Lecture 2. Viscosity solutions of the Hamilton-Jacobi equation on a non-compact manifold: Uniqueness

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Introduction

The setting is the same as in Lecture 1, namely M is a connected complete Riemannian manifold.
Assume that L is a Tonelli Lagrangian. Its associated Hamiltonian $H : T^*M \to \mathbb{R}$ is defined by

$$H(x,p) = \sup_{v \in T_xM} p(v) - L(x,v).$$

If $u : M \to \mathbb{R}$ is a function bounded below by a Lipschitz function, we have said in the first lecture that its Lax-Oleinik evolution

$$U(t,x) = T_t^- u(x),$$

is continuous on $[0,+\infty[\times M$, with $U(0,x) = u(x)$, everywhere on M, and is also a viscosity solution, on $]0,+\infty[\times M$, of

$$\frac{\partial u}{\partial t}(t,x) + H\big(x, \frac{\partial u}{\partial x}(t,x)\big) = 0.$$

We will address the problem: if a function $V : [0,+\infty[\times M \to \mathbb{R}$ shares the properties of U given above is it necessarily equal to U?

Crash Course on Viscosity

We will now introduce the notion of viscosity solutions and their properties.
We do it by recalling the parts of

Albert Fathi, *Weak KAM from a PDE point of view: viscosity solutions of the Hamilton-Jacobi equation and Aubry set*, Proc. Roy. Soc. Edinburgh Sect. A, 120 **(2012)** 1193–1236

that are relevant here.

Again, the setting is the same as in Lecture 1, namely M is a connected complete Riemannian manifold.

We will suppose that $H : T^*M \to \mathbb{R}$ is a continuous function, which we will call the Hamiltonian.

Examples

1) If $L : TM \to \mathbb{R}$ is a Tonelli Lagrangian, its associated Hamiltonian $H : T^*M \to \mathbb{R}$ is defined by

$$H(x,p) = \sup_{v \in T_x M} p(v) - L(x,v).$$

2) The Hamiltonian $H_V : T^*M \to \mathbb{R}$ is defined by

$$H(x,p) = \frac{1}{2}\|p\|_x^2 + V(x).$$

It is the Hamiltonian associated to the Lagrangian $L_V : TM \to \mathbb{R}$ defined by

$$L_V(x,v) = \frac{1}{2}\|v\|_x^2 - V(x).$$

The Hamiltonian H_V is a good example to keep in mind.

Hamilton-Jacobi equation

The (stationary) Hamilton-Jacobi equation associated to H is the equation

$$H(x, d_x u) = c,$$

where c is some constant.

A classical solution of the Hamilton-Jacobi equation $H(x, d_x u) = c$ on the open subset U of M is a C^1 map $u : U \to \mathbb{R}$ such that $H(x, d_x u) = c$, for each $x \in U$.

Usually, one deals only with the case $H(x, d_x u) = 0$, since it is possible to reduce the general case to that case by replacing the Hamiltonian H by H_c, defined by $H_c(x, p) = H(x, p) - c$.

Evolutionary Hamilton-Jacobi equation

The evolutionary Hamilton-Jacobi equation associated to the Hamiltonian H is the equation

$$\frac{\partial u}{\partial t}(t,x) + H\big(x, \frac{\partial u}{\partial x}(t,x)\big) = 0.$$

A classical solution to this evolutionary Hamilton-Jacobi equation on the open subset W of $\mathbb{R} \times T^*M$ is a C^1 map $u : W \to \mathbb{R}$ such that

$$\frac{\partial u}{\partial t}(t,x) + H\big(x, \frac{\partial u}{\partial x}(t,x)\big) = 0,$$

for each $(t,x) \in W$.
The evolutionary form can be reduced to the stationary form by introducing the Hamiltonian $\hat{H} : T^*(\mathbb{R} \times M)$ defined by

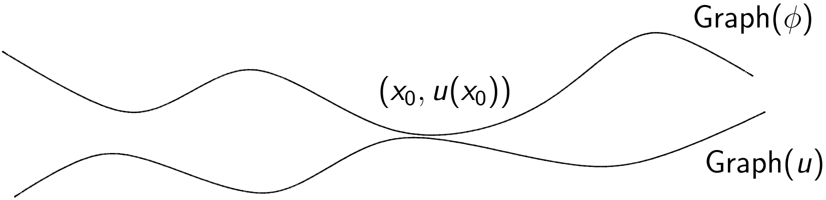
$$\hat{H}(t,x,s,p) = s + H(x,p),$$

where $(t,x) \in \mathbb{R} \times M$, and $(s,p) \in T^*_{(t,x)}(\mathbb{R} \times M) = \mathbb{R} \times T^*_x M$.

It is generally impossible to find C^1-solutions to the Hamilton-Jacobi equation. One has to define a notion of weak solution. For the Hamilton-Jacobi equation, at least when H is convex in the momentum, the most successful notion of weak solution is the notion of *viscosity solution*.

Viscosity Subsolution

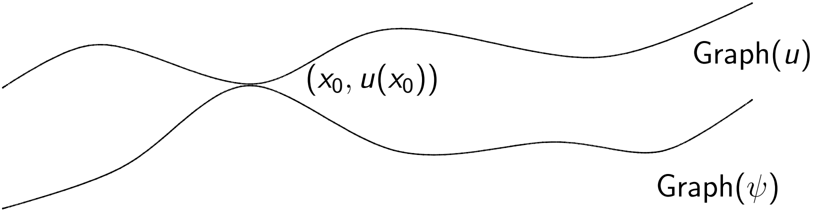
A function $u : V \to \mathbb{R}$ is a *viscosity subsolution* of $H(x, d_x u) = c$ on the open subset $V \subset M$, if for every C^1 function $\phi : V \to \mathbb{R}$, with $\phi \geq u$ everywhere, at every point $x_0 \in V$ where $u(x_0) = \phi(x_0)$ we have $H(x_0, d_{x_0}\phi) \leq c$.



Subsolution: $\phi \geq u, u(x_0) = \phi(x_0) \Rightarrow H(x_0, d_{x_0}\phi) \leq c$.

Viscosity Supersolution

A function $u : V \to \mathbb{R}$ is a *viscosity supersolution* of $H(x, d_x u) = c$ on the open subset $V \subset M$, if for every C^1 function $\psi : V \to \mathbb{R}$, with $u \geq \psi$ everywhere, at every point $x_0 \in V$ where $u(x_0) = \psi(x_0)$ we have $H(x_0, d_{x_0}\psi) \geq c$.



Supersolution: $\psi \leq u, u(x_0) = \psi(x_0) \Rightarrow H(x_0, d_{x_0}\psi) \geq c$.

Viscosity Solution

A function $u : V \to \mathbb{R}$ is a *viscosity solution* of $H(x, d_x u) = c$ on the open subset $V \subset M$, if it is both a subsolution and a supersolution.

In the sequel of this lecture, we will concentrate on viscosity solutions of the evolutionary Hamilton-Jacobi equation

$$\frac{\partial u}{\partial t}(t,x) + H\big(x, \frac{\partial u}{\partial x}(t,x)\big) = 0.$$

We will mainly address the problem of uniqueness of the solution on $[0, T[\times M$ for a given initial condition on $\{0\} \times M$.

Some facts about viscosity solutions

We enumerate a first facts facts about viscosity subsolutions, supersolutons, and solutions.

- A C^1 function is a viscosity subsolution of the Hamilton-Jacobi equation if and only if it is a classical solution.
- If the viscosity subsolution u (resp. supersolution, solution) of the Hamilton-Jacobi equation $H(x, d_x u) = c$ is differentiable at x_0, then $H(x, d_{x_0} u) \leq c$ (resp. $H(x, d_{x_0} u) \geq c$, $H(x, d_{x_0} u) = c$).
- (Stability) Suppose that $v_n : M \to \mathbb{R}$ is a sequence of continuous functions converging uniformly on compact subsets to $v : M \to \mathbb{R}$. If, for each n, the function v_n is a viscosity subsolution (resp. supersolution, solution) of $H(x, d_x u) = 0$, then v is a viscosity subsolution (resp. supersolution, solution) of $H(x, d_x u) = 0$.

- If $H(x,p)$ in convex in the momentum variable, then a locally Lipschitz function u is a viscosity subsolution of $H(x, d_x u) = c$ if and only if $H(x, d_x u) \leq c$ almost everywhere.
- If $H(x,p)$ in convex in the momentum variable and the two locally Lipschitz function $u_1, u_2 : O \to \mathbb{R}$ are viscosity subsolutions of $H(x, d_x u) = c$, on the open subset $O \subset M$, then so is $\min(u_1, u_2)$.
- If $H(x,p)$ in convex in the momentum variable, and $u : O \to \mathbb{R}$ is a locally Lipschitz viscosity subsolution of $H(x, d_x u) = c$, defined on the open subset $O \subset M$, then for any $\epsilon > 0$ we can find a C^∞ function $v : O \to \mathbb{R}$ which is a viscosity subsolution of $H(x, d_x v) = c + \epsilon$ on O and such that $\sup_{x \in O} |v(x) - u(x)| \leq \epsilon$.

To give further properties we need to introduce:

Definition 1 (Coercive)

A continuous function $H : T^*M \to \mathbb{R}$ is said to be coercive above every compact subset, if for each compact subset $K \subset M$ and each $c \in \mathbb{R}$ the set $\{(x, p) \in T^*M \mid x \in K, H(x, p) \leq c\}$ is compact.

It is not difficult to see that H is coercive if and only if for each compact subset $K \subset M$, we have $\lim_{\|p\|_x \to \infty} H(x, p) = +\infty$, the limit being uniform in $x \in K$.

Theorem 2

Suppose that $H : T^*M \to \mathbb{R}$ is coercive above every compact subset, and $c \in \mathbb{R}$. Then a viscosity subsolution of $H(x, d_x u) = c$ is necessarily locally Lipschitz, and therefore satisfies $H(x, d_x u) \leq c$ almost everywhere.

Note however that the Hamiltonian

$$\hat{H}(t,x,s,p) = s + H(x,p),$$

which give rise to the evolutionary Hamilton-Jacobi equation is never coercive even if H is coercive, since s can $\to -\infty$.
Therefore, it is difficult to assume (or obtain) a priori that a viscosity subsolution of the evolutionary Hamilton-Jacobi equation is locally Lipschitz.
In fact, if U is a viscosity subsolution of

$$\frac{\partial U}{\partial t}(t,x) + H\big(U, \frac{\partial u}{\partial x}(t,x)\big) = 0,$$

and $\rho : [0,+\infty[\to \mathbb{R}$ which is continuous and non-increasing, then $V(x,s) = U(x,s) + \rho(s)$ is a viscosity subsolution of the same equation.
At this point, it is useful to note that the Hamiltonian

$$\tilde{H}(t,x,s,p) = |s| + H(x,p),$$

is coercive, if H is.

The main ingredient to prove uniqueness properties for viscosity solutions is the following one:

Theorem 3

Let $H : T^*M \to \mathbb{R}$ be any continuous Hamiltonian on the manifold M. Suppose that $u : M \to \mathbb{R}$ is a viscosity subsolution of $H(x, d_x u) = c_1$, and $v : M \to \mathbb{R}$ is a viscosity supersolution of $H(x, d_x v) = c_2$. Assume further that either u or v is locally Lipschitz on M. If $u - v$ has a local maximum, then necessarily $c_2 \leq c_1$.

Note that, if at x_0 the difference $u - v$ vanishes, then x_0 is a local maximum of $u - v$ if and only if $v \geq u$ in a neighborhood of x_0.

Because this Theorem 3 needs at least one of the functions to be Lipschitz, to apply it to the evolution case, we will need to approximate subsolutions by subsolutions which are Lipschitz.

Lax-Oleinik and Viscosity

We now explain the relationship between the Lax-Oleinik semi-group and viscosity solutions.

Theorem 4

Assume $L : TM \to \mathbb{R}$ is a Tonelli Lagrangian, and H is its associated Hamiltonian.
If $u : M \to \mathbb{R}$ is continuous, and $U(t,x) = T_t^- u(x)$ is finite on $]0,\tau[\times M$, for some $\tau \in]0,+\infty]$, where T_t^- is the Lax-Oleinik semi-group obtained from L.
Then U is a (continuous) viscosity solution of

$$\frac{\partial U}{\partial t}(t,x) + H(x, \frac{\partial U}{\partial x}(t,x)) = 0,$$

on the open subset $]0,\tau[\times M$.

Proof

We first note that we know from the first lecture that U is continuous on $]0,\tau[\times M$.
We then show the so-called domination inequality

$$U(b,\gamma(b)) - U(a,\gamma(a)) \leq \int_a^b L(\gamma(s),\dot\gamma(s))\,ds, \tag{1}$$

for every curve $\gamma : [a,b] \to M$.

In fact, by the definition and the semi-group property of T_t^-, we have

$$\begin{aligned} U(b,\gamma(b)) - U(a,\gamma(a)) &= T_b^- u(\gamma(b)) - T_a^- u(\gamma(a)) \\ &= T_{b-a}^- [T_a^- u](\gamma(b)) - T_a^- u(\gamma(a)) \\ &\leq \int_a^b L(\gamma(s),\dot\gamma(s))\,ds. \end{aligned}$$

Next, we prove that $U(t,x) = T_t^- u(x)$ is a viscosity subsolution.
Suppose $\phi \geq U$, with ϕ of class C^1 and $\phi(t_0, x_0) = U(t_0, x_0)$, where $t_0 > 0$.
Fix $v \in T_{x_0}M$, and pick a C^1 curve $\gamma : [0, t_0] \to M$ such that $(\gamma(t_0), \dot\gamma(t_0)) = (x_0, v)$.
If $0 \leq t \leq t_0$, by the domination inequality (1), we have

$$U(t_0, \gamma(t_0)) - U(t, \gamma(t)) \leq \int_t^{t_0} L(\gamma(s), \dot\gamma(s))\, ds,$$

for all $t \in [0, t_0]$.
Since $\phi \geq U$, with equality at $(t_0, x_0) = (t_0, \gamma(t_0))$, we obtain

$$\phi(t_0, \gamma(t_0)) - \phi(t, \gamma(t)) \leq \int_t^{t_0} L(\gamma(s), \dot\gamma(s))\, ds,$$

for all $t \in [0, t_0]$.

Dividing both sides of this last inequality

$$\phi(t_0, \gamma(t_0)) - \phi(t, \gamma(t)) \leq \int_t^{t_0} L(\gamma(s), \dot\gamma(s))\, ds,$$

by $t_0 - t > 0$, and letting $t \to t_0$, we get

$$\frac{\partial \phi}{\partial t}(t_0, x_0) + \frac{\partial \phi}{\partial x}(t_0, x_0)(v) \leq L(x_0, v).$$

Since this is true for all $v \in T_{x_0}M$, and

$$H(x_0, \frac{\partial \phi}{\partial x}(t_0, x_0)) = \sup_{v \in T_{x_0}M} \frac{\partial \phi}{\partial x}(t_0, x_0)(v) - L(x_0, v),$$

we obtain

$$\frac{\partial \phi}{\partial t}(t_0, x_0) + H(x_0, \frac{\partial \phi}{\partial x}(t_0, x_0)) \leq 0.$$

This finishes to show that $U(t, x) = T_t^- u(x)$ is a viscosity subsolution.

To prove that $U(t,x) = T_t^- u(x)$ is a supersolution, we consider $\psi \leq U$, with ψ of class C^1.
Suppose $U(t_0,x_0) = \psi(t_0,x_0)$, with $t_0 > 0$.
As we saw in lecture 1, we can find a $y \in M$ such that $U(t_0,x_0) = T_{t_0}^- u(x_0) = u(y) + h_{t_0}(y,x_0)$. By Tonelli's theorem, we can find a curve $\gamma : [0,t_0] \to M$, with $\gamma(t_0) = x_0, \gamma(0) = y$, and whose action is precisely $h_{t_0}(y,x_0)$.
Therefore

$$U(t_0,x_0) = T_{t_0}^- u(x_0) = u(\gamma(0)) + \int_0^{t_0} L(\gamma(s),\dot\gamma(s))\,ds.$$

Since $U(0,\gamma(0)) = u(\gamma(0))$, this can be rewritten as

$$U(t_0,x_0) - U(0,\gamma(0)) = \int_0^{t_0} L(\gamma(s),\dot\gamma(s))\,ds.$$

We have thus obtained

$$U(t_0, x_0) - U(0, \gamma(0)) = \int_0^{t_0} L(\gamma(s), \dot\gamma(s))\, ds. \qquad (2)$$

Applying twice the domination inequality (1)

$$U(b, \gamma(b)) - U(a, \gamma(a)) \leq \int_a^b L(\gamma(s), \dot\gamma(s))\, ds,$$

which is valid for every $a, b \in [0, t_0]$, we obtain for every $t \in [0, t_0]$

$$U(t_0, x_0) - U(t, \gamma(t)) \leq \int_t^{t_0} L(\gamma(s), \dot\gamma(s))\, ds$$

$$U(t, \gamma(t)) - U(0, \gamma(0)) \leq \int_0^t L(\gamma(s), \dot\gamma(s))\, ds.$$

Adding these two inequalities, we get in fact the equality (2). Therefore both inequalities must be equalities. Hence

$$\forall t \in [0, t_0], U(t_0, \gamma(t_0)) - U(t, \gamma(t)) = \int_t^{t_0} L(\gamma(s), \dot\gamma(s))\, ds.$$

Since $\psi \leq U$, with equality at (t_0, x_0), from this last equality

$$\forall t \in [0, t_0], U(t_0, \gamma(t_0)) - U(t, \gamma(t)) = \int_t^{t_0} L(\gamma(s), \dot{\gamma}(s))\, ds$$

we obtain

$$\psi(t_0, \gamma(t_0)) - \psi(t, \gamma(t)) \geq \int_t^{t_0} L(\gamma(s), \dot{\gamma}(s))\, ds, \text{ for every } t \in [0, t_0].$$

Dividing by $t_0 - t > 0$, and letting $t \to t_0$, we get

$$\frac{\partial \psi}{\partial t}(t_0, x_0) + \frac{\partial \psi}{\partial x}(t_0, x_0)(\dot{\gamma}(t_0)) \geq L(x_0, \dot{\gamma}(t_0)).$$

By definition of L, we have

$$L(x_0, \dot{\gamma}(t_0)) \geq \frac{\partial \psi}{\partial x}(t_0, x_0)(\dot{\gamma}(t_0)) - H(x_0, \frac{\partial \psi}{\partial x}(t_0, x_0)).$$

It follows that

$$\frac{\partial \psi}{\partial t}(t_0, x_0) + \frac{\partial \psi}{\partial x}(t_0, x_0)(\dot{\gamma}(t_0)) \geq \frac{\partial \psi}{\partial x}(t_0, x_0)(\dot{\gamma}(t_0)) - H(x_0, \frac{\partial \psi}{\partial x}(t_0, x_0)).$$

Therefore

$$\frac{\partial \psi}{\partial t}(t_0, x_0) + H(x_0, \frac{\partial \psi}{\partial x}(t_0, x_0)) \geq 0. \qquad \square$$

Approximation by Lipschitz subsolutions

As we said, we will mainly address the problem of uniqueness of the solution on $[0, T[\times M$ for a given initial condition on $\{0\} \times M$.
For this we will need to local approximate by Lipschitz functions.
Under a coercivity condition on H, we now show how to locally approximate viscosity subsolutions of the evolutionary Hamilton-Jacobi equation

$$\frac{\partial U}{\partial t}(t,x) + H(x, \frac{\partial U}{\partial x}(t,x)) = 0,$$

with U defined on an open subset of $\mathbb{R} \times M$, by Lipschitz viscosity subsolutions.
These results are well-known when M is the Euclidean space, see for example:
Hitoshi Ishii, *A Short Introduction to Viscosity Solutions and the Large Time Behavior of Solutions of Hamilton-Jacobi Equations* in Y. Achdou et al., Hamilton-Jacobi Equations: Approximations, Numerical Analysis and Applications, Springer LNM **2074**, **(2013)**.

Sup-convolution in one variable

The usefulness of sup-convolution to improve regularity of viscosity subsolutions is already well established. Our treatment follows closely the work of Hitoshi Ishii cited above, which deals with the case of the Euclidean space.

Assume $u : V \to \mathbb{R}$, is a continuous function, where V is an open subset of $\mathbb{R} \times M$.

Assume $K \subset V$ is compact subset. By continuity of u and compactness of K, we can find an open subset $O_1 \supset K$ whose closure $\bar{O}_1$is compact and contained in V and set

$$m = \sup_{\bar{O}_1} |u| < +\infty.$$

Again by compactness of K, we can find $\delta > 0$ and an open neighborhood $O_2 \subset O_1$ of K, with compact closure $\bar{O}_2 \subset O_1$ such that

$$[t-\delta, t+\delta] \times \{x\} \subset O_1, \text{ for every } (t,x) \in \bar{O}_2.$$

Since

$$[t-\delta, t+\delta] \times \{x\} \subset O_1, \text{ for every } (t,x) \in \bar{O}_2,$$

and u is defined on O_1, for $\epsilon > 0$, we can define $u_\epsilon : O_2 \to \mathbb{R}$ by

$$u_\epsilon(t,x) = \max_{s \in [-\delta, +\delta]} u(t+s, x) - \frac{s^2}{\epsilon} \tag{3}$$

Note that u_ϵ is continuous by continuity of u and compactness of $[-\delta, +\delta]$. We summarize the properties of u_ϵ in the following proposition.

Proposition 5

(1) *For every* $\epsilon > 0$*, we have* $u_\epsilon \geq u$*.*

(2) *For every* $0 < \epsilon < \epsilon'$*, we have* $u_\epsilon < u_{\epsilon'}$*.*

(3) *If* $(t,x) \in O_2$*, and* $s_\epsilon \in [-\delta, +\delta]$ *is such that* $u_\epsilon(t,x) = u(t+s_\epsilon, x) - (s_\epsilon)^2/\epsilon$*, then* $|s_\epsilon| \leq \sqrt{2\epsilon m}$*.*

(4) $u_\epsilon \to u$ *uniformly on* O_2*, when* $\epsilon \to 0$*.*

(5) *If* $(t,x), (t',x) \in O_2$*, with* $|t-t'| < \delta - \sqrt{2\epsilon m}$*, then*

$$|u_\epsilon(t',x) - u_\epsilon(t,x)| \leq \frac{2\sqrt{2\epsilon m} + |t-t'|}{\epsilon}|t-t'|.$$

In particular, we have

$$|u_\epsilon(t',x) - u_\epsilon(t,x)| \leq \frac{\sqrt{2\epsilon m} + \delta}{\epsilon}|t-t'|.$$

Moreover, for every $x \in X$*, the map* $t \mapsto u_\epsilon(t,x)$ *is Lipschitz on every connected component of* $O_2 \cap \{x\} \times \mathbb{R}$ *with Lipschitz constant* $\leq 2\sqrt{2m/\epsilon}$*.*

Proof

Part (1) $u_\epsilon \geq u$ and (2) $u_\epsilon < u_{\epsilon'}$, for $0 < \epsilon < \epsilon'$, are obvious from the definition of u_ϵ.

For part (3) $|s_\epsilon| \leq \sqrt{2\epsilon m}$, if $u_\epsilon(t,x) = u(t+s_\epsilon, x) - (s_\epsilon)^2/\epsilon$, we notice that

$$u_\epsilon(t,x) = u(t+s_\epsilon, x) - \frac{(s_\epsilon)^2}{\epsilon} \geq u(t,x).$$

Therefore

$$\frac{(s_\epsilon)^2}{\epsilon} \leq u(t+s_\epsilon, x) - u(t,x) \leq 2 \sup_{O_1} |u| = 2m.$$

For part (4) $u_\epsilon \to u$ uniformly on O_2, when $\epsilon \to 0$, note that by part (3), we have

$$\sup_{(x,t)\in O_2} |u_\epsilon(t,x) - u(t,x)| \leq \sup_{(t,x)\in \bar{O}_2, |s| \leq \sqrt{2\epsilon m}} |u(t+s,x) - u(t,x)|.$$

By compactness of $\bar{O}_2$ and continuity of u, the right hand side of the inequality above tends to 0 as $\epsilon \to 0$.

To prove part (5) namely:
If $(t,x),(t',x)\in O_2$, with $|t-t'|<\delta-\sqrt{2\epsilon m}$, then

$$|u_\epsilon(t',x)-u_\epsilon(t,x)|\leq\frac{2\sqrt{2\epsilon m}+|t-t'|}{\epsilon}|t-t'|,$$

we choose s_ϵ such that $u_\epsilon(t,x)=u(t+s_\epsilon,x)-(s_\epsilon)^2/\epsilon$. v By part (3), we have $|s_\epsilon|\leq\sqrt{2\epsilon m}$. Therefore, we get

$$|s_\epsilon+t-t'|\leq|s_\epsilon|+|t-t'|\leq\sqrt{2\epsilon m}+\delta-\sqrt{2\epsilon m}=\delta.$$

Hence, by the definition of u_ϵ, we obtain

$$\begin{aligned}u_\epsilon(t',x)&\geq u(t'+(s_\epsilon+t-t'),x)-\frac{(s_\epsilon+t-t')^2}{\epsilon}\\&=u(t+s_\epsilon,x)-\frac{(s_\epsilon+t-t')^2}{\epsilon}.\end{aligned}$$

Subtracting this last inequality from the equality $u_\epsilon(t,x) = u(t+s_\epsilon,x) - (s_\epsilon)^2/\epsilon$ yields

$$\begin{aligned} u_\epsilon(t,x) - u_\epsilon(t',x) &\leq \frac{(s_\epsilon + t - t')^2}{\epsilon} - \frac{(s_\epsilon)^2}{\epsilon} \\ &= \frac{(2s_\epsilon + t - t')(t-t')}{\epsilon} \\ &\leq \frac{2|s_\epsilon| + |t-t'|}{\epsilon}|t-t'| \\ &\leq \frac{2\sqrt{2\epsilon m} + |t-t'|}{\epsilon}|t-t'|, \end{aligned}$$

where we used $|s_\epsilon| \leq \sqrt{2\epsilon m}$, for the last inequality. By symmetry, we obtain

$$|u_\epsilon(t',x) - u_\epsilon(t,x)| \leq \frac{\sqrt{2\epsilon m} + |t-t'|}{\epsilon}|t-t'|. \tag{4}$$

To finish the proof of part (5), we must show that,for t, t', x with $[t, t'] \times \{x\} \subset O_2$, we have

$$|u_\epsilon(t', x) - u_\epsilon(t, x)| \leq \frac{2\sqrt{2\epsilon m}}{\epsilon}|t - t'|.$$

For $\eta < \delta - \sqrt{2\epsilon m}$, pick $t = t_0 < t_1 < \cdots < t_n = t'$, with $|t_{i+1} - t_i| \leq \eta$. By applying (4) for t_i, t_{i+1} instead of t, t', we get

$$\begin{aligned}|u_\epsilon(t_{i+1}, x) - u_\epsilon(t_i, x)| &\leq \frac{2\sqrt{2\epsilon m} + |t_{i+1} - tt_i|}{\epsilon}|t_{i+1} - t_i,| \\ &\leq \frac{2\sqrt{2\epsilon m} + \eta}{\epsilon}|t_{i+1} - t_i,|.\end{aligned}$$

Adding the inequalities, we obtain

$$|u_\epsilon(t', x) - u_\epsilon(t, x)| \leq \frac{2\sqrt{2\epsilon m} + \eta}{\epsilon}|t - t'|.$$

We can then let $\eta \to 0$, to conclude that

$$|u_\epsilon(t', x) - u_\epsilon(t, x)| \leq \frac{2\sqrt{2\epsilon m}}{\epsilon}|t - t'| = 2\sqrt{\frac{2m}{\epsilon}}|t - t'|.$$

This finishes the proof of (5), and also of the Proposition. □

Proposition 6

Let $H : T^*M \to \mathbb{R}$ be a continuous Hamiltonian. If $u : V \to \mathbb{R}$ is a continuous function, defined on the open subset $V \subset \mathbb{R} \times M$, which is a viscosity subsolution on V of

$$\frac{\partial u}{\partial t}(t,x) + H\big(x, \frac{\partial u}{\partial x}(t,x)\big) = 0. \tag{5}$$

Then for every compact subset $K \subset V$, we can find a sequence of continuous functions $u_n : K \to \mathbb{R}$, such that $u_n \to u$ uniformly on K, and for every n, the function u_n is a viscosity subsolution on the interior $\mathring{K}$ of K, not only of the same evolutionary Hamilton-Jacobi equation (5), but also of

$$|\partial_t u_n(t,x)| + H(x, \partial_x u_n(t,x)) = C\sqrt{n},$$

for some $C < +\infty$.
In particular if H is coercive above each compact subset of M, then the u_n's are locally Lipschitz on $\mathring{K}$.

Proof

As was done above, we choose an open subset $O_1 \supset K$ whose closure $\bar{O}_1$ is compact and contained in V and we set

$$m = \sup_{\bar{O}_1} |u| < +\infty,$$

then we find $\delta > 0$ and an open neighborhood $O_2 \subset O_1$ of K, with compact closure $\bar{O}_2 \subset O_1$ such that

$$[t - \delta, t + \delta] \times \{x\} \subset O_1, \text{ for every } (t, x) \in \bar{O}_2.$$

We then set $\hat{u}_n = u_{1/n} : O_2 \to \mathbb{R}$, where $u_{1/n}$ is defined, for by (3) with $\epsilon = 1/n$, for $1/n < \delta$, namely

$$\hat{u}_n(t, x) = \max_{s \in [-\delta, +\delta]} u(x, +s) - ns^2.$$

By part (4) of Proposition 5, we indeed get the uniform convergence of $\hat{u}_n$ to u.
We will now check the fact that $\hat{u}_n$ is a viscosity subsolution of both Hamilton-Jacobi equations on O_2.

Assume $(t_0, x_0) \in O_2$, and that $\varphi : V \to \mathbb{R}$ is C^1 is such that $\hat{u}_n \leq \varphi$ with equality at (t_0, x_0). By (5) Proposition 5, we know that $t \mapsto \hat{u}_n(x, t)$ is locally Lipschitz with local Lipschitz constant $\leq 2\sqrt{2mn}$. This implies

$$|\partial_t \varphi(t_0, x_0)| \leq 2\sqrt{2mn}. \tag{6}$$

We now choose $s_n \in [-\delta, +\delta]$ such that

$$u(t_0 + s_n, x_0) - ns_n^2 = \hat{u}_n(t_0, x_0) = \varphi(t_0, x_0).$$

Since $(t_0, x_0) \in O_2$, we can find $\eta > 0$, such that $(t_0 + s, x_0) \in O_2$, for $|s| < \eta$. By the definition of $\hat{u}_n$, for $(t, x) \in O_2$, we have

$$\hat{u}_n(t, x) = \max_{s \in [-\delta, +\delta]} u(x, t + s) - ns^2,$$

Therefore, for $|s| < \eta$, we get

$$u(t_0 + s + s_n, x_0) - ns_n^2 \leq \hat{u}_n(t_0 + s, x_0) \leq \varphi(t_0 + s, x_0).$$

Subtracting from this inequality the equality $u(t_0 + s_n, x_0) - ns_n^2 = \varphi(t_0, x_0)$, we obtain

$$u(y, t_0 + s + s_n) - u(t_0 + s_n, x_0) \leq \varphi(t_0 + s, y) - \varphi(t_0, x_0), \text{ for } |s| < \eta.$$

The last inequality, for $|s| < \eta$, can be rewritten as

$$u(y, t_0 + s + s_n) \leq \varphi(t_0 + s, y) - \varphi(t_0, x_0) + u(t_0 + s_n, x_0).$$

Since this inequality is an equality at $s = 0$ and u is a viscosity subsolution of

$$\frac{\partial u}{\partial t}(t, x) + H\big(x, \frac{\partial u}{\partial x}(t, x)\big) = 0,$$

we must have

$$\partial_t \varphi(t_0, x_0) + H(x_0, \partial_x \varphi(t_0, x_0)) \leq 0. \tag{7}$$

Therefore $\hat{u}_n$ is a viscosity subsolution of

$$\frac{\partial \hat{u}_n}{\partial t}(t, x) + H\big(x, \frac{\partial \hat{u}_n}{\partial x}(t, x)\big) = 0.$$

Using using the already established inequalities (6)

$$|\partial_t \varphi(t_0, x_0)| \leq 2\sqrt{2mn}$$

and (7)

$$\partial_t \varphi(t_0, x_0) + H(x_0, \partial_x \varphi(t_0, x_0)) \leq 0,$$

we also obtain

$$|\partial_t \varphi(t_0, x_0)| + H(x_0, \partial_x \varphi(t_0, x_0)) \leq 4\sqrt{2mn}.$$

Therefore u_n is a viscosity solution of

$$\left|\frac{\partial \hat{u}_n}{\partial t}(t, x)\right| + H\big(x, \frac{\partial \hat{u}_n}{\partial x}(t, x)\big) = C\sqrt{n},$$

with $C = 4\sqrt{2m}$. $\square$

Corollary 7

Let $H : T^*M \to \mathbb{R}$ be a continuous Hamiltonian, which is coercive above each compact subset of M and convex in the momentum p. Let $u : V \to \mathbb{R}$ be a continuous functions defined on the open subset $V \subset \mathbb{R} \times M$ which is viscosity subsolution of the evolutionary Hamilton-Jacobi equation

$$\partial_t u(t,x) + H(x, \partial_x u(t,x)) = 0.$$

For every open set $V' \subset V$ whose closure $\bar{V}'$ is compact and contained in V, we can approximate uniformly u on V' by a C^∞ solution of the same evolutionary Hamilton-Jacobi equation.

Proof By Proposition 6 above, we can make a first approximation by a viscosity solution $u_1 : V' \to \mathbb{R}$ of

$$\partial_t u(t,x) + H(x, \partial_x u(t,x)) = 0,$$

which is locally Lipschitz on V'.

Therefore the function $u_2 : V' \to \mathbb{R}, (t,x) \to u_1(t,x) - \epsilon t$ is a locally Lipschitz viscosity subsolution of

$$\partial_t u + H(x, \partial_x u) = -\epsilon.$$

Note also that the variable t is bounded on the compact subset $\bar{V}'$ of $\mathbb{R} \times M$.
Hence by choosing appropriately ϵ, we can assume u_2 uniformly as close to u_1 as we wish. We can now consider the Hamiltonian $\hat{H} : T^*(\mathbb{R} \times M)$ defined by

$$\hat{H}(t,s,x,p) = s + H(x,p),$$

where we use the identification
$T^*(\mathbb{R} \times M) = T^*\mathbb{R} \times T^*M = \mathbb{R} \times \mathbb{R} \times T^*M.$

With this identification, we get that the function u_2 is a locally Lipschitz viscosity subsolution of

$$\hat{H}(t, x, Du(t, x)) = -\epsilon.$$

Since the Hamiltonian $\hat{H}(t, s, x, p)$ is convex in the momentum (s, p), we can now invoque one of the properties of viscosity subsolutions we mentioned, and approximate uniformly u_2 on V' by a C^∞ viscosity subsolution $u_3 : V' \to \mathbb{R}$ of

$$\hat{H}(t, x, Du(t, x)) = 0.$$

This means that u_3 is a uniform approximation of u, which is a viscosity subsolution of the evolutionary Hamilton-Jacobi equation

$$\partial_t u(t, x) + H(x, \partial_x u(t, x)) = 0. \ \square$$

Corollary 8

Let $H : T^*M \to \mathbb{R}$ be a continuous Hamiltonian, which is coercive above each compact subset of M and convex in the momentum p. If $u_1 : V \to \mathbb{R}$ and $u_2 : V \to \mathbb{R}$ are continuous functions defined on the open subset $V \subset \mathbb{R} \times M$, which are both viscosity subsolutions of

$$\partial_t u + H(x, \partial_x u) = 0, \tag{8}$$

then $\min(u_1, u_2)$ is also a viscosity subsolution on V of the same equation.

Proof Since H is convex, from of the general properties that we recalled, the corollary is well known when u_1 and u_2 are locally Lipschitz.The result follows from this case and the stability of viscosity solutions using the approximation result obtained in Proposition 6. □

Maximum principle

Theorem 9

Let $H : T^*M \to \mathbb{R}$ be a continuous Hamiltonian, which is coercive above each compact subset of M and convex in the momentum p. Suppose $a < b \in \mathbb{R}$. and $K \subset M$ is a compact subset.
If the continuous functions $u, v : [a, b] \times K \to \mathbb{R}$ are respectively a subsolution and a supersolution, on $]a, b[\times \mathring{K}$, of the evolutionary Hamilton-Jacobi equation

$$\partial_t u(t, x) + H(x, \partial_x u(t, x)) = 0,$$

then the maximum of $u - v$ on $[a, b] \times K$ is achieved on $[a, b] \times \partial K \cup \{a\} \times K$.

Proof

It is not difficult to realize that, by the approximation result of Proposition 6, we can assume that the viscosity subsolution u is locally Lipschitz in $\mathring{K}\times]a,b[$.
As usual in proofs of that form of the maximum principle, for $\epsilon, \delta > 0$, we introduce the function $u_{\epsilon,\delta} : [a,b[\times K \to \mathbb{R}$ defined by

$$u_{\epsilon,\delta}(t,x) = u(t,x) - \epsilon(t-a) - \frac{\delta}{b-t}.$$

Note that $u_{\epsilon,\delta}(t,x) \to -\infty$, as $t \to b$, and $u_{\epsilon,\delta} \leq u$.
Moreover, since $t \mapsto -\epsilon(t-a) - \delta/(b-t)$ is C^1, with derivative $t \mapsto -\epsilon - \delta/(b-t)^2 \leq -\epsilon$, the function $u_{\epsilon,\delta}$ is, on $]a,b[\times\mathring{K}$, a viscosity subsolution of

$$\partial_t u_{\epsilon,\delta} + H(x, \partial_x u_{\epsilon,\delta}) = -\epsilon.$$

.

We now recall Theorem (3) in the form that will apply here

Theorem 10

Suppose that $u :]a, b[\times \mathring{K} \to \mathbb{R}$ *is a viscosity subsolution of*

$$\partial_t u + H(x, \partial_x u) = c_1,$$

and $v :]a, b[\times \mathring{K} \to \mathbb{R}$ *is a viscosity supersolution of*

$$\partial_t v + H(x, \partial_x v) = c_2.$$

Assume further that either u *or* v *is locally Lipschitz on* $]a, b[\times \mathring{K}$*. If* $u - v$ *has a local maximum, then necessarily* $c_2 \leq c_1$*.*

The function $u_{\epsilon,\delta}$ is locally Lipschitz on $]a,b[\times\mathring{K}$ and a viscosity **subsolution** of

$$\partial_t u_{\epsilon,\delta} + H(x, \partial_x u_{\epsilon,\delta}) = -\epsilon,$$

The function v is a viscosity **supersolution** on $]a,b[\times\mathring{K}$ of

$$\partial_t u(t,x) + H(x, \partial_x u(t,x)) = 0,$$

Since $-\epsilon < 0$, by the Theorem above, we conclude that $u_{\epsilon,\delta} - v$ cannot have a local maximum in $]a,b[\times\mathring{K}$.
Since $u_{\epsilon,\delta}(t,x) \to -\infty$, as $t \to b$, the function $u_{\epsilon,\delta} - v$ attains its maximum at a point in $[a,b[\times\partial K \cup \{a\} \times K$. Using that $u_{\epsilon,\delta} \leq u$, we obtain

$$u_{\epsilon,\delta} - v \leq \max_{[a,b[\times\partial K\cup\{a\}\times K} u_{\epsilon,\delta} - v \leq \max_{[a,b]\times\partial K\cup\{a\}\times K} u - v,$$

everywhere on $K \times [a,b[$. Letting $\delta, \epsilon \to 0$, we obtain

$$u - v \leq \max_{[a,b]\times\partial K\cup\{a\}\times K} u - v,$$

on $K \times [a,b[$. Continuity of both u and v yields

$$\max_{K\times[a,b]} u - v \leq \max_{[a,b]\times\partial K\cup\{a\}\times K} u - v. \ \square$$

Recall that our goal is to give the conditions under which we have uniqueness of viscosity solution with the same initial condition. Therefore, we have to remove in the previous theorem the possibility that $u - v$ achieves on $[a, b] \times \partial K$ its maximum on $[a, b] \times K$

This will be done in the next theorem, which is a generalization of Proposition A.2, page 80 in

NAOYUKI ICHIHARA & HITOSHI ISHII, *Asymptotic Solutions of Hamilton-Jacobi Equations with Semi-Periodic Hamiltonians*, Communications in Partial Differential Equations, 33 (**2008**) 784–807.

Note, however, that Proposition A.2, page 80 is established, in the paper above, under a uniform continuity of the Hamiltonian (condition (A1)). This is not really necessary as can be seen from their proof.

Theorem 11

Let $H : T^*M \to \mathbb{R}$ be a continuous Hamiltonian, which is coercive above each compact subset of M and convex in the momentum p. For $a < b \in \mathbb{R}$, assume the continuous functions $u, v : [a, b] \times M \to \mathbb{R}$ are respectively a viscosity subsolution and a viscosity supersolution of

$$\frac{\partial U}{\partial t}(t,x) + H(x, \frac{\partial U}{\partial x}(t,x)) = 0,$$

on $]a, b[\times M$. If there exists a continuous function $\varphi : [a, b] \times M \to \mathbb{R}$ which is a viscosity subsolution of the same equation such that

$$v(t,x) - \varphi(t,x) \to +\infty,$$

as (t,x) leaves every compact subset of $M \times [a, b]$, then

$$\sup_{[a,b]\times M} u - v \leq \sup_{\{a\}\times M} u - v.$$

Proof

We adapt to our context a argument that can be found in the proof of Proposition A.2 of the work Naoyuki Ichihara & Hitoshi Ishii cited above.
If $\sup_{M\times\{a\}} u - v = +\infty$, there is nothing to prove. If not subtracting $\sup_{M\times\{a\}} u - v$ from u, we see that we can assume $u \le v$ on $M \times \{a\}$. We will show that

$$\min(u, A+\varphi) \le v, \text{ or every } A < +\infty. \tag{9}$$

If we let $A \to +\infty$, then $\min(u, A+\varphi) \to u$ pointwise. Therefore, it will follow from the inequality (9) above that we indeed have $u \le v$ everywhere.

It remains to prove inequality (9)

$$\min(u, A+\varphi) \leq v, \text{ or every } A < +\infty.$$

Fix $(t_0, x_0) \in M \times [a, b]$ and $A \in \mathbb{R}$.

Since $v(t,x) - \varphi(t,x) \to +\infty$ as (t,x) leaves every compact subset of $M \times [a, b]$, we can find a compact subset $K \in M$ such that $x_0 \in K$ and $v - \varphi \geq A$ outside $\mathring{K} \times [a, b]$.

In particular, we get $\min(u, A+\varphi) - v \leq A + \varphi - v \leq 0$ on $\partial K \times [a, b]$.

Since, we also have $\min(u, A+\varphi) - v \leq u - v \leq 0$ on $K \times \{a\}$, we conclude from the last Theorem 9 that

$$\begin{aligned} \min(u, A+\varphi)(t_0, x_0) - v(t_0, x_0) &\leq \max_{K \times [a,b]} [\min(u, A+\varphi) - v] \\ &\leq \max_{[a,b] \times \partial K \cup \{a\} \times K} [\min(u, A+\varphi) - v] \\ &\leq 0. \ \square \end{aligned}$$

Case of the Hamiltonian associated to a Tonelli Lagrangian

In fact, the Hamiltonian H associated to a Lagrangian on the Riemannian manifold M satisfies the following (Tonelli) conditions:

(1*) The function H is C^2.

(2*) (C^2 strict convexity in the fibers) For every $(x, p) \in T^*M$, the second derivative along the fibers $\partial^2 H/\partial p^2(x, p)$ is positive strictly definite.

(3*) (Uniform superlinearity) For every $K \geq 0$, we have

$$C^*(K) = \sup_{(x,p)\in T^*M} H(x, p) - K\|p\|_x < \infty.$$

(4*) (Uniform boundedness in the fibers) For every $R \geq 0$, we have

$$A^*(R) = \sup\{H(x, p) \mid \|p\| \leq R\} < +\infty.$$

We note that the uniform superlinearity implies that such a Hamiltonian is coercive.

Lemma 12

Let H be the Hamiltonian associated to a Tonelli Lagrangian L on M. Suppose $\varphi : [0, t] \times M \to \mathbb{R}$ is a (globally) Lipschitz function whose Lipschitz constant is $\leq \lambda$.
Fix $x_0 \in M$, and $\epsilon > 0$, the function $\Phi : [0, t] \times M \to \mathbb{R}$ defined by

$$\Phi(x, s) = \varphi(x, s) - (A^*(\lambda + \epsilon) + \lambda)s - \epsilon d(x_0, x)$$

is a viscosity subsolution of

$$\partial_t \Phi + H(x, \partial_x \Phi) = 0, \tag{10}$$

on $]0, t[\times M$.
Moreover, we have $\varphi(x, s) - \Phi(x, s) \to +\infty$ as (x, s) leaves every compact subset of $[0, t] \times M$.

Proof

The map Φ is Lipschitz, with Lipschitz constant $\leq \lambda + \epsilon$ in x. Hence, almost everywhere, we have $\|\partial_x \Phi(s, x)\|_x \leq \lambda + \epsilon$. By the definition of A^*, we get

$$H(x, \partial_x \Phi) \leq A^*(\lambda + \epsilon) \text{ a.e.}$$

We also have almost everywhere $\partial_t \Phi = \partial_t \varphi - A^*(\lambda + \epsilon) - \lambda$. Since φ has Lipschitz constant $\leq \lambda$, we obtain

$$\partial_t \Phi \leq -A^*(\lambda + \epsilon) \text{ a.e.}$$

Therefore

$$\partial_t \Phi + H(x, \partial_x \Phi) \leq 0 \text{ a.e.}$$

Since H is convex in p, this implies that Φ is a viscosity subsolution.
The last part follows from the inequality

$$\Phi(s, x) - \varphi(s, x) \leq \|A^*(\lambda + \epsilon)\| t - \epsilon d(x_0, x). \square$$

Last Lemma 12 together with Theorem 11 clearly imply the following corollary

Corollary 13 (Uniqueness)

Assume that H is the Hamiltonian associated to a Tonelli Lagrangian L on M .
If $u, v : [0, t] \times M \to \mathbb{R}$ are both continuous functions which are bounded below by Lipschitz functions and are both viscosity solutions of

$$\partial_t U + H(x, \partial_x U) = 0$$

on $]0, t[\times M$, then $\sup_{[0,t]\times M} u - v = \sup_{M\times\{0\}} u - v$.
Moreover, if $u = v$ on $M \times \{0\}$, then $u = v$ everywhere.

Theorem 14

Assume that H is the Hamiltonian associated to a Tonelli Lagrangian L on M.
Suppose $u : M \to \mathbb{R}$ is continuous function which is bounded below by a Lipschitz function. Define $U : [0, +\infty[\times M \to \mathbb{R}$ by

$$U(t, x) = T_t^- u(x).$$

For every $t > 0$, this Lax-Oleinik evolution is the only continuous function $V : [0, t] \times M \to \mathbb{R}$, bounded below by a Lipschitz function, which is a viscosity solution of

$$\partial_t U + H(x, \partial_x U) = 0$$

on $]0, t[\times M$, with $V(x, 0) = u(x)$, for every $x \in M$.

Proof In the last lecture we showed that U is continuous and bounded below by a Lipschitz function.
We also showed at the beginning of this lecture that it was a viscosity solution.
Therefore this is a consequence of the Uniqueness Corollary 13. $\square$