{1}{2}}$$

(see I. Fernandez-Corbaton, M. Fruhnert and C. Rockstuhl (2016))

Measure of chirality

Lemma

(a) D achiral implies $\chi(\mathcal{F}) = 0$ *(see observation)*

(b) Let σ_j *be the singular values of* $\mathcal{F}$*, then*

$$\chi(\mathcal{F}) \leq \|\mathcal{F}\|_{HS} = \sqrt{\sum_j \sigma_j^2}\,.$$

(c) If D does not scatter fields of one helicity, then $\chi(\mathcal{F}) = \|\mathcal{F}\|_{HS}$

("$\Leftarrow$" holds, if D satisfies reciprocity relation)

(see T.Arens, F. Hagemann, F.H., A. Kirsch (2017))

Sketch of proof of last statement

By orthogonality

$$\chi(\mathcal{F})^2 = \|\mathcal{F}\|_{HS}^2 - 2\sum_{j=1}^{\infty}\left(\sigma_j^{++}\sigma_j^{--} + \sigma_j^{+-}\sigma_j^{-+}\right)$$

Thus, " $=$ " implies
either $\quad \mathcal{F}^{++} = 0$ or $\mathcal{F}^{--} = 0 \quad$ and $\quad \mathcal{F}^{+-} = 0$ or $\mathcal{F}^{-+} = 0$.
By reciprocity, i.e. $A \cdot E_\infty(\widehat{x}, \widehat{y}, B) = B \cdot E_\infty(\widehat{x}, \widehat{y}, A)$, follows

$$\left(\mathcal{F}A, B\right)_{L^2(\mathbb{S}^2)} = \cdots = \left(\mathcal{F}\overline{B(-.)}, \overline{A(-.)}\right)_{L^2(\mathbb{S}^2)} .$$

For $A \in V^+$ and $B \in V^-$ we conclude from $\mathcal{F}^{+-} = 0$ and

$$(\mathcal{F}^{-+}A, B)_{L^2(\mathbb{S}^2)} = \cdots = (\mathcal{F}^{+-}\overline{B(-.)}, \overline{A(-.)})_{L^2(\mathbb{S}^2)}$$

that $\mathcal{F}^{-+} = 0$ and vice versa. □

Shape Design Problem

Find D with

$$\chi(\mathcal{F}) = \|\mathcal{F}\|_{HS} \quad \text{or} \quad \operatorname{argmax}_{\partial D} \chi(\mathcal{F})\,.$$

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Lemma

(a) $\chi_{HS}(\mathcal{F}) \leq \chi(\mathcal{F})$

(b) $\chi(\mathcal{F}) = 0 \;\Rightarrow\; \chi_{HS}(\mathcal{F}) = 0$

(c) $\chi(\mathcal{F}) = \|\mathcal{F}\|_{HS} \;\Leftrightarrow\; \chi_{HS}(\mathcal{F}) = \|\mathcal{F}\|_{HS}$

(d) $\chi^2_{HS}(\mathcal{F})$ *differentiable w.r.t. h, if* $\chi_{HS}(\mathcal{F}) \notin \{0, \|\mathcal{F}\|_{HS}\}$

(F. Hagemann (2019))