An introduction to the mathematical theory of quantum fluids

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Quantum Hydrodynamical (QHD) systems are fluid dynamical models connotated by quantum effects. They describe many physical phenomena, for instance in the theory of Superfluidity of in Bose-Einstein condensation; in general they describe gases or fluids where quantum effects are relevant even at a macroscopic scale.

From the mathematical point of view the QHD system consists in a compressible, inviscid fluid with a stress tensor depending on the gradient of the mass density, hence encoding dispersive effects in the system. More generally, the QHD systems can be also ascribed in the more general class of the so-called Euler-Korteweg systems describing capillarity phenomena in diffuse interfaces.

In this course I will present a rigorous mathematical theory to study finite energy weak solutions for such systems. Most of the results exposed here are fruit of a long term collaboration with Pierangelo Marcati.

The first part of the course will be dedicated to an overview of the main physical features of the phenomena we want to describe with the QHD and analogue systems [7,8]. I will briefly recall the classical theory for nonlinear Schrödinger (NLS) equations [4].

I will discuss the analogy between solutions to NLS equations and to QHD systems, first formally by using the WKB approach then rigorously by introducing the polar factorisation method.

Then I will present a global existence result for finite energy weak solutions to a dissipative QHD system [2]. This model is particularly relevant not only for its applications - it describes electron transport in a semiconductor device [5], where the dissipation is given by a relaxation term due to electron collisions in the device. In fact the dissipative term can be seen as a model term describing the interaction of the quantum fluid with the environment (a reservoir). This is the case for example when we describe Superfluidity (or Bose-Einstein condensates) at finite temperatures [6]. For this purpose I will introduce the Landau-Khalatnikov two-fluid model [8]. In this theory the fluid is composed by two different flows, the first one describing the superfluid (given by a QHD-type system) and the second one describing the normal flow by a classical compressible viscous fluid. After some discussions on the interaction terms between the two flows I will provide you some partial results on those models [3].

Finally I will present some recent results, obtained in collaboration with Pierangelo Marcati and Hao Zheng, on the one dimensional case.

References

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