Hyperbolic and dispersive free boundary problems

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This course is devoted to the analysis of several free boundary problems arising in wave-structure interactions, as well as in several other physical situations. We shall introduce several general mathematical tools that are very important in many free boundary and/or initial boundary value problems, and discuss the particular role of dispersion, which is still to be fully understood for such problems.

1 Free boundary hyperbolic systems

We shall consider here two general families of free boundary hyperbolic problems. In both cases, a 2×2 quasilinear hyperbolic system is considered in a moving domain $(\underline{x}(t), \infty)$,

$$\begin{cases} \partial_t U + A(U)\partial_x U = 0 & \text{in } (\underline{x}(t), \infty) \text{ for } t \in (0,T), \\ U_{|_{t=0}} = u_0^{\rm in}(x) & \text{in } (\underline{x}(0), \infty), \end{cases} \tag{1}$$

where U is an $\mathbb{R}^2$-valued function and $A(U)$ a 2×2 symmetric matrix whose eigenvalues satisfy some hyperbolicity conditions. The two families of free boundary value problems considered here are given by (1) together with a boundary condition and an evolution equation for the free boundary $\underline{x}(t)$.

- *Free boundary problem with boundary condition of "kinematic" type.* We consider here a general boundary condition of the form

$$\underline{\nu} \cdot U_{|_{x=\underline{x}(t)}} = g(t) \text{ on } (0,T), \tag{2}$$

  with $\underline{\nu} \in \mathbb{R}^2$ a fixed vector, and an evolution equation for the free boundary which is of "kinematic" type,

$$\dot{\underline{x}} = \mathcal{X}\big(U_{|_{x=\underline{x}(t)}}\big), \tag{3}$$

  with $\mathcal{X}$ a smooth function. Many physical problems can be reduced to such a problem, such as : waves created by a lateral piston, stability of one dimensional Lax shocks or undercompressive shocks for incompressible isentropic gases.

- *Free boundary with a fully nonlinear boundary equation.* Here, the boundary equation *and* the evolution equation for the free boundary are given by a single vectorial equation

$$U = U_{\rm i} \quad \text{on} \quad \underline{x}(t), \tag{4}$$

where $U_{\rm i}$ is a known function. Seen as an initial boundary value problem, (1) with boundary condition (4) is overdetermined; this overdetermination provides the evolution equation for the boundary. Finding the contact line between a floating object and the surface of a fluid governed by the nonlinear shallow water equations is a problem that falls into this family of free boundary problems which is more singular than the previous one.

We will show how to handle these two families of free boundary value problems and introduce several objects than play a crucial role in many free boundary value problems (as well as for hyperbolic initial boundary value problems): choice of a good diffeomorphism $\varphi(t,\cdot):(0,\infty)\to(\underline{x}(t),\infty)$ in order to work in a fixed domain, importance of Alinhac's good unknown to deal with the dependence of the transformed equations with respect to this diffeomorphism, Lopatinskiĭ condition for the stability of the boundary conditions, construction of Kreiss symmetrizers in order to get full regularity of the trace of the solution at the boundary, etc. The fact that we are dealing with one dimensional problems will allow for a simple introduction of these important concepts.

2 Nonlinear dispersive initial boundary value problems

Initial boundary value problems (and also transmission problems) in dimension $d=1$ for 2×2 hyperbolic systems are well understood. However, for many applications, and especially for the description of surface water waves, dispersive pertubations of hyperbolic systems must be considered. For instance, water waves in shallow water can be described at leading order by the nonlinear shallow water equations

$$\begin{cases}\partial_t\zeta+\partial_x q=0,\\ \partial_t q+\partial_x\big(\dfrac{1}{h}q^2\big)+gh\partial_x\zeta=0,\end{cases}\tag{5}$$

where ζ is the surface elevation above the rest state, $h=h_0+\zeta$ (h_0 being the depth at rest), g is the acceleration of gravity and q is the horizontal discharge (or equivalently, the vertical integral of the horizontal velocity). A better approximation is achieved by including the *dispersive* effects; the nonlinear shallow water equations (5) are then replaced by the *Boussinesq equations*,

$$\begin{cases}\partial_t\zeta+\partial_x q=0,\\ (1-\frac{H_0^2}{3}\partial_x^2)\partial_t q+\partial_x\big(\dfrac{1}{h}q^2\big)+gh\partial_x\zeta=0.\end{cases}\tag{6}$$

Similar dispersive perturbation of hyperbolic systems also arise in other context, for instance if one includes capillarity in the description of compressible gases. For such problems, a general theory for the initial boundary value problem is still missing. We shall describe here a transmission problem for (6) arising in the description of wave-structure interactions. We shall insist on the differences and similarities with respect to the standard hyperbolic case, and focus our attention on a new phenomenon, namely, the apparition of a *dispersive boundary layer*.

3 Free boundary problems for nonlinear dispersive problems

In the previous section, we considered nonlinear dispersive initial boundary value problems, which is of course a key ingredient for the understanding of *free boundary problems* in this setting. We shall mention several open problems and, if time permits, we will describe the analysis a a particular and important problem: the shoreline problem. This problems consists in allowing the water depth to vanish in the nonlinear shallow water equations (5) or in dispersive perturbations of this problem. Finding the points where the depth vanishes is a free boundary problem (in the case of the nonlinear shallow water equations, it is equivalent to the vacuum problem for the incompressible Euler equations). Here again, emphasis will be put on several points of general interest such as Hardy type inequalities, conormal estimates, elliptic regularization for fully nonlinear problems, etc.

References

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