Steklov eigenvalues for Maxwell's equations

Samuel Cogar

Rutgers University

samuel.cogar@rutgers.edu

November 25, 2020
University of Padova, Italy

Research supported by AFOSR and the NDSEG Fellowship

Overview

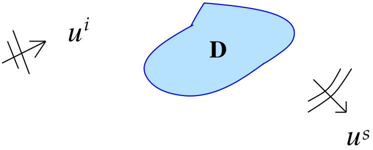
Steklov eigenvalues for the Helmholtz equation

Steklov eigenvalues for Maxwell's Equations

Trace class Steklov eigenvalues

Steklov eigenvalues for the Helmholtz equation

Scattering by an inhomogeneous medium $(d = 2, 3)$



$$\nabla \cdot A \nabla u + k^2 n u = 0 \text{ in } \mathbb{R}^d$$

$$u = u^i + u^s \text{ in } \mathbb{R}^d$$

$$\lim_{r \to \infty} r^{(d-1)/2} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0$$

- A is a $d \times d$ matrix function with $A = I$ outside of some ball
- n is the index of refraction with $n = 1$ outside of some ball
- $u^i(x) = e^{ikx \cdot \hat{d}}$ is a plane wave incident field with propagation direction $\hat{d}$
- D is a bounded domain outside of which $A = I$ and $n = 1$
- The last equation is the **Sommerfeld radiation condition**.

Generating eigenvalue problems

- When the scattered field u^s is measured far away from the scatterer, it can be recorded as the **far field pattern** $u_\infty(\hat{x}, \hat{d})$ with **observation direction** $\hat{x}$ and **incident direction** $\hat{d}$.
- Eigenvalues can be generated by comparing $u_\infty(\hat{x}, \hat{d})$ with the far field pattern $u_{\lambda,\infty}(\hat{x}, \hat{d})$ for some artificial **auxiliary problem** depending on a parameter λ.
- We generate eigenvalue problems by asking the following question:

For what values of λ do there exist nontrivial functions g such that

$$\int_{\mathbb{S}^{d-1}} \left[u_\infty(\hat{x}, \hat{d}) - u_{\lambda,\infty}(\hat{x}, \hat{d}) \right] g(\hat{d})\, ds(\hat{d}) = 0 \text{ for all } \hat{x}?$$

- In other words, for what values of λ can we construct special incident fields for which the **physical problem** and the **auxiliary problem** produce essentially the same scattering response?

Steklov eigenvalues

Let $B \supseteq D$. If for some $\lambda \in \mathbb{C}$ we consider the auxiliary problem of finding a radiating solution $u^s_\lambda \in H^1_{\text{loc}}(\mathbb{R}^d \setminus \overline{B})$ such that

$$\Delta u^s_\lambda + k^2 u^s_\lambda = 0 \text{ in } \mathbb{R}^d \setminus \overline{B},$$
$$\frac{\partial u^s_\lambda}{\partial \nu} + \lambda u^s_\lambda = -\frac{\partial u^i}{\partial \nu} - \lambda u^i \text{ on } \partial B,$$

then we obtain the eigenvalue problem of finding $\lambda \in \mathbb{C}$ and a nontrivial $w \in H^1(B)$ satisfying

$$\nabla \cdot A \nabla w + k^2 n w = 0 \text{ in } B,$$
$$\frac{\partial w}{\partial \nu_A} + \lambda w = 0 \text{ on } \partial B.$$

Such values of λ are called **Steklov eigenvalues**.

Cakoni, Colton, Meng, and Monk, Stekloff eigenvalues in inverse scattering, *SIAM J. Appl. Math.*, 2016.

Operator theory

- Properties of Steklov eigenvalues can be established by reformulating the problem in terms of **operator theory**.

- Define $T_{A,n} : L^2(\partial B) \to L^2(\partial B)$ such that $T_{A,n} h := w_h|_{\partial B}$, where $w_h \in H^1(B)$ satisfies

$$\nabla \cdot A \nabla w_h + k^2 n w_h = 0 \text{ in } B,$$
$$\frac{\partial w_h}{\partial \nu} = h \text{ on } \partial B.$$

- It can be shown that λ is a Steklov eigenvalue if and only if $-\lambda^{-1}$ is an eigenvalue of the operator $T_{A,n}$!

Properties of the eigenvalues

Discreteness

Stekloff eigenvalues are **discrete** in the complex plane.
Proof: The operator $T_{A,n}$ is compact.

Existence

Infinitely many Steklov eigenvalues **exist** whenever

1) both A and n are real-valued (i.e. the material is non-absorbing), or
2) the auxiliary domain B and the coefficients A and n are infinitely smooth.

In the first case, all eigenvalues are real.
Proof: For 1), the operator $T_{A,n}$ is compact and self-adjoint.For 2), Agmon's theory of non-selfadjoint boundary value problems can be applied.

Properties of the eigenvalues

Continuous dependence

Steklov eigenvalues **depend continuously** on the coefficients A and n.

Proof: The mapping $(A, n) \mapsto T_{A,n}$ is continuous.

Detection

Steklov eigenvalues can be **detected** from measured scattering data.

Proof: The Linear Sampling Method (LSM) or Generalized Linear Sampling Method (GLSM) can be applied.

Detection with LSM

Linear Sampling Method

1. Choose a rectangular region in the complex plane (or an interval on the real line if all eigenvalues are known to be real), and construct a grid of λ-values.
2. Compute the auxiliary far field pattern $u_{\lambda,\infty}(\hat{x},\hat{d})$ (for various $\hat{x}$ and $\hat{d}$) for each λ in the grid.
3. For each λ in the grid, approximately solve the integral equation

$$\int_{\mathbb{S}^{d-1}} \left[u_{\infty}(\hat{x},\hat{d}) - u_{\lambda,\infty}(\hat{x},\hat{d})\right] g(\hat{d})\, ds(\hat{d}) = \frac{1}{4\pi} e^{-ik\hat{x}\cdot z} \text{ for all } \hat{x}$$

   for the function $g_{\lambda} = g$, where z is a point in D.
4. Define an indicator function $I(\lambda) = \|g_{\lambda}\|$, which can be shown to be large when λ is near an eigenvalue and small otherwise. Seek eigenvalues as locations of peaks in the plot of $I(\lambda)$ versus λ!

Example: Detection with LSM

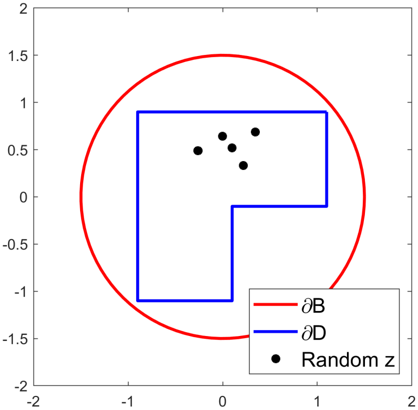


Figure: D is an L-shaped domain and B is a disk that contains D. We have chosen $A = I$ and $n = 4$, and $k = 1$. Random points z are chosen for LSM, and the results will be averaged.

Example: Detection with LSM

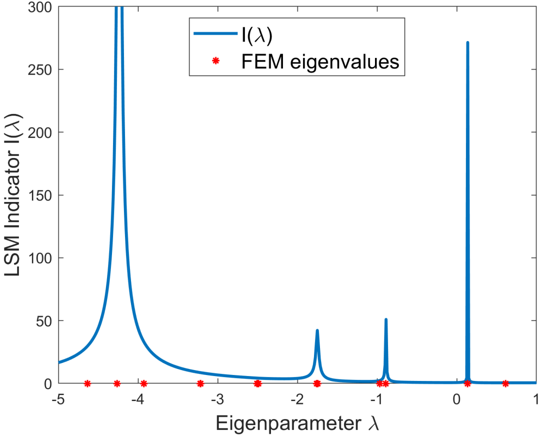


Figure: A plot of $I(\lambda)$ versus λ, compared with the exact Steklov eigenvalues obtained with a finite element method. Some eigenvalues are detected well, while others are not.

Steklov eigenvalues for Maxwell's Equations

Maxwell's equations

The standard **Maxwell's equations** for a linear medium may be written as

$$\begin{aligned}
\mu\frac{\partial \boldsymbol{\mathcal{H}}}{\partial t} + \nabla_x \times \boldsymbol{\mathcal{E}} &= \mathbf{0}, \\
\epsilon\frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} - \nabla_x \times \boldsymbol{\mathcal{H}} &= -\sigma\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{J}}_e, \\
\nabla_x \cdot (\epsilon\boldsymbol{\mathcal{E}}) &= \rho, \\
\nabla_x \cdot (\mu\boldsymbol{\mathcal{H}}) &= 0,
\end{aligned}$$

where

- μ and ϵ are the **magnetic permeability** and **electric permittivity**,
- σ is the **conductivity**,
- $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{E}}$ are the **magnetic field** and the **electric field**,
- $\boldsymbol{\mathcal{J}}_e$ and ρ are the **external current density** and **charge density**.

Time-harmonic Maxwell's equations

With $\boldsymbol{\mathcal{H}}(x,t) = e^{-i\omega t}\mathbf{H}(x)$ and $\boldsymbol{\mathcal{E}}(x,t) = e^{-i\omega t}\mathbf{E}(x)$ for a **frequency** ω, the fields $\mathbf{H}$ and $\mathbf{E}$ satisfy

$$\begin{aligned} \nabla \times \mathbf{E} - i\omega\mu\mathbf{H} &= \mathbf{0}, \\ \nabla \times \mathbf{H} + (i\omega\epsilon - \sigma)\mathbf{E} &= \mathbf{J}_e, \\ \nabla \cdot (\epsilon\mathbf{E}) &= \rho, \\ \nabla \cdot (\mu\mathbf{H}) &= 0. \end{aligned}$$

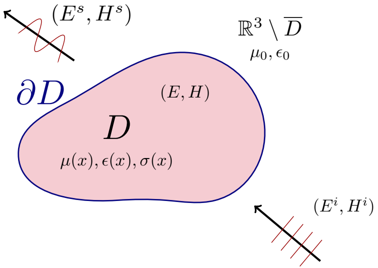
With the vacuum permeability and permittivity given by μ_0 and ϵ_0, we can define the **relative permeability and permittivity** as

$$\mu_r = \frac{\mu}{\mu_0} \text{ and } \epsilon_c = \frac{\epsilon}{\epsilon_0} + i\frac{\sigma}{\omega\epsilon_0}$$

and the **wave number** as $k = \omega\sqrt{\epsilon_0\mu_0}$. The problem may be written in terms of the electric field as

$$\nabla \times {\mu_r}^{-1}\nabla \times \mathbf{E} - k^2\epsilon_c\mathbf{E} = i\omega\mu_0\mathbf{J}_e.$$

Time-harmonic electromagnetic scattering (with $\mu_r = 1$)



$$\nabla \times \nabla \times \mathbf{E} - k^2 \epsilon \mathbf{E} = \mathbf{0} \text{ in } \mathbb{R}^3$$

$$\mathbf{E} = \mathbf{E}^i + \mathbf{E}^s \text{ in } \mathbb{R}^3$$

$$\lim_{r \to \infty} \left(\nabla \times \mathbf{E}^s \times x - ikr\mathbf{E}^s \right) = \mathbf{0}$$

- For simplicity we write ϵ to represent the relative permittivity (equal to 1 outside of D).
- The incident field is an electromagnetic plane wave $\mathbf{E}^i(x) = pe^{ikx \cdot \hat{d}}$ with **polarization** p and **incident direction** $\hat{d}$.
- The last equation is the **Silver-Müller radiation condition**.

Electromagnetic Steklov eigenvalues

- When the scattered field $\mathbf{E}^s$ is measured far away from the scatterer, it can be recorded as the **electric far field pattern** $\mathbf{E}_\infty(\hat{x}, \hat{d}; p)$.
- As before, eigenvalues can be generated by comparing $\mathbf{E}_\infty(\hat{x}, \hat{d}; p)$ with the far field pattern $\mathbf{E}_{\lambda,\infty}(\hat{x}, \hat{d}; p)$ for some artificial **auxiliary problem** depending on a parameter λ.
- The electromagnetic analogue of the Steklov problem is to find a radiating field $\mathbf{E}^s_\lambda$ such that

$$\nabla \times \nabla \times \mathbf{E}^s_\lambda - k^2 \mathbf{E}^s_\lambda = \mathbf{0} \text{ in } \mathbb{R}^3 \setminus \overline{B},$$
$$\nu \times \nabla \times \mathbf{E}^s_\lambda - \lambda \mathbf{E}^s_{\lambda,T} = -\nu \times \nabla \times \mathbf{E}^i + \lambda \mathbf{E}^i_T \text{ on } \partial B,$$

where $\mathbf{F}_T := (\nu \times \mathbf{F}) \times \nu$ is the **tangential component** of the vector field $\mathbf{F}$.

Electromagnetic Steklov eigenvalues

- The comparison with this auxiliary problem leads to the **electromagnetic Steklov eigenvalue problem** of finding $\lambda \in \mathbb{C}$ and a nontrivial $\mathbf{w}$ satisfying

$$\nabla \times \nabla \times \mathbf{w} - k^2 \epsilon \mathbf{w} = \mathbf{0} \text{ in } B,$$
$$\nu \times \nabla \times \mathbf{w} - \lambda \mathbf{w}_T = \mathbf{0} \text{ on } \partial B.$$

Camaño, Lackner, and Monk, Electromagnetic Stekloff eigenvalues in inverse scattering, *SIAM J. Math. Anal.*, 2017.

- We will call such a value of λ a **standard electromagnetic Steklov eigenvalue**.
- In principle, we can use operator theory like before to establish properties of the eigenvalues, but it turns out that the eigenvalues have **degenerate behavior**.

Standard Steklov eigenvalues

- It was shown in [Camaño, Lackner, Monk, 2017] that in a simple case the eigenvalues accumulate at $-\infty$ **and** at 0.
- In particular, the eigenvalues are **not discrete** in $\mathbb{C}$, meaning that the operator representing them cannot be compact as in the scalar case!
- In a pair of papers, M. Halla showed that the **essential spectrum** consists of only the point 0 and that the eigenvalues are discrete in $\mathbb{C} \setminus \{0\}$. He also showed that infinitely many eigenvalues exist when all of the coefficients are real-valued.

Halla, Electromagnetic Stekloff eigenvalues: Approximation analysis, to appear in *ESAIM*, 2020.

Halla, Electromagnetic Stekloff eigenvalues: Existence and behavior in the self-adjoint case, arXiv, 2019.

Modified Steklov eigenvalues

- Instead of studying the degenerate standard problem, the authors in [Camaño, Lackner, Monk, 2017] altered the auxiliary problem to produce the **modified Steklov eigenvalue problem**

$$\nabla \times \nabla \times \mathbf{w} - k^2 \epsilon \mathbf{w} = \mathbf{0} \text{ in } B,$$
$$\nu \times \nabla \times \mathbf{w} - \lambda \mathcal{S} \mathbf{w}_T = \mathbf{0} \text{ on } \partial B,$$

  where $\mathcal{S}$ is a **projection operator**.

- For surface divergence free $\mathbf{h}$, if we define the operator $\mathcal{T}_\epsilon$ such that $\mathcal{T}_\epsilon \mathbf{h} := \mathcal{S}\mathbf{u}_T|_{\partial B}$, where $\mathbf{u}$ satisfies

$$\nabla \times \nabla \times \mathbf{u} - k^2 \epsilon \mathbf{u} = \mathbf{0} \text{ in } B,$$
$$\nu \times \nabla \times \mathbf{u} = \mathbf{h} \text{ on } \partial B,$$

  then the operator $\mathcal{T}_\epsilon$ relates to the eigenvalues as before and is compact!

The operator $\mathcal{S}$

- A given tangential vector field $\boldsymbol{\xi}$ on ∂B can be written as

$$\boldsymbol{\xi} = \sum_{m=1}^{\infty} \left[\boldsymbol{\xi}_m^{(1)} \nabla_{\partial B} Y_m + \boldsymbol{\xi}_m^{(2)} \vec{\nabla}_{\partial B} \times Y_m \right],$$

  where $\{Y_m\}$ is an orthonormal set of functions on ∂B (eigenfunctions of the Laplace-Beltrami operator $\Delta_{\partial B}$).
- The operator $\mathcal{S}$ is defined such that

$$\mathcal{S}\boldsymbol{\xi} := \sum_{m=1}^{\infty} \boldsymbol{\xi}_m^{(2)} \vec{\nabla}_{\partial B} \times Y_m,$$

  and hence the operator projects $\boldsymbol{\xi}$ onto its **surface divergence free component**.
- This property combined with **regularity results for Maxwell's equations** imply compactness of $\mathcal{T}_\epsilon$.

Properties of modified Steklov eigenvalues

- The presence of $\mathcal{S}$ results in a **discrete** set of eigenvalues in $\mathbb{C}$, as in the scalar case.
- It can also be shown that the eigenvalues **exist** in the self-adjoint case, **depend continuously** on ϵ, and may be **detected** with LSM.
- However, **no existence results** are available when ϵ is not real-valued.
- Other properties (such as sensitivity to changes in ϵ) have been investigated numerically.

C., Colton, and Monk, Eigenvalue problems in inverse electromagnetic scattering theory, ch. 5 of *Maxwell's Equations: Analysis and Numerics*, de Gruyter, 2019.

Trace class Steklov eigenvalues

Existence for complex coefficients

- There are no existence results when the coefficients are generally complex-valued and nonsmooth.
- Like the modified Steklov eigenvalues from [Camaño, Lackner, Monk], we can **alter the auxiliary problem** to prove this type of result.
- For $\delta \geq 0$, the **electromagnetic δ-Steklov eigenvalue problem** is to find $\lambda \in \mathbb{C}$ and a nontrivial $\mathbf{w}$ satisfying

$$\nabla \times \nabla \times \mathbf{w} - k^2 \epsilon \mathbf{w} = \mathbf{0} \text{ in } B,$$
$$\nu \times \nabla \times \mathbf{w} - \lambda \mathcal{S}_\delta \mathbf{w}_T = \mathbf{0} \text{ on } \partial B,$$

where $\mathcal{S}_\delta$ is a **smoothing projection operator**.

C., Analysis of a trace class Stekloff eigenvalue problem arising in inverse scattering, *SIAM J. Appl. Math.*, 2020.

C., Existence and stability of electromagnetic Stekloff eigenvalues with a trace class modification, to appear in *Inverse Probl. Imaging*, 2020.

The operator $\mathcal{S}_\delta$

- With $\{\mu_m\}_{m=0}^{\infty}$ the sequence of **eigenvalues of the Laplace-Beltrami operator** $\Delta_{\partial B} := -\nabla_{\partial B} \cdot \nabla_{\partial B}$, we define $\mathcal{S}_\delta$ as

$$\mathcal{S}_\delta \boldsymbol{\xi} := \sum_{m=1}^{\infty} \mu_m^{-\delta} \boldsymbol{\xi}_m^{(2)} \vec{\nabla}_{\partial B} \times Y_m.$$

- In particular, we see that $\mathcal{S}_0 = \mathcal{S}$.
- In addition to projecting onto the surface divergence free component of $\boldsymbol{\xi}$, the operator $\mathcal{S}_\delta$ also **smooths** the vector field.
- When B is chosen to be the unit ball in $\mathbb{R}^3$, the eigenvalues $\{\mu_m\}$ and the eigenfunctions $\{Y_m\}$ can be easily computed (the latter being the **spherical harmonics**).

An operator formulation

In this case the correct operator is defined by $\mathcal{T}_{\epsilon}^{(\delta)}\mathbf{h} := \mathcal{S}_{\delta/2}\mathbf{u}_T|_{\partial B}$, where $\mathbf{u}$ satisfies

$$\begin{aligned} \nabla \times \nabla \times \mathbf{u} - k^2\epsilon\mathbf{u} &= \mathbf{0} \text{ in } B, \\ \nu \times \nabla \times \mathbf{u} &= \mathcal{S}_{\delta/2}\mathbf{h} \text{ on } \partial B. \end{aligned}$$

Lemma

If $\delta > 1$, then the operator $\mathcal{T}_{\epsilon}^{(\delta)}$ is a **trace class operator**, meaning that there exists a sequence $\{\mathcal{T}_N\}$ of operators such that $\text{rank}(\mathcal{T}_N) \leq N$ and

$$\sum_{N=1}^{\infty} \left\| \mathcal{T}_{\epsilon}^{(\delta)} - \mathcal{T}_N \right\| < \infty.$$

Lidski's theorem and existence of eigenvalues

Lidski's Theorem

If $\mathcal{T}$ is a trace class operator on a Hilbert space X such that $\mathcal{T}$ has finite-dimensional nullspace and Im$(\mathcal{T}g, g)_X \geq 0$ for each $g \in X$, then $\mathcal{T}$ has **infinitely many eigenvalues**.

- The operator $\mathcal{T}_\epsilon^{(\delta)}$ is trace class whenever $\delta > 1$.
- The operator can be assumed to be injective under mild conditions.
- The common assumption Im$(\epsilon) \geq 0$ ensures the nonnegativity condition.

Existence Theorem

If $\delta > 1$, then infinitely many electromagnetic δ-Steklov eigenvalues exist.

Stability of eigenvalues

Suppose that $\tilde{\epsilon}$ and ϵ are piecewise continuously differentiable.

Stability with respect to permittivity

If $\tilde{\epsilon}$ is fixed and ϵ is a perturbed permittivity for which $\|\tilde{\epsilon} - \epsilon\|_{L^\infty(B)}$ is sufficiently small, then there exist constants $s \in (0, \frac{1}{2})$ and $C_{s,\tilde{\epsilon}} > 0$ such that

$$\left\| \mathcal{T}_{\tilde{\epsilon}}^{(\delta)} - \mathcal{T}_{\epsilon}^{(\delta)} \right\| \leq C_{s,\tilde{\epsilon}} \|\tilde{\epsilon} - \epsilon\|_{L^{3/s}(B)} .$$

- The constant $s \in (0, \frac{1}{2})$ depends on the regularity of $\tilde{\epsilon}$ (see [Bonito,Guermond,Luddens,2013] and [Ciarlet,2020]).

Bonito, Guermond, and Luddens, Regularity of the Maxwell equations in heterogenous media and Lipschitz domains, *J. Math. Anal. Appl.*, 2013.

Ciarlet, On the approximation of electromagnetic fields by edge finite elements. Part 3: Sensitivity to coefficients, *SIAM J. Math. Anal.*, 2020.

Stability of eigenvalues

- The stability result follows from factoring the solution operator in the form

$$\mathcal{T}_{\epsilon}^{(\delta)} = \mathbf{V}_{\epsilon}\mathbf{M}_{\tilde{\epsilon},\epsilon}\mathbf{W}_{\tilde{\epsilon}}.$$

- Regularity properties of $\mathbf{W}_{\tilde{\epsilon}}$ lead to the appearance of the $L^{3/s}(B)$-norm, but the same regularity does not appear to hold for $\mathbf{V}_{\epsilon}$, which limits the desired control over $\tilde{\epsilon} - \epsilon$ to the $L^{\infty}(B)$-norm.
- It is not known if these results may be extended to the case $\mu \neq 1$, as different regularity results must be used.
- Finally, it can also be shown that the δ-Steklov eigenvalues converge to the standard Steklov eigenvalues as $\delta \to 0^+$.

Questions?