Lecture 1. On point finiteness of the Lax-Oleinik semi-group

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Introduction

We consider a connected manifold M endowed with a **complete** Riemannian metric.
We will denote by $\|\cdot\|_x$ the induced norm on either T_xM or T_x^*M, the fiber above x of the tangent TM or cotangent T^*M bundle of M.
We will denote bien d the Riemannian distance on M obtained from the Riemannian metric. Due to the completeness of the Riemannian metric, bounded sets for d are relatively compact. Therefore the distance d is also complete.
The canonical projections from the tangent and cotangent bundle are denoted by $\pi : TM \to M$ and $\pi^* : T^*M \to M$.
If $\gamma : [a, b] \to M$ is an absolutely continuous curve, its Riemannian length $\ell_g(\gamma)$ is

$$\ell_g(\gamma) = \int_a^b \|\dot\gamma(s)\|_{\gamma(s)}\, ds.$$

Tonelli Lagrangian

We will assume that $L : TM \to \mathbb{R}$ is a Tonelli Lagrangian (with respect to the Riemannian metric), i.e. L is at least C^2 and satisfies:

(a) (Uniform superlinearity) For every $K \geq 0$, we have

$$C(K) = \sup_{(x,v)\in TM} K\|v\|_x - L(x,v) < \infty.$$

(b) (Uniform boundedness in the fibers) For every $R \geq 0$, we have

$$A(R) = \sup\{L(x,v) \mid \|v\|_x \leq R\} < +\infty \;;$$

(c) (C^2 strict convexity in the fibers) for every $(x,v) \in TM$, the second derivative $\partial^2 L/\partial v^2(x,v)$ is positive strictly definite.

A and C are both non-decreasing as functions of $R \in [0+\infty[$.
Note also that (a) and (b) imply

$$\forall (x,v) \in TM, L(x,v) \geq K\|v\|_x - C(K). \tag{1}$$

$$\forall (x,v) \in TM, L(x,v) \leq A(\|v\|_x). \tag{2}$$

Example

1) The easiest example of a Tonelli Lagrangian is $L_0 : TM \to \mathbb{R}$ defined by pause

$$L_0(x, v) = \frac{1}{2}\|v\|_x^2.$$

In fact, in this case

$$A_0(R) = \sup\{L_0(x, v) \mid \|v\|_x \leq R\} = \frac{R^2}{2},$$

$$C_0(K) = \sup_{(x,v)\in TM} K\|v\|_x - L_0(x, v) = \sup_{(x,v)\in TM} K\|v\|_x - \frac{1}{2}\|v\|_x^2 = \frac{K^2}{2}.$$

2) Let $V : M \to \mathbb{R}$ be a C^r function, with $r \geq 2$, the Lagrangian $L_V : TM \to \mathbb{R}$ defined by

$$L_V(x, v) = \frac{1}{2}\|v\|_x^2 - V(x)$$

is a Tonelli Lagrangian is and only if V is bounded.

Action and Lax-Oleinik

We recall that the action $\mathbb{L}(\gamma)$ of an absolutely continuous curve $\gamma : [a, b] \to M$ is defined by

$$\mathbb{L}(\gamma) = \int_a^b L(\gamma(s), \dot\gamma(s))\, ds.$$

By the superlinearity of L, the action is always bounded below by $-C(0)(b-a)$.
Once the action of a curve is defined, we can introduce the Lax-Oleinik semi-group $T_t^-, t \geq 0$ on the space of real valued functions on M.
Namely, if $u : M \to \mathbb{R}$ is given, for $t > 0$, we define $T_t^- u$ by

$$T_t^- u(x) = \inf_\gamma u(\gamma(0)) + \int_0^t L(\gamma(s), \dot\gamma(s))\, ds,$$

where the infimum is taken over all absolutely continuous curves $\gamma : [0, t] \to M$ with $\gamma(t) = x$. We also set $T_0^- u(x) = u(x)$.

Since the action of curves is bounded from below, if u is also bounded from below, the function $T_t^- u$ will be finite everywhere on M.

In particular, if u is continuous and M compact, the Lax-Oleinik evolution of u is finite.

As we already know in that case $T_t^- u$ is also continuous.

When M is not compact, there are cases where $T_t^- u$ assumes everywhere the value $-\infty$.

An example is provided below.

Moreover, even when $T_t^- u$ is finite, it is not clear that $T_t^- u$ is continuous (however, upper semi-continuity is easy to establish).

Main Theorem for today

Our goal is to prove:

Theorem 1

Let $u : M \to \mathbb{R}$ be a continuous function.
Assume that $T^-_{t_0}u(x_0)$ is finite for some $t_0 > 0$ and some $x_0 \in M$.
Then $T^-_t u(x)$ is finite and continuous (even locally Lipschitz) on $[0, t_0[\times M$.
Moreover, the function U defined by $U(t,x) = T^-_t u(x)$ is a viscosity solution of $\partial_t U + H(x, \partial_x U) = 0$ on $]0, t_0[\times M$, where H is the Hamiltonian associated to L.

The Hamiltonian H, associated to L, is the function $H : T^*M \to \mathbb{R}$ defined by

$$H(x,p) = \sup_{v \in T_x M} p(v) - L(x,v).$$

In fact, today we will concentrate on the first part of the theorem, and leave the viscosity aspects to the second lecture.

Minimizers and Extremals. Euler-Lagrange Equation and Flow

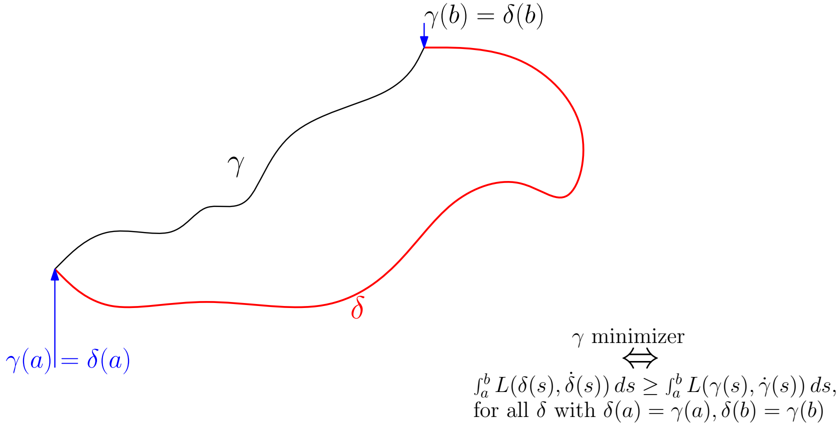
Although we assume familiarity with minimizers, extremals and Euler-Lagrange Equation for the Lagrangian L, we now sketch some of the definition and properties.

Definition 2 (Minimizer)

A minimizier (for L) is a curve $\gamma : [a, b] \to M$ such that

$$\mathbb{L}(\delta) = \int_a^b L(\delta(s), \dot{\delta}(s))\, ds \geq \mathbb{L}(\gamma) = \int_a^b L(\gamma(s), \dot{\gamma}(s))\, ds,$$

for every curve $\delta : [a, b] \to M$, with $\delta(a) = \gamma(a), \delta(b) = \gamma(b)$.


$\gamma(b) = \delta(b)$
γ
δ
$\gamma(a) = \delta(a)$
γ minimizer
$\Longleftrightarrow$
$\int_a^b L(\delta(s), \dot{\delta}(s))\, ds \geq \int_a^b L(\gamma(s), \dot{\gamma}(s))\, ds$,
for all δ with $\delta(a) = \gamma(a), \delta(b) = \gamma(b)$

- Minimizers play a crucial role in Aubry-Mather theory.
- Minimizers (like all minimums of a function) must be critical points for the action functional $\mathbb{L}$.

These critical points are called extremals.

- More precisely, an extremal (for L) is a curve $\gamma : [a, b] \to M$ such that the derivative $D_\gamma \mathbb{L}|\mathcal{E}_\gamma$ at γ vanishes, with

$$\mathcal{E}_\gamma = \{\delta : [a, b] \to M \mid \delta(a) = \gamma(a), \delta(b) = \gamma(b)\}.$$

- By classical theory of Calculus of Variations, the curve γ is an extremal if and only if it satisfies Euler-Lagrange equation, given in local coordinates by

$$\frac{d}{dt}\left[\frac{\partial L}{\partial v}(\gamma(t), \dot{\gamma}(t))\right] = \frac{\partial L}{\partial x}(\gamma(t), \dot{\gamma}(t)). \tag{3}$$

This last ODE (3) defines a second order equation on M. Therefore there exists a flow φ_t on TM, called the Euler-Lagrange flow, such that $\gamma : [a, b] \to M$ is an extremal if and only if its speed curve $s \mapsto (\gamma(t), \dot{\gamma}(t))$ is an orbit of φ_t.

Moreover, for any $(x, v) \in TM$, the projected curve $\gamma_{x,v}(t) = \pi\varphi_t(x, v)$, where $\pi : TM \to M$ is the canonical projection, is an extremal with $(\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t)) = \varphi_t(x, v)$.

We know recall Tonelli's theorem.

Theorem 3 (Tonelli)

For every $a, b \in \mathbb{R}$, with $a < b$, and every $x, y \in M$, there exists a minimizer $\gamma : [a, b] \to M$, with $\gamma(a) = x, \gamma(b) = y$. Any such minimizer γ is as smooth as L and is a solution of the Euler-Lagrange equation.

Actions and Speed Estimates

For the properties of the Lax-Oleinik semi-group, for a non-compact M, we need a control of the action and speed of minimizers.

Lemma 4

Let $\gamma : [a, b] \to M$ be absolutely continuous curve. For every $K \in [0, \infty[$, we have

$$\mathbb{L}(\gamma) \geq K\ell_g(\gamma) - C(K)(b-a) \geq Kd(\gamma(a), \gamma(b)) - C(K)(b-a) \quad (4)$$

$$d(\gamma(a), \gamma(b)) \leq \ell_g(\gamma) \leq \frac{\mathbb{L}(\gamma) + C(K)(b-a)}{K}. \quad (5)$$

Moreover, if $\gamma : [a, b] \to M$ is a minimizer, we have

$$\mathbb{L}(\gamma) \leq (b-a)A\left(\frac{d(\gamma(a), \gamma(b))}{b-a}\right). \quad (6)$$

$$\frac{\ell_g(\gamma)}{b-a} \leq \frac{A(d(\gamma(a), \gamma(b))/(b-a)) + C(K)}{K} \quad (7)$$

Proof

We integrate the superlinear inequality (1)

$$L(\gamma(s),\dot\gamma(s)) \geq K\|\dot\gamma(s)\|_{\gamma(s)} - C(K),$$

to obtain

$$\mathbb{L}(\gamma) \geq K\ell_g(\gamma) - C(K)(b-a),$$

from which the first two inequalities (4) and (5) follow easily.
To bound the action of a minimizer (6), call $\delta : [a,b] \to M$ a geodesic from $\gamma(a)$ to $\gamma(b)$, with $\ell_g(\delta) = d(\gamma(a),\gamma(b))$.
The speed of a geodesic has constant norm. But integrating the speed yields the length, hence

$$\|\dot\delta(t)\|_{\delta(t)} = d(\gamma(a),\gamma(b))/(b-a), \text{ for } t \in [a,b].$$

Therefore by the uniform boundedness inequality

$$L(\delta(t),\dot\delta(t)) \leq A\left(d(\gamma(a),\gamma(b))/(b-a)\right), \text{ for every } t \in [a,b],$$

and again by integration

$$\mathbb{L}(\gamma) \leq \mathbb{L}(\delta) \leq (b-a)A\left(\frac{d(\gamma(a),\gamma(b))}{b-a}\right). \square$$

Energy

To get further estimates, we need the concept of energy of a Lagrangian.
Recall that the Energy $E : TM \to \mathbb{R}$ is defined by

$$E(x,v) = \langle \partial_v L(x,v), v\rangle - L(x,v) = \sup_{u \in T_x M} \langle \partial_v L(x,v), u\rangle - L(x,u).$$

As is well known, the Energy E is constant along any solution of the Euler-Lagrange equation, in particular, along any minimizer.

Definition 5

We define the functions α, β on $[0, +\infty[$ by

$$\alpha(R) = \inf\{E(x,v) \mid (x,v) \in TM, \text{ with } \|v\|_x \geq R\},$$
$$\beta(R) = \sup\{E(x,v) \mid (x,v) \in TM, \text{ with } \|v\|_x \leq R\}.$$

Lemma 6

There two functions α, β are finite-valued and non-decreasing. Moreover, we have $\alpha(R) \to +\infty$, as $R) \to +\infty$, and

$$\alpha(\|v\|_x) \leq E(x, v) \leq \beta(\|v\|_x), \text{ for every } (x, v) \in TM.$$

The fact that α, β are non-decreasing and the last inequalities are obvious from the definitions of α and β.
To prove the rest of the Lemma, we use convexity of $L(x, v)$ in v

$$L(x, v + u) - L(x, v) \geq \langle \partial_v L(x, v), u \rangle. \tag{8}$$

Setting $u = v$ and subtracting $L(x, v)$ from both sides, we get

$$L(x, 2v) - 2L(x, v) \geq \langle \partial_v L(x, v), u \rangle - L(x, v) = E(x, v).$$

Since $L(x, v) \geq -C(0)$ and $L(x, 2v) \leq A(2\|v\|_x)$, we obtain

$$E(x, v) \leq A(2\|v\|_x) + 2C(0).$$

Therefore $\beta(R) \leq A(2R) + 2C(0) < +\infty$.

Taking $u = -v$ in the convexity inequality (8), we get

$$L(x,0) - L(x,v) \geq -\langle \partial_v L(x,v), v \rangle.$$

Hence, since $A(0) = \sup_{x \in M} L(x,0)$

$$E(x,v) = \langle \partial_v L(x,v), v \rangle - L(x,v) \geq -L(x,0) \geq -A(0),$$

Therefore $\alpha \geq \inf_{(x,v) \in TM} E(x,v) \geq -A(0)$ is finite-valued.

It remains to show that $\alpha(R) \to +\infty$, as $R \to +\infty$.
Assume $\|v\|_x \geq R > 1$. Use the convexity inequality (8), with $u = \frac{v}{\|v\|_x} - v = \frac{1-\|v\|_x}{\|v\|_x} v$, to yield

$$\begin{aligned} L\left(x, \frac{v}{\|v\|_x}\right) - L(x,v) &\geq \langle \partial_v L(x,v), \frac{1-\|v\|_x}{\|v\|_x} v \rangle \\ &= \frac{1-\|v\|_x}{\|v\|_x} \langle \partial_v L(x,v), v \rangle. \end{aligned}$$

Since the norm of $v/\|v\|_x$ is equal to 1, we obtain

$$\frac{\|v\|_x - 1}{\|v\|_x} \langle \partial_v L(x,v), v \rangle - L(x,v) \geq -A(1).$$

If in this last inequality

$$\frac{\|v\|_x - 1}{\|v\|_x}\langle \partial_v L(x,v), v\rangle - L(x,v) \geq -A(1)$$

we multiply both sides by $\dfrac{\|v\|_x}{\|v\|_x - 1} = 1 + \dfrac{1}{\|v\|_x - 1} > 0$, we get

$$\langle \partial_v L(x,v), v\rangle - \left(1 + \frac{1}{\|v\|_x - 1}\right) L(x,v) \geq -\frac{\|v\|_x}{\|v\|_x - 1} A(1).$$

From which we obtain

$$E(x,v) = \langle \partial_v L(x,v), v\rangle - L(x,v) \geq \frac{L(x,v)}{\|v\|_x - 1} - \frac{\|v\|_x}{\|v\|_x - 1} A(1).$$

We now fix $K > A(1)$. Since $\|v\|_x - 1 > 0$ and $L(x,v) \geq K\|v\|_x - C(K) \geq K\|v\|_x - |C(K)|$, we see that

$$E(x,v) \geq [K - A(1)]\frac{\|v\|_x}{\|v\|_x - 1} - \frac{|C(K)|}{\|v\|_x - 1}.$$

From this inequality

$$E(x,v) \geq [K - A(1)]\frac{\|v\|_x}{\|v\|_x - 1} - \frac{|C(K)|}{\|v\|_x - 1},$$

using $K > A(1)), \|v\|_x \geq R > 1$, and the fact that $x/(x-1) \geq 1$, for $x > 1$, we obtain

$$E(x,v) \geq K - A(1) - \frac{|C(K)|}{R-1}.$$

Therefore $\alpha(R) \geq K - A(1) - \dfrac{|C(K)|}{R-1}$.

Keeping K fixed and letting $R \to +\infty$ shows that

$$\liminf_{R\to+\infty} \alpha(R) \geq K - A(1).$$

Since $K > A(1)$ is arbitrary, we get $\lim_{R\to+\infty} \alpha(R) = +\infty$. $\square$

The function η

We define the function η on $[0, +\infty[$, by

$$\eta(R) = \sup\{\rho \geq 0 \mid \alpha(\rho) \leq \beta(R)\}.$$

The function η is indeed well-defined since $\alpha \leq \beta$, which also implies $\eta(R) \geq R$. From its very definition, the function η is non-decreasing. It is finite-valued, since $\alpha(R) \to +\infty$, when $R \to +\infty$.

Lemma 7

For every curve $\gamma : [a, b] \to M$ which satisfies the Euler-Lagrange equation, we have

$$\sup_{t \in [a,b]} \|\dot{\gamma}(t)\|_{\gamma(t)} \leq \eta \left(\inf_{t \in [a,b]} \|\dot{\gamma}(t)\|_{\gamma(t)} \right).$$

Therefore

$$\sup_{t \in [a,b]} \|\dot{\gamma}(t)\|_{\gamma(t)} \leq \eta[\ell_g(\gamma)/(b-a)].$$

Proof

Consider a solution $\gamma : [a, b] \to M$ of the Euler-Lagrange equation. Define $s_{\min}$ and $s_{\max}$ by

$$\|\dot\gamma(s_{\min})\|_{\gamma(s_{\min})} = \inf_{t\in[a,b]} \|\dot\gamma(t)\|_{\gamma(t)}, \|\dot\gamma(s_{\max})\|_{\gamma(s_{\max})} = \sup_{t\in[a,b]} \|\dot\gamma(t)\|_{\gamma(t)}.$$

By last Lemma 6 , and the conservation of Energy along solutions of the Euler-Lagrange equation, we get

$$\begin{aligned}\beta(\|\dot\gamma(s_{\min})\|_{\gamma(s_{\min})}) &\geq E(\gamma(s_{\min}), \dot\gamma(s_{\min}))\\ &= E(\gamma(s_{\max}), \dot\gamma(s_{\max}))\\ &\geq \alpha(\|\dot\gamma(s_{\max})\|_{\gamma(s_{\max})}).\end{aligned}$$

The definition of η yields

$$\|\dot\gamma(s_{\max})\|_{\gamma(s_{\max})} \leq \eta(\|\dot\gamma(s_{\min})\|_{\gamma(s_{\min})}).$$

It remains to prove the last inequality. Since η is non-decreasing, this follows from

$$(b-a) \inf_{t\in[a,b]} \|\dot\gamma(t)\|_{\gamma(t)} \leq \int_a^b \|\dot\gamma(s)\|_{\gamma(s)}\, ds = \ell_g(\gamma). \quad \square$$

We now combine Lemma 4 and Lemma 7 to obtain:

Proposition 8

Suppose $S \subset M$ and $t_0 > 0$. Any minimizer $\gamma : [a, b] \to M$ such that $\gamma(a), \gamma(b) \in S$ and $b - a \geq t_0$ satisfies

$$\sup_{t \in [a,b]} \|\dot{\gamma}(t)\|_{\gamma(s)} \leq \eta \left[\frac{A(\mathrm{diam}(S)/t_0) + C(K)}{K} \right],$$

for every $K \geq 0$, where $\mathrm{diam}(S) = \sup\{d(x, y) \mid x, y \in S\}$ is the diameter of S for the Riemannian distance d on M.

In particular, the set of minimizers $\gamma : [a, b] \to M$ such that $\gamma(a), \gamma(b) \in S$ and $b - a \geq t_0$ is equi-Lipschitz.

Proof

From inequality (6) in Lemma 4, for $K > 0$, we obtain

$$\frac{\ell_g(\gamma)}{b-a} \leq \frac{A(d(\gamma(a),\gamma(b))/(b-a)) + C(K)}{K}.$$

Since $b - a > t_0, d(\gamma(a),\gamma(b)) \leq \mathrm{diam}(S)$ and A is non-decreasing, we get

$$\frac{\ell_g(\gamma)}{b-a} \leq \frac{A\left(\mathrm{diam}(S)/t_0\right) + C(K)}{K}.$$

From Lemma 7, we have

$$\sup_{t\in[a,b]} \|\dot{\gamma}(t)\|_{\gamma(t)} \leq \eta[\ell_g(\gamma)/(b-a)].$$

Since η is non-decreasing, combining the last two inequalities, we conclude that

$$\sup_{t\in[a,b]} \|\dot{\gamma}(t)\|_{\gamma(s)} \leq \eta\left[\frac{A(\mathrm{diam}(S)/t_0) + C(K)}{K}\right]. \ \square$$

The minimal action

Definition 9 (Minimal action h_t)

For $x, y \in M$, and $t > 0$, we define the minimal action $h_t(x, y)$ to join x to y in time t by

$$h_t(x, y) = \inf_{\gamma} \int_0^t L(\gamma(s)\dot{\gamma}(s))\, ds,$$

where the infimum is taken over all absolutely continuous curves $\gamma : [0, t] \to M$, with $\gamma(0) = x$ and $\gamma(t) = y$.

By Tonelli's theorem 3, the infimum in the definition of $h_t(x, y)$ is always attained by a minimizer which is as smooth as the Lagrangian.

It is convenient to denote by $\mathcal{H}$ the function defined on $]0, +\infty[\times M \times M$ by

$$\mathcal{H}(t, x, y) = h_t(x, y).$$

Example

1) For the Tonelli Lagrangian $L_0 : TM \to \mathbb{R}$ defined by $L_0(x, v) = \frac{1}{2}\|v\|_x^2$, we have

$$h_t^0(x, y) = \frac{d(x, y)^2}{2t}.$$

2) For the Tonelli Lagrangian $L_V : TM \to \mathbb{R}$ defined by

$$L_V(x, v) = \frac{1}{2}\|v\|_x^2 - V(x),$$

where $V : M \to \mathbb{R}$ is a bounded C^r function, with $r \geq 2$, we have

$$\frac{d(x, y)^2}{2t} - \sup V \leq h_t^V(x, y) \leq \frac{d(x, y)^2}{2t} - \inf V.$$

Properties of minimal action

The properties of the h_t's that we will use are the following ones:

(a) For every $K \in [0, \infty[$, $t > 0$ and every $x, y \in M$, we have:

$$Kd(x,y) - C(K)t \leq h_t(x,y) \leq tA\left(\frac{d(x,y)}{t}\right). \qquad (9)$$

(b) (semi-group property) For every $t, t' > 0$ and every $x, y \in M$, we have:

$$h_{t+t'}(x,y) = \inf_{z \in M} h_t(x,z) + h_{t'}(z,y).$$

(c) The function $\mathcal{H}$ is locally Lipschitz, and locally semi-concave, on $]0, +\infty[\times M \times M$.

(d) If we fix $x \in M$, the function $\mathcal{H}_x :]0, +\infty[\times M \to \mathbb{R}$, defined by

$$\mathcal{H}_x(t,y) = \mathcal{H}(t,x,y) = h_t(x,y),$$

is a viscosity solution of

$$\partial_t \mathcal{H}_x + H(y, \partial_y \mathcal{H}_x) = 0.$$

Sketch of proofs

Part (a) follows from the similar inequalities (4) and (6) given in Lemma 4.

Part (b) is left to the reader.

Part (c). It suffices to prove the local properties of $\mathcal{H}$ on any compact subset of the form $[t_0, T_0] \times S \times S$, with S a compact subset of M. In the compact case, to prove the locally Lipschitz (or semi-concave) property, the ingredients were:

- Minimizers exist (Tonelli's theorem valid even for non-compact manifolds)
- Speeds of minimizers, for a time bounded away for 0, are uniformly bounded if the endpoints are in the compact set S, which we obtained Proposition 8.

Part (d) The proof is the same as in the compact case, since it is a local argument which uses only domination properties and existence of minimizers. More on this in the next lecture.

The Lax-Oleinik semi-group

We now come back to the definition of the (negative) Lax-Oleinik semi-group $T_t^-, t \geq 0$.

If $u : M \to [-\infty, +\infty]$ is a function and $t > 0$, the function $T_t^- u : M \to [-\infty, +\infty]$ is defined by

$$T_t^- u(x) = \inf_{\gamma} u(\gamma(0)) + \int_0^t L(\gamma(s), \dot{\gamma}(s))\, ds.$$

Using that

$$h_t(x, y) = \inf_{\gamma} \int_0^t L(\gamma(s)\dot{\gamma}(s))\, ds,$$

where the infimum is taken over all absolutely continuous curves $\gamma : [0, t] \to M$, with $\gamma(0) = x$ and $\gamma(t) = y$. We can equivalently define $T_t^- u$ by

$$T_t^- u(x) = \inf_{y \in M} u(y) + h_t(y, x).$$

We also set $T_0^- u = u$.

First properties of the Lax-Oleinik semi-group

Let $u : M \to [-\infty, +\infty]$ be a function, we have:

(a) $T_t^- u(x) \leq u(x) + A(0)t$, for $x \in M$ and every $t \geq 0$.

(b) If $u < +\infty$ at one point in M, then $T_t^- u < +\infty$ everywhere $t > 0$.

(c) If $u = -\infty$ at one point in M, then $T_t^- u = -\infty$ everywhere, for $t > 0$.

(d) $T_t^-(u + c) = T_t^-(u) + c$, for $c \in \mathbb{R}$.

(e) If $u \leq v$ everywhere , then $T_t^- u \leq T_t^- v$.

(f) $-\|u - v\|_\infty + T_t^- v \leq T_t^- u \leq T_t^- v + \|u - v\|_\infty$.

(g) (semi-group property) $T_{t+t'}^- = T_t^- \circ T_{t'}^-$ for $t, t' \geq 0$.

Sketch of Proof

From the property of h_t (9), we have
$h_t(x,x) \leq tA(d(x,x)/t) = A(0)t$. Therefore
$T_t^- u(x) \leq u(x) + h_t(x,x) \leq u(x) + A(0)t$. This proves (a).

Note that from the definition of $T_t^- u$

$$T_t^- u(x) = \inf_{y \in M} u(y) + h_t(y,x),$$

we get $T_t^- u(x) \leq u(y) + h_t(y,x)$, for every $y \in M$. Since $h_t(y,x)$ is finite everywhere, this proves (b) and (c).

(d) and (e) are clear from the definition of $T_t^- u$.

(f) is a consequence of (d) and (e) since
$-\|u-v\|_\infty + tv \leq u \leq v + \|u-v\|_\infty$.

(g) is a consequence of the semi-group property of h_t. $\square$

Example with $T_t^- u$ finite everywhere

Proposition 10

If $\theta : M \to \mathbb{R}$ is a Lipschitz function, then $T_t^- \theta$ is everywhere finite-valued for $t \geq 0$.

Moreover the function $\Theta : [0, +\infty[\times \to \mathbb{R}$ defined by

$$\Theta(t, x) = T_t^- \theta(x),$$

is bounded below by a (globally) Lipschitz function on $[0, +\infty[\times M$.

Therefore, for any $u : M \to \mathbb{R}$ which is bounded below by a Lipschitz function, we also have that $T_t^- u$ is everywhere finite-valued for $t \geq 0$, and the function $U(t, x) = T_t^- u(x)$ is everywhere bounded below by a (globally) Lipschitz function on $[0, +\infty[\times M$.

Proof

Call K a Lipschitz constant for θ.

Hence $\theta(y) \geq \theta(x) - Kd(x,y)$.

By the superlinearity (9), we also have
$h_t(y,x) \geq Kd(y,x) - C(K)t$.

Adding the two inequality yields $\theta(y) + h_t(y,x) \geq \theta(x) - C(K)t$.

Therefore

$$\Theta(t,x) = T_t^- \theta(x) = \inf_{y \in M} \theta(y) + h_t(y,x) \geq \theta(x) - C(K)t.$$

Hence, the function Θ is everywhere bounded below by the finite-valued Lipschitz function $t \mapsto \theta(x) - C(K)t$ on $[0, +\infty[\times M$.

Moreover, if $u \geq \theta$, we know that $T_t^- u \geq T_t^- \theta$. $\square$

Example with $T_t^- u$ not necessarily finite everywhere

For the Tonelli Lagrangian $L_0 : TM \to \mathbb{R}$ defined by $L_0(x, v) = \frac{1}{2}\|v\|_x^2$, we know that $h_t^0(y, x) = d(y, x)^2/2t$. Therefore, if $u : M \to \mathbb{R}$, we get

$$T_t^- u(x) = \inf_{y \in M} u(y) + \frac{d(y, x)^2}{2t}.$$

Fix some $x_0 \in M$, and consider the case $u_\alpha(x) = -\alpha d(x_0, x)^2$, where $\alpha > 0$. In this case,

$$T_t^- u_\alpha(x) = \inf_{y \in M} -\alpha d(x_0, y)^2 + \frac{d(y, x)^2}{2t}.$$

If M is not compact, then $T_t^- u_\alpha = -\infty$ everywhere, for $-\alpha + \dfrac{1}{2t} > 0$ or equivalently $t > \dfrac{1}{2\alpha}$, and $T_t^- u_\alpha$ is finite everywhere, for $t < \dfrac{1}{2\alpha}$. Note also that $T_{1/2\alpha}^- u_\alpha(x_0) = 0$.

Pointwise finiteness of the Lax-Oleinik semi-group

Proposition 11 (Pointwise Finiteness)

Let $u : M \to \mathbb{R}$ be a continuous function. Assume that $T^-_{t_0} u(x_0)$ is finite for some $t_0 > 0$ and some $x_0 \in M$, then we have:

(a) *For every $t < t_0$, and every $x \in M$, the value $T^-_t u(x)$ is finite.*

(b) *For every $t < t_0$, and every $x \in M$, we can find $y \in M$ such that $T^-_t u(x) = u(y) + h_t(y, x)$.*

(c) *For any compact subset $\mathfrak{K} \subset M$, and any $t'_0 \in]0, t_0[$, we can find $R < +\infty$, such that for any $x \in \mathfrak{K}$, any $t \in]0, t'_0]$, and any $y \in M$, if $T^-_t u(x) = u(y) + h_t(y, x)$, then $d(x, y) \leq R$.*

(d) *For any compact subset $\mathfrak{K} \subset M$ and any $\epsilon > 0$, we can find a $\delta > 0$ such that $T^-_t u(x) = u(y) + h_t(y, x)$, with $x \in \mathfrak{K}$ and $t \leq \delta$ implies $d(x, y) \leq \epsilon$.*

Theorem 12

Let $u : M \to \mathbb{R}$ be a continuous function.
Assume that $T^-_{t_0}u(x_0)$ is finite for some $t_0 > 0$ and some $x_0 \in M$.
The function U defined by $U(t,x) = T^-_t u(x)$ is finite, continuous on $[0, t_0[\times M$ and locally Lipschitz (even locally semi-concave) on $]0, t_0[\times M$.
Moreover, the function U is a viscosity solution of $\partial_t U + H(x, \partial_x U) = 0$ on $]0, t_0[\times M$.

Proof assuming Proposition 11 above

It suffices to show the properties of U on any set of the form $[0, t_0'] \times \mathfrak{K}$, with $t_0' \in]0, t_0[$ and $\mathfrak{K} \subset M$ compact.
By part (c) of the Pointwise Finiteness Proposition 11, we can find $R > 0$ such that for $x \in \mathfrak{K}$ and $t \in]0, t_0']$, we have

$$U(t,x) = T^-_t u(x) = \inf_{y \in \bar{V}_R(\mathfrak{K})} u(y) + h_t(y,x),$$

since the map $(t,x,y) \mapsto h_t(x,y)$ is locally Lipschitz on $]0, +\infty[\times M \times M$, and $\bar{V}_R(\mathfrak{K})$ is compact, we indeed see that U is locally Lipschitz on $]0, +\infty[\times M$.

Moreover, since $(t,x,y)\mapsto h_t(x,y)$ is locally semi-concave on $]0,+\infty[\times M\times M$, then again by the compactness of $\bar V_R(\mathfrak{K})$, we also get the local semi-concavity of U.
To show the continuity at $t=0$, fix $\epsilon>0$. By part (b) and (d), there exists $\delta>0$ such that for $x\in\mathfrak{K}$ and $t\in]0,\delta]$, we can find $y_{x,t}\in M$, such that

$$T_t^- u(x)=u(y_{x,t})+h_t(y_{x,t},x),\ \text{with}\ d(y_{x,t},x)\leq\epsilon.$$

By the inequalities (9) on h_t, we also have

$$u(y_{x,t})-C(0)t\leq T_t^- u(x)\leq u(x)+A(0)t.$$

Therefore, if we set

$$\rho(\epsilon)=\sup\{|u(y)-u(x)|\mid x\in\mathfrak{K},d(x,y)\leq\epsilon\},$$

for $t\leq\delta$, we obtain

$$\sup_{x\in\mathfrak{K}}|T_t^- u(x)-u(x)|\leq\rho(\epsilon)+t\max\left[|A(0)|,|C(0)|\right]$$

Hence for $t \leq \min(\delta, \epsilon)$, we get

$$\sup_{x \in \mathfrak{K}} |T_t^- u(x) - u(x)| \leq \rho(\epsilon) + \epsilon \max\left[|A(0)|, |C(0)|\right].$$

Since u is continuous and $\mathfrak{K}$ is compact, we get

$$\rho(\epsilon) = \sup\{|u(y) - u(x)| \mid x \in \mathfrak{K}, d(x, y) \leq \epsilon\} \to 0, \text{ when } \epsilon \to 0.$$

Therefore U is continuous on $[0, t_0'] \times \mathfrak{K}$.

The fact that U is a viscosity solution is standard since (c) holds, see next lecture. $\square$

Proof of the Pointwise Finiteness Proposition 11

By the semi-group property $T^-_{t_0} = T^-_{t_0-t} T^-_t$, we get

$$T^-_{t_0} u(x_0) \leq T^-_t u(x) + h_{t_0-t}(x, x_0),$$

for every $x \in M$ and any $t < t_0$. Therefore

$$T^-_t u(x) \geq T^-_{t_0} u(x_0) - h_{t_0-t}(x, x_0),$$

from which part (a), about finiteness of $T^-_t u(x)$ for $t < t_0$ follows. Moreover, by continuity of $h_{t_0-t}(x, y)$, for $t < t_0$, the value $T^-_t u(x)$ is uniformly bounded below on any compact subset of $[0, t_0[\times M$.

Fix now a compact subset $\mathfrak{K} \subset M$ and $t_0' \in]0, t_0[$. Using the fact that $T^-_t u(x) \leq u(x) + A(0)t$, and the boundedness by below that we just obtained, we can find a finite constant α such that

$$-\alpha \leq T^-_t u(x) \leq \alpha,$$

for every $t \in [0, t_0']$ and every $x \in \bar{V}_1(\mathfrak{K}) = \{x \in M \mid d(x, \mathfrak{K}) \leq 1\}$.

Consider the set

$$S = \{(x, y, t) \mid u(y) + h_t(y, x) \leq \alpha + 1, x \in \mathfrak{K}, y \in M, t \in [0, t_0']\}.$$

Claim *For every $\epsilon > 0$, we can find $R_\epsilon \geq 0$ such that for any $(x, y, t) \in S$, we have $d(x, y) \leq \epsilon + R_\epsilon t$.*

Without loss of generality, to prove the claim, we can suppose that $\epsilon \leq 1$ and that $(x, y, t) \in S$ is such $d(x, y) > \epsilon$.
Let us consider a minimizing curve $\gamma : [0, t] \to M$ with $\gamma(0) = y$ and $\gamma(t) = x$.
It suffices to show that there exists a constant $\bar{R}_\epsilon$ which does not depend on the particular $(x, y, t) \in S$ such that for $\|\dot{\gamma}(s_0)\|_{\gamma(s_0)} \leq \bar{R}_\epsilon$ for some $s_0 \in [0, t]$.
In fact, by Lemma 7, this implies that $\|\dot{\gamma}(s)\|_{\gamma(s)} \leq \eta(\bar{R}_\epsilon)$, and therefore we would obtain

$$d(x, y) \leq \ell_g(\gamma) \leq R_\epsilon t, \text{ with } R_\epsilon = \eta(\bar{R}_\epsilon).$$

Denote by $\bar{B}(x,\epsilon)$ the closed ball of center x and radius ϵ. Since $d(x,y) > \epsilon$, the curve $\gamma(s)$ must cross the boundary of $\bar{B}(x,\epsilon)$ at some time $t' \in]0,t[$.
We set $y' = \gamma(t')$. Note that $d(x,y') = \epsilon$.
Since γ is a minimizer we have $h_t(y,x) = h_{t'}(y,y') + h_{t-t'}(y',x)$.
Therefore we get

$$\begin{aligned} u(y) + h_t(y,x) &= u(y) + h_{t'}(y,y') + h_{t-t'}(y',x) \\ &\geq T^-_{t'} u(y') + h_{t-t'}(y',x), \end{aligned}$$

hence, since $(x,y,t) \in S$, we get

$$T^-_{t'} u(y') + h_{t-t'}(y',x) \leq u(y) + h_t(y,x) \leq \alpha + 1.$$

Note also that $T^-_{t'} u(y') \geq -\alpha$, since $t' \in]0, t'_0]$ and $y' \in \bar{B}(x,\epsilon) \subset \bar{V}_1(\mathfrak{K})$.
Combining the two inequalities, we get

$$h_{t-t'}(y',x) \leq 2\alpha + 1.$$

Since $\bar{\gamma} = \gamma|[t', t]$ is a minimizer joining y' to x whose action is $h_{t-t'}(y', x) \leq 2\alpha + 1$, by Lemma 4, for every $K > 0$, we have

$$\epsilon = d(x, y') \leq \ell_g(\bar{\gamma}) \leq \frac{2\alpha + 1 + C(K)(t - t')}{K}, \tag{10}$$

We now show that there is a positive constant $\rho_\epsilon > 0$ independent of the choice of $(x, y, t) \in S$ such that $t - t' \geq \rho_\epsilon$. Since $d(x, y') = \epsilon$, choosing K_ϵ such that

$$\frac{2\alpha + 1}{K_\epsilon} = \frac{\epsilon}{2},$$

from the inequality (10), we obtain that

$$\frac{C(K_\epsilon)(t - t')}{K_\epsilon} \geq \frac{\epsilon}{2}.$$

This finishes the proof of the existence of ρ_ϵ.

Using now $t - t' \geq \rho_\epsilon$ and the inequality (10) with $K = 1$, we get

$$\ell_g(\bar{\gamma}) \leq \bar{R}_\epsilon(t - t')$$

where

$$\bar{R}_\epsilon = \frac{2\alpha + 1}{\rho_\epsilon} + C(1).$$

This implies, using $\bar{\gamma} = \gamma|[t', t]$, that there exists some $s_0 \in [t', t]$ such that $\|\dot{\gamma}(s_0)\|_{\gamma(s_0)} \leq \bar{R}_\epsilon$.
This finishes the proof of the claim:
<u>Claim</u> *For every $\epsilon > 0$, we can find $R_\epsilon \geq 0$ such that for any $(x, y, t) \in S$, we have $d(x, y) \leq \epsilon + R_\epsilon t$,*
we can prove easily the remaining parts (b), (c) and (d).

For part (b), we note that $x \in \mathfrak{K}$, we can find a sequence $y_n \in M, \geq 1$ such that

$$T_t^- u(x) \leq u(y_n) + h_t(y_n, x) \leq T_t^- u(x) + \frac{1}{n}.$$

If moreover $t \leq t_0'$, we know, by choice of α that $T_t^- u(x) \leq \alpha$. Hence

$$T_t^- u(x) \leq u(y_n) + h_t(y_n, x) \leq T_t^- u(x) + \frac{1}{n} \leq \alpha + \frac{1}{n}.$$

Since

$$S = \{(x, y, t) \mid u(y) + h_t(y, x) \leq \alpha + 1, x \in \mathfrak{K}, y \in M, t \in [0, t_0']\},$$

we conclude $(x, y_n, t) \in S$.
Therefore by the claim, we obtain $d(x, y_n) \leq 1 + R_1 t \leq 1 + R_1 t_0'$.
Hence the sequence y_n is bounded in M. Extracting if necessary, we can assume that $y_n \to y$.
Obviously, by continuity, we have $T_t^- u(x) = u(y) + h_t(y, x)$.

To prove (c), consider $x \in \mathfrak{K}, y \in M$ and $t \leq t_0'$ such that $T_t^- u(x) = u(y) + h_t(y, x)$. Since $T_t^- u(x) \leq \alpha$, for $x \in \mathfrak{K}$ and $t \in [0, t_0']$, we have $(t, x, y) \in S$. By the claim, we get

$$d(x, y) \leq 1 + R_1 t \leq 1 + R_1 t_0' = R.$$

To prove (d), consider again $x \in \mathfrak{K}, y \in M$ and $t \leq t_0'$ such that $T_t^- u(x) = u(y) + h_t(y, x)$. Again as above $(t, x, y) \in S$, hence

$$d(x, y) \leq \frac{\epsilon}{2} + R_{\frac{\epsilon}{2}} t.$$

Therefore, if we set

$$\delta = \min \left[t_0', \frac{\epsilon}{2R_{\frac{\epsilon}{2}}} \right],$$

for $t \leq \delta$, we get

$$d(x, y) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \ \square$$