Lecture 3. Topology of the set of singularities of a solution of the Hamilton-Jacobi Equation

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We will present results which are a joint work with Piermarco Cannarsa & Wei Cheng.

A version appeared in:

Cannarsa, Piermarco; Cheng, Wei; Fathi, Albert, *On the topology of the set of singularities of a solution to the Hamilton-Jacobi equation. C. R. Math. Acad. Sci. Paris 355 (2017), no. 2, 176–180.*

This article is in Open Access.

A major building block of the proof is

 P. Cannarsa, & W. Cheng, *Generalized Characteristics and Lax-Oleinik Operators: Global Theory, Calc. Var. Partial Differential Equations 56 (2017)*

where they use in a very clever way the POSITIVE Lax-Oleinik operator.

In this lecture, a singularity for a real valued function u is a point where u is not differentiable.

We will denote by $\text{Sing}(u)$ the set of singularities of u.

We address the problem of propagation of singularities (to be explained below) and the topology of the set of singularities $\text{Sing}(u)$, when u is a viscosity solution of the Hamilton-Jacobi equation, under the "usual" (i.e. Tonelli) regularity of the Hamiltonian.

We will state and prove our results for the stationary case on a compact manifold M, but they are also valid in the evolution case even on non-compact manifolds provided we assume the (appropriate) Tonelli regularity in that framework.

We can choose on M any Riemannian metric. it is automatically complete since M is compact.

To simplify notations, we will say that a Hamiltonian $H : T^*M \to \mathbb{R}$ is a Tonelli Hamiltonian, if it is the Hamiltonian associated to a Tonelli Lagrangian $L : TM \to \mathbb{R}$.

In fact, a function $H : T^*M, (x, p) \mapsto H(x, p)$, where M is a compact manifold, is a **Tonelli Hamiltonian** if and olnly it satisfies the following conditions:

1) H is C^2.
2) **(C^2 Strict Convexity)** At every $(x, p) \in T^*M$, the second partial derivative $\partial^2_{pp}H(x, p)$ is definite > 0. In particular $H(x, p)$ is strictly convex in p.
3) **(Superlinearity)** $H(x, p)/\|p\|_x \to +\infty$, as $\|p\|_x \to +\infty$.

Note here that uniform boundedness is automatic when M is compact, since in that case $\{(x, p) \in T^*M \mid \|p\|_x \leq R\}$ is also compact, for every $R \geq 0$.

The Lagrangian $L : TM \to \mathbb{R}$, from whom to get H, is given by the same type of formula

$$L(x, v) = \sup_{p \in T^*_x M} p(v) - H(x, p).$$

Main result

We consider $u : M \to \mathbb{R}$ a viscosity solution of the (stationary) Hamilton-Jacobi equation

$$H(x, d_x u) = c[0]. \tag{1}$$

Although there may be several solutions u for (1), the constant $c[0]$ is unique. Dynamicists call c[0] the Mañé critical value. In PDE, it is denoted by $\bar{H}(0)$, where $\bar{H}$ is the homogenized Hamiltonian. One of our main results is the following:

Theorem 1

The space $\text{Sing}(u)$ *is locally path connected. In fact it is even locally contractible, i.e. for every* $x \in \text{Sing}(u)$ *and every neighborhood V of x in* $\text{Sing}(u)$, *we can find a neighborhood W of x in* $\text{Sing}(u)$ *such that* $W \subset V$, *and W is null-homotopic in V.*

Local path connectedness

We now comment on the local path connectedness. This is a very strong result that was not expected. It is much stronger than propagation of singularities, which it of course implies. However, as we will see this result will be obtained from a strong form of propagation of singularities.

Since H is Tonelli, the viscosity solution u is a *semi-concave* function (i.e. locally the sum of a concave and a smooth function) on M, one should expect the set $\mathrm{Sing}(u)$ to look locally as the set of singularities of a concave function.

To fix the ideas, let us consider the singularities of a "generic" concave function $f : \mathbb{R} \to \mathbb{R}$. Hence the derivative f' should be a "generic" non-increasing function. The set of singularities for f is the set of jumps of f'. This set of jumps is of course countable but in the "generic" case it should have non-isolated points. But a countable locally connected set in a metric space has only isolated points.

A consequence of our theorem is therefore:

Viscosity solutions of Hamilton-Jacobi Equations for Tonelli Hamiltonians form a very small subset of the set of semi-concave functions

Propagation of singularities

Our results will be obtained from a strong form of propagation of singularities. In fact, people have been studying propagation of singularities for a long time.

If $u : M \to \mathbb{R}$ is a Lipschitz function and $x \in \text{Sing}(u)$, we say that the singularity $x \in \text{Sing}(u)$ propagates, if there exists a **non-constant** path $\gamma : [0, 1] \to \text{Sing}(u)$ with $\gamma(0) = x$.

Previous work in this area was done by several people for semi-concave functions, and then specialized to solutions of Hamilton-Jacobi equations.

Among the people who work in this area of propagation of singularities, we would like to mention Paolo Albano, Piermarco Cannarsa, Wei Cheng, Marco Mazzola, Carlo Sinestrari, and Yifeng Yu, since it is their contributions that motivated our work.

I will not survey these previous works, since you can find a good survey in the paper

P. Cannarsa, M. Mazzola, & C. Sinestrari, *Global Propagation of singularities for time dependent Hamilton-Jacobi equations* DCDS, **35** (2015), 4225–4239.

In this work, the authors give a description of the state of the art in 2015. They also note that, in general, the problem of propagation of singularities (at least for the evolution case) has a negative answer if H is allowed only Lipschitz dependence in (t, x) (Example 5.6.7, in the book by Cannarsa and Sinestrari). They conclude this survey stating that (in 2015):

> Nevertheless, establishing whether genuine singularities propagate indefinitely or not remains a largely open problem.

Background

We now recall some of the characterization and features of viscosity solutions of the Hamilton-Jacobi equation for Tonelli Hamiltonians. Since we mostly spoke of evolutionary Hamilton-Jacobi equation, let us state some facts.

- (Lipschitzianity) Since a Tonelli Hamiltonian is coercive, a viscosity solution $u : M \to \mathbb{R}$ of $H(x, d_x u) = c$ is necessarily Lipschitz.

- (Uniqueness) Since any continuous function on a compact space is bounded below, by what we did in Lecture 2, if $v : M \to \mathbb{R}$ is continuous, then $V(t, x) = T_t^- v(x)$ is the only continuous function $V : [0, +\infty[\times M \to \mathbb{R}$, with $V(0, x) = v(x)$, which is a viscosity solution, on $]0, +\infty[\times M$, of

$$\partial_t V + H(x, \partial_x V) = 0.$$

- The continuous function $u : M \to \mathbb{R}$ is a (stationary) viscosity solution (resp. subsolution, supersolution) of $H(x, d_x u) = c$ if and only if $U(t, x) = u(x) - ct$ is a viscosity solution of $\partial_t U + H(x, \partial_x U) = 0$, on $]0, +\infty[\times M$.
  This is a simple exercise from the definitions.

- Using the property above and the uniqueness result for the evolutionary Hamilton-Jacobi equation, it follows that $u : M \to \mathbb{R}$ is a (stationary) viscosity solution of $H(x, d_x u) = c$ if and only if $T_t^- u = u - ct$, for $t \geq 0$ (or equivalently $u = T_t^- u + ct$ for $t \geq 0$).

This last property can be restated as "stationary solutions of the Hamilton-Jacobi equation are fixed points, modulo constants, of the Lax-Oleinik semi-group".

Characterization of viscosity solutions

Since u is a viscosity solution of $H(x, d_x u) = c$ if and only if $T_t^- u = u - ct$, for $t \geq 0$, we obtain the following characterization of viscosity solutions

Theorem 2

A function $u : M \to \mathbb{R}$ is a viscosity solution of the Hamilton-Jacobi equation $H(x, d_x u) = c$, where H is a Tonelli Hamiltonian, if and only if it satisfies the following two conditions:

(1) *for every curve $\gamma : [a, b] \to M$, we have*

$$u(\gamma(b)) - u(\gamma(a)) \leq \int_a^b L(\gamma(s), \dot{\gamma}(s)) + c \, ds = \mathbb{L}(\gamma) + c(b-a);$$

(2) *For every $x \in M$, there exists a curve $\gamma_x :\,] -\infty, 0] \to M$, with $\gamma_x(0) = x$, and*
$$u(\gamma_x(0)) - u(\gamma_x(-t)) = \int_{-t}^0 L(\gamma_x(s), \dot{\gamma}(s)) + c \, ds, \text{ for all } t > 0.$$

Assume that $u : M \to \mathbb{R}$ is a viscosity solution of $H(x, d_x u) = c$. It satisfies:

(1) for all curves $\gamma : [a, b] \to M$, we have

$$u(\gamma(b)) - u(\gamma(a)) \leq \mathbb{L}(\gamma) + c(b - a).$$

Therefore any curve $\gamma : [a, b] \to M$ that satisfies the equality in inequality (1) above is a minimizer.

A curve satisfying the **equality** in (1) is called calibrating (in fact we should say (u, L, c)-calibrating).

This implies that any curve $\gamma_x :\,] - \infty, 0] \to M$, with $\gamma_x(0) = x$, which satisfies

(2) $u(\gamma_x(0)) - u(\gamma_x(-t)) = \int_{-t}^{0} L(\gamma_x(s), \dot{\gamma}(s)) + c \, ds$, for all $t > 0$, is a minimizer.

A curve γ_x satisfying (2) is called a backward characteristic ending at x.
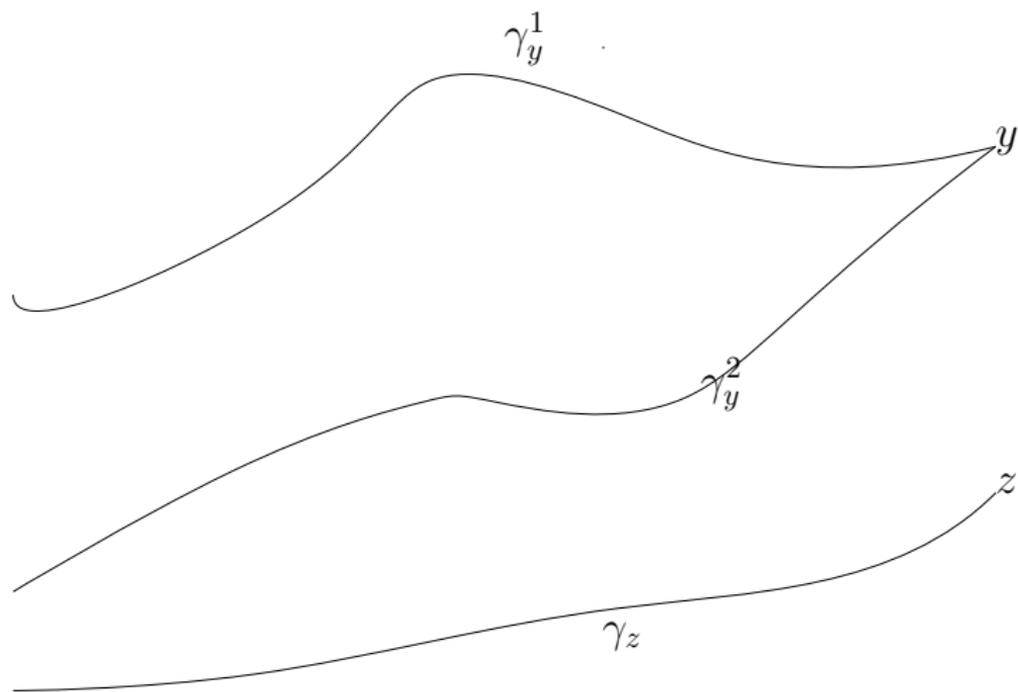
You may have one or several characteristics ending at a given point.
In the figure two characteristics end at y and only one ends at z.

Calibration, characteristics and differentiabilty

The relation between differentiability and calibrated curves is given by:

Theorem 3 (Differentiability)

Assume $u : M \to \mathbb{R}$ is a viscosity solution of $H(x, d_x u) = c$.

(i) *If $\gamma : [a, b] \to M$ is calibrated, and u is differentiable at $\gamma(t)$, with $t \in [a, b]$, then*
$$d_{\gamma(t)} u = \frac{\partial L}{\partial v}(\gamma(t), \dot{\gamma}(t)).$$

(ii) *Moreover, for every $t \in]a, b[$, the function u is indeed differentiable at $\gamma(t)$.*

Note that in the case $L(x, v) = \frac{1}{2}\|v\|^2 - V(x)$, we have $\partial L / \partial v = v$, or more accurately $\partial L / \partial v(x, v) = \langle v, \cdot \rangle$, since differentials (or derivative) are linear maps.
In this case, the relation above becomes $\dot{\gamma}(t) = \operatorname{grad} u(\gamma(t))$.

In the general case, since L is strictly convex in v and superlinear, for a given $x \in M$, the map $v \mapsto \partial L / \partial v(x, v)$ is bijective. In fact, its inverse is given by $p \mapsto \partial H / \partial p(x, p)$.

Therefore we can define the Lagrangian gradient $\text{grad}_L u$ of u by

$$d_x u = \frac{\partial L}{\partial v}(x, \text{grad}_L u(x)), \text{ or } \text{grad}_L u(x) = \frac{\partial H}{\partial p}(x, d_x u),$$
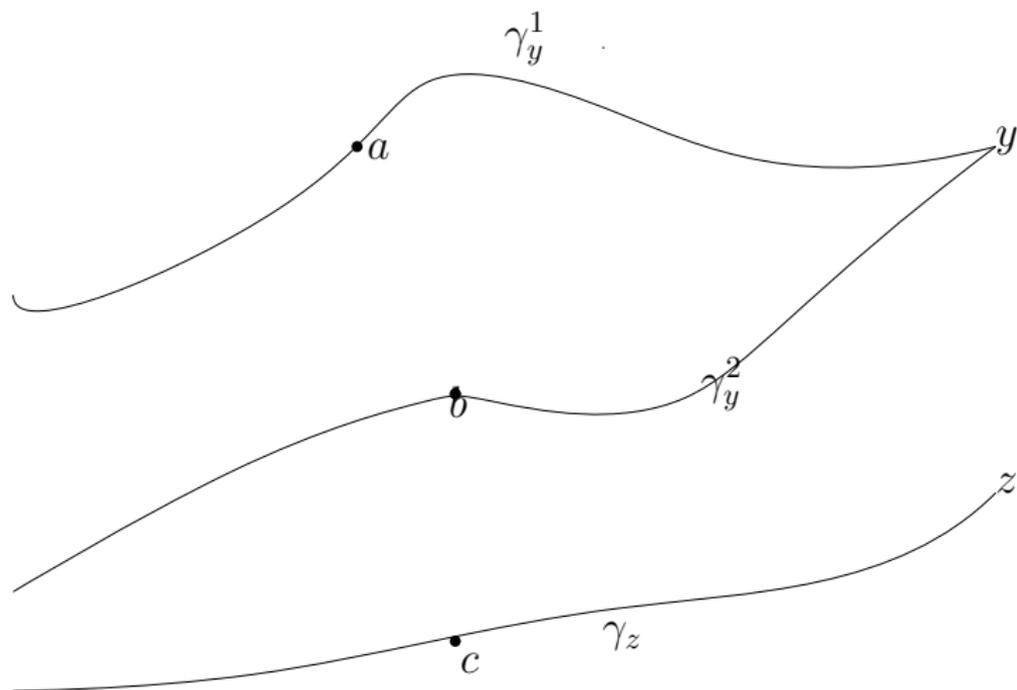
at every x where the derivative $d_x u$ exists.

For a calibrated curve $\gamma : [a, b] \to M$, we can illustrate the relation between differentiability and calibration by:
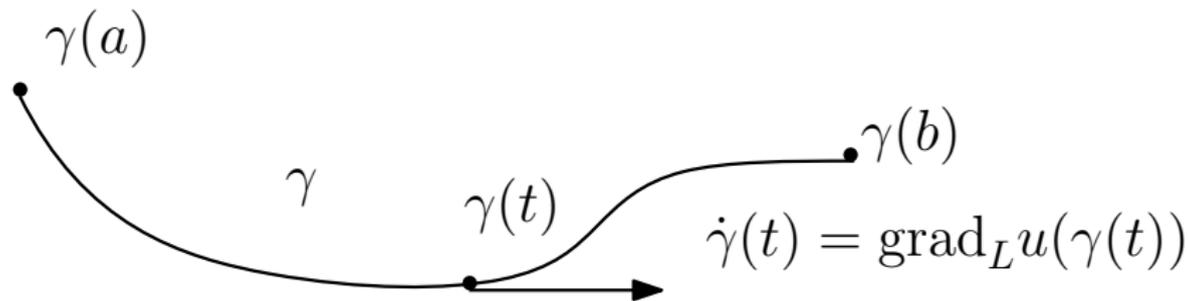
The relation between differentiability and characteristics is given by:

▶ The function u is differentiable at x if and only if there is a *unique* backward characteristic ending at x

In the figure, the function u is differentiable at a, b, c or z, but it is not differentiable at y.

Assuming $c[0] = 0$

As we already said, there is a unique constant c such that $H(x, d_x u) = c$ has a global viscosity solution $u : M \to \mathbb{R}$.

In fact, if $u_1, u_2 : M \to \mathbb{R}$ are respectively solutions of $H(x, d_x u_1) = c_1$ and $H(x, d_x u) = c_2$, using $T_t^- u_i = u_i - c_i t$, for $t \geq 0, i = 1, 2$, and $\| T_t^- u_2 - T_t^- u_1 \|_\infty \leq \| u_2 - u_1 \|_\infty$, we get

$$|c_2 - c_1| t \leq 2 \| u_2 - u_1 \|_\infty, \text{ for all } t \geq 0.$$

Hence $c_2 = c_1$.

This unique constant is what we denoted by $c[0]$.

Replacing H by $H - c[0]$, we can assume that $c[0] = 0$ to simplify notations.

Therefore in the sequel we assume:

Standing assumption

There exits a global viscosity solution $u : M \to \mathbb{R}$ of the Hamilton-Jacobi equation

$$H(x, d_x u) = 0.$$

The Aubry set $\mathcal{I}(u)$ and the path components of $\text{Sing}(u)$

Definition 4 (Aubry Set)

The Aubry set $\mathcal{I}(u)$ is defined as the set of points $x \in M$ for which there exists a calibrating curve $\gamma :] - \infty, +\infty[\to M$ with $\gamma(0) = x$.

The set $\mathcal{I}(u)$ is a closed subset of M.
Moreover, by the relation between calibrating curve and differentiability, the viscosity solution is differentiable at every point of $\mathcal{I}(u)$.
Therefore the set of singularities $\text{Sing}(u)$ of u is disjoint from the Aubry set $\mathcal{I}(u)$.

Theorem 5 (Homotopy Theorem)

For every connected component C of the open set $M \setminus \mathcal{I}(u)$, the intersection $C \cap \text{Sing}(u)$ is path connected.
In fact, the inclusion $\text{Sing}(u) \subset M \setminus \mathcal{I}(u)$ is a homotopy equivalence

Global propagation of singularities

As we already said our results rely on a strong global version of propagation of singularities which we now state as the main lemma.

Lemma 1 (Main Lemma)

Assume that $u : M \to \mathbb{R}$ is a a viscosity solution of $H(x, d_x u) = 0$, where $H : T^*M \to \mathbb{R}$ is a Tonelli Hamiltonian.
We can find a continuous function $F : M \times [0, +\infty[\to M$ such that:

(a) *for all $x \in M$, we have $F(x, 0) = x$;*

(b) *if $F(x, s) \notin \text{Sing}(u)$, for some $s > 0$, and $x \in M$, then the curve $\sigma \mapsto F(x, \sigma)$ is calibrated (for u) on $[0, s]$;*

(c) *if there exists a calibrated curve $\gamma : [0, s] \to M$, with $\gamma(0) = x$, then $\sigma \mapsto F(x, \sigma) = \gamma(\sigma)$, for every $\sigma \in [0, s]$.*

The properties (a), (b), and (c) stated above imply easily:

(1) $F(\mathrm{Sing}(u) \times]0, +\infty[) \subset \mathrm{Sing}(u)$.

(2) If $F(x, s)$ never enters $\mathrm{Sing}(u)$, then $x \in \mathcal{I}(u)$, and $s \mapsto F(x, s), s \in [0, \infty[$ is the forward calibrating curve through x.

(3) If $x \notin \mathcal{I}(u)$, then $F(x, s) \notin \mathcal{I}(u)$, for every $s \in [0, \infty[$. In particular, for every connected component C of $M \setminus \mathcal{I}(u)$, we have $F(C \times [0, +\infty[) \subset C$

Before proceeding further, we note that this function F shows not only that we have a propagation of singularities in infinite time, which depends continuously on the singularity, but also that we can extend this propagation to a propagation of the other points *into* $\mathrm{Sing}(u)$.

We show how to obtain the Homtopy Theorem from the map F. It is convenient to introduce the cut time function $\tau : M \to [0, +\infty]$ for u, where $\tau(x)$ is the supremum of the $t \geq 0$ such that there exists a calibrated curve $\gamma : [0, t] \to M$, with $\gamma(0) = x$. The properties of τ are:

(i) $\tau(x) = 0$ for $x \in \text{Sing}(u)$;

(ii) $\tau(x) = +\infty$ if and only if $x \in \mathcal{I}(u)$;

(iii) the function τ is upper semi-continuous.

For example, to prove (i), we note that if $\tau(x) > 0$, then we can find a calibrated curve $\gamma : [0, s] \to M$, with $s > 0$, and $\gamma(0) = 0$. If we extend this curve to $]-\infty, 0]$ by a backward characteristic ending at x, we find a calibrated curve $\gamma :]-\infty, s]$, with $\gamma(0) = x$, since $0 \in]-\infty, s[$ the function u is differentiable at x. This implies (i).

Part (ii) and (iii) result from the fact that a limit of calibrated curves is calibrated.

We now recall part (b) of the main lemma.

> If $F(x, s) \notin \mathrm{Sing}(u)$, for some $s > 0$, and $x \in M$, then the curve $\sigma \mapsto F(x, \sigma)$ is calibrated (for u) on $[0, s]$.

This clearly implies that $F(x, s) \in \mathrm{Sing}(u)$, for $s > \tau(x)$.

We now show that for every connected component C of the open set $M \setminus \mathcal{I}(u)$, the intersection $C \cap \mathrm{Sing}(u)$ is path connected.

Since τ is upper semi-continuous and finite on $M \setminus \mathcal{I}(u)$, we can find a continuous function $\alpha : M \setminus \mathcal{I}(u) \to]0, +\infty[$, with $\alpha > \tau$ on $M \setminus \mathcal{I}(u)$.

Since $\alpha(z) > \tau(z)$, note that $F(z, \alpha(z)) \in \mathrm{Sing}(u)$ for every $z \in M \setminus \mathcal{I}(u)$.
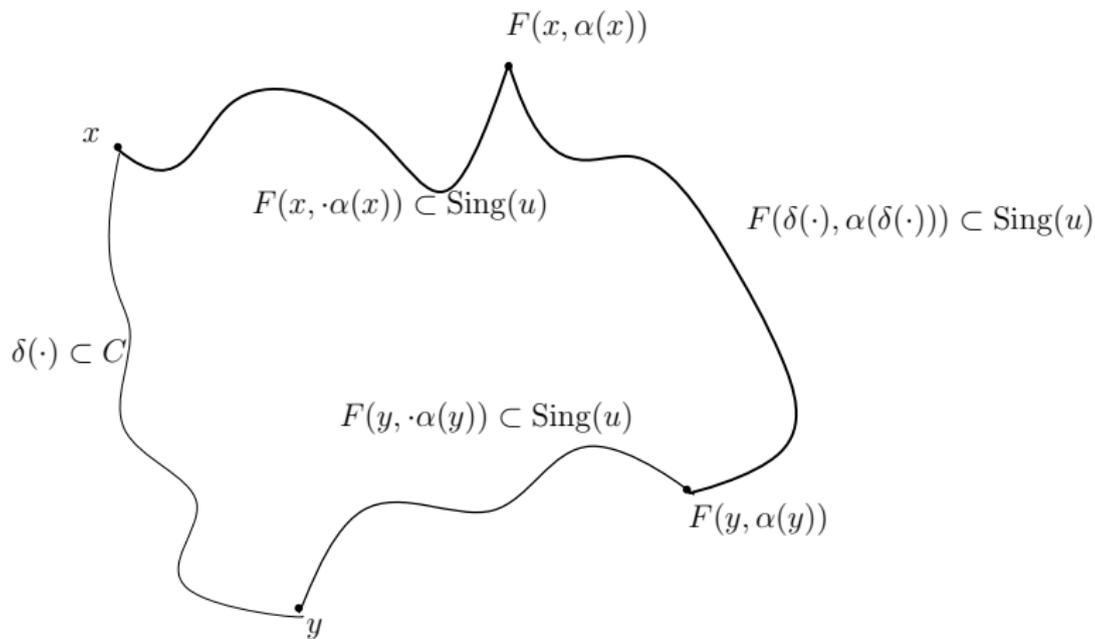
If $x, y \in C \cap \text{Sing}(u)$, since C is open and connected in M, we can find a path $\delta : [0,1] \to C$ with $\delta(0) = x, \delta(1) = y$.

Note that the path $t \mapsto F(\delta(t), \alpha(\delta(t))), t \in [0,1]$ is a continuous path in $C \cap \text{Sing}(u)$ joining $F(x, \alpha(x))$ to $F(y, \alpha(y))$.

Hence $F(x, \alpha(x))$ and $F(y, \alpha(y))$ are in the same path component of $C \cap \text{Sing}(u)$.

But if $z \in C \cap \text{Sing}(u)$, the path $t \mapsto F(z, t), t \in [0, \alpha(z)]$ is a continuous path in $C \cap \text{Sing}(u)$ joining z and $F(z, \alpha(z))$.

It follows that $x, F(x, \alpha(x)), F(y, \alpha(y))$ and y are all in the same path component of $C \cap \text{Sing}(u)$.

Both paths $s \mapsto F(x, s\alpha(x))$, and $s \mapsto F(x, s\alpha(x))$ are contained in $\mathrm{Sing}(u)$, since $x, y \in \mathrm{Sing}(u)$. Moreover, by the definition of α, for every $t \in [0, 1]$, we have $F(\delta(t), \alpha(\delta(t))) \in \mathrm{Sing}(u)$

The homotopy equivalence

To show that the inclusion is a $\mathrm{Sing}(u) \subset M \setminus \mathcal{I}(u)$ is a homotopy equivalence, we introduce the map
$G : M \setminus \mathcal{I}(u) \times [0,1] \to M \setminus \mathcal{I}(u)$ defined by

$$G(x, t) = F(x, t\alpha(x)).$$

The properties of G are:
(1) $G(x, 0) = x$, for all $x \in M \setminus \mathcal{I}(u)$.
(2) $G(\mathrm{Sing}(u) \times [0,1]) \subset \mathrm{Sing}(u)$.
(3) $G(x, 1) \in \mathrm{Sing}(u)$, for all $x \in M \setminus \mathcal{I}(u)$.
This shows clearly that the time one map $x \mapsto G(x, 1)$ which sends
$M \setminus \mathcal{I}(u)$ to $\mathrm{Sing}(u)$ is a homotopy inverse of the inclusion
$\mathrm{Sing}(u) \subset M \setminus \mathcal{I}(u)$.

The proof of the local connectedness is a localisation of the arguments given above.

This will be done after the proof of the Main Lemma 5.

We now proceed to a first reduction of the Main Lemma 5, which we now recall.

Lemma 2 (Main Lemma)

Assume that $u : M \to \mathbb{R}$ is a a viscosity solution of $H(x, d_x u) = 0$, where $H : T^*M \to \mathbb{R}$ is a Tonelli Hamiltonian.
We can find a continuous function $F : M \times [0, +\infty[\to M$ such that:

(a) *for all $x \in M$, we have $F(x, 0) = x$;*

(b) *if $F(x, s) \notin \mathrm{Sing}(u)$, for some $s > 0$, and $x \in M$, then the curve $\sigma \mapsto F(x, \sigma)$ is calibrated (for u) on $[0, s]$;*

(c) *if there exists a calibrated curve $\gamma : [0, s] \to M$, with $\gamma(0) = x$, then $\sigma \mapsto F(x, \sigma) = \gamma(\sigma)$, for every $\sigma \in [0, s]$.*

In fact, it suffices to construct F on $M \times [0, t]$, for some $t > 0$, and then to extend by induction on the intervals of the form $[nt, (n + 1)t]$ by

$$F(x, s) = F(F(x, nt), s - nt), \text{ for } s \in [nt, (n + 1)t].$$

To make sure that properties (a), (b), and (c) of the main lemma hold, it suffices to impose the following properties for F on $M \times [0, t]$:

(a') for all $x \in M$, we have $F(x, 0) = x$;

(b') if $F(x, s) \notin \mathrm{Sing}(u)$, for some $s \in \,]0, t]$ and some $x \in M$, then the curve $\sigma \mapsto F(x, \sigma)$ is calibrated (for u) on $[0, s]$;

(c') if there exists a calibrated curve $\gamma : [0, s] \to M$, with $\gamma(0) = x$, then $\sigma \mapsto F(x, \sigma) = \gamma(\sigma)$, for every $\sigma \in [0, \min(s, t)]$.

For the proof of existence of F on $M \times [0, t]$, for some $t > 0$, we do as Piermarco Cannarsa and Wei Cheng did: we use the positive Lax-Oleinik semi-group T_t^+ on the space of continuous function $\mathcal{C}^0(M, \mathbb{R})$.

For $w \in \mathcal{C}^0(M, \mathbb{R})$, the function $T_t^+ w$ is defined by $T_0^+ w = w$, and for $t > 0$

$$T_t^+ w(x) = \sup_\gamma w(\gamma(t)) - \int_0^t L(\gamma(s), \dot{\gamma}(s)) \, ds,$$

where the supremum is taken over all paths $\gamma : [0, t] \to M$, with $\gamma(0) = x$. By Tonelli's theorem this supremum is always attained by a curve which is necessarily minimizer, and therefore a C^2 curve. We will denote by $\hat{\gamma}_{x,t} : [0, t] \to M$, a curve such that $\hat{\gamma}_{x,t}(0) = x$

$$T_t^+ w(x) = w(\hat{\gamma}_{x,t}(t)) - \int_0^t L(\hat{\gamma}_{x,t}(s), \dot{\hat{\gamma}}_{x,t}(s)) \, ds.$$

An important property, which follows from the first variation formula, of such a curve $\hat{\gamma}_{x,t}$ realizing the equality

$$T_t^+ w(x) = w(\hat{\gamma}_{x,t}(t)) - \int_0^t L(\hat{\gamma}_{x,t}(s), \dot{\hat{\gamma}}_{x,t}(s))\, ds.$$

is

$$\frac{\partial L}{\partial v}(x, \dot{\hat{\gamma}}_{x,t}(0)) \in D^- T_t^+ w(x),$$

$$\frac{\partial L}{\partial v}(\hat{\gamma}_{x,t}(t), \dot{\hat{\gamma}}_{x,t}(t)) \in D^+ w(\hat{\gamma}_{x,t}(t)),$$

where $D^- T_t^+ w(x)$ is the set of lower-differentials of $T_t^+ w$ at x, and $D^+ w(\hat{\gamma}_{x,t}(t))$ is the set of upper-differentials of w at $\hat{\gamma}_{x,t}(t)$.

In particular, if $T_t^+ u$ is differentiable at x, we have

$$d_x T_t^+ u = \frac{\partial L}{\partial v}(x, \dot{\hat{\gamma}}_{x,s}(0)).$$

We can now invoke the following result of Patrick Bernard:

Theorem 6 (Patrick Bernard)

If u is a viscosity solution of $H(x, d_x u) = 0$, where H is a Tonelli Hamiltonian, then there exists a $t > 0$ such that for every $s \in]0, t]$ the function $T_s^+ u$ is $C^{1,1}$.

Therefore for every $x \in M$ and every $s \in]0, t]$, we have

$$d_x T_s^+ u = \frac{\partial L}{\partial v}(x, \dot{\hat{\gamma}}_{x,s}(0)),$$

and the starting speed of any minimizer $\hat{\gamma}_{x,s}$ is imposed

$$\dot{\hat{\gamma}}_{x,s}(0) = \operatorname{grad}_L T_s^+(x).$$

Since a minimizer is completely determined by its starting position and speed, we see that $\hat{\gamma}_{x,s}$ is unique, and we can therefore set

$$F(x, s) = \hat{\gamma}_{x,s}(s), \text{ for } x \in M, \text{ and } s \in]0, t].$$

The uniqueness of $\hat{\gamma}_{x,s}$, and some compactness properties of this set of curves show that F is continuous. It can also be shown that $F(x, s) \to x$ uniformly in x as $s \to 0$.

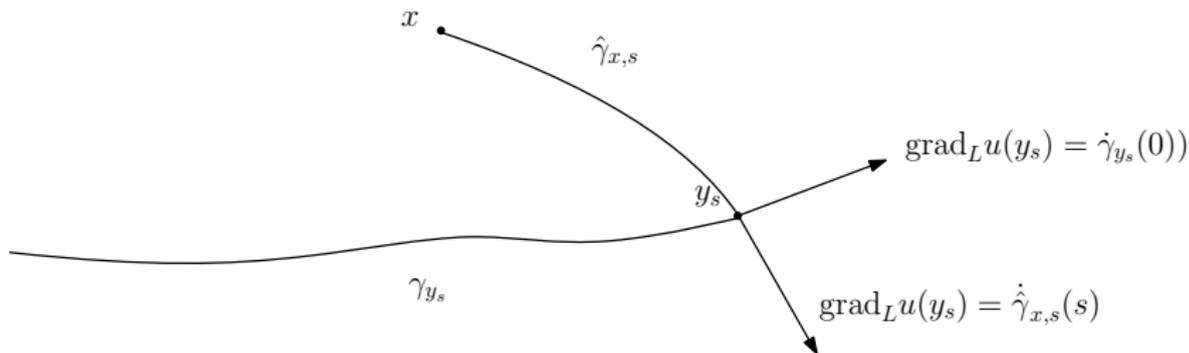This proves property (a') for F.

Note that from what we said above, we have

$$\frac{\partial L}{\partial v}(\hat{\gamma}_{x,s}(s), \dot{\hat{\gamma}}_{x,s}(s)) \in D^+ u(\hat{\gamma}_{x,s}(s)).$$
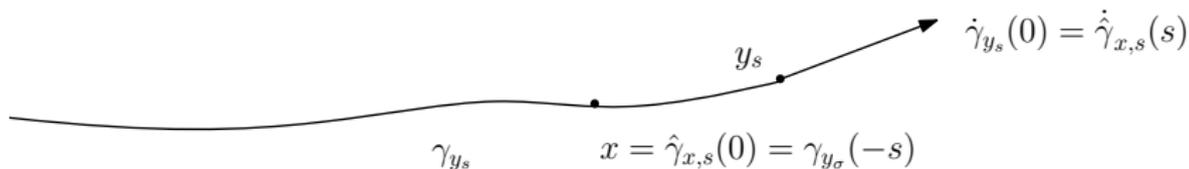
In particular, if u is differentiable at $F(x, s) = \hat{\gamma}_{x,s}(s) = y_s$ for some $s > 0$, then we get

$$d_{y_s} u = \frac{\partial L}{\partial v}(y_s, \dot{\hat{\gamma}}_{x,s}(s)) = \frac{\partial L}{\partial v}(\hat{\gamma}_{x,s}(s), \dot{\hat{\gamma}}_{x,s}(s)).$$

But we already know that $d_{y_s} u = \frac{\partial L}{\partial v}(y_s, \dot{\gamma}_{y_s}(0))$, where $\gamma_{y_s} :] - \infty, 0] \to M$ is a backward characteristic ending at y_s.

Therefore $\dot{\gamma}_{y_s}(0) = \dot{\hat{\gamma}}_{x,s}(s) = \mathrm{grad}_L\, u(y_s)$. By uniqueness of minimizers going through the same point with the same speed, we obtain $\hat{\gamma}_{x,s}(\sigma) = \gamma_{y_s}(\sigma - s)$, for all $\sigma \in [0, s]$.

In particular $x = \hat{\gamma}_{x,s}(0) = \gamma_{y_s}(-s)$. Therefore, if $F(x, s) \notin \mathrm{Sing}(u)$, there exists a calibrated curve $\gamma : [0, s] \to M$ with $\gamma(0) = x$ (namely $\gamma(\sigma) = \gamma_{y_s}(\sigma - s)$).

Therefore to finish proving properties (b') and (c'), we have to show:

If there exists a calibrated curve $\gamma : [0, s] \to M$ with $\gamma(0) = x$, then $F(x, \sigma) = \gamma(\sigma)$, for all $\sigma \in [0, \min(t, s)]$.

We first show that, for $\sigma \in [0, s]$, we have

$$T_\sigma^+ u(x) = u(x) = u(\gamma(\sigma)) - \int_0^\sigma L(\gamma(z), \dot{\gamma}(z)) \, dz. \qquad (2)$$

In fact, by the domination property of u and the calibration of γ, for any curve $\delta : [0, \sigma] \to M$, starting at x, we have

$$u(\delta(\sigma)) - u(x) \leq \int_0^\sigma L(\delta(z), \dot{\delta}(z)) \, dz, \text{ with equality for } \delta = \gamma|[0, \sigma].$$

This is equivalent to

$$u(\gamma(\sigma)) - \int_0^\sigma L(\gamma(z), \dot{\gamma}(z)) \, dz = u(x) \geq u(\delta(\sigma)) - \int_0^\sigma L(\delta(z), \dot{\delta}(z)) \, dz,$$

which clearly implies (2), by the definition of $T_\sigma^+ u(x)$.

Therefore we can take $\gamma_{x,\sigma} = \gamma|[0, \sigma]$, which yields $F(x, \sigma) = \gamma_{x,\sigma}(\sigma) = \gamma(\sigma)$, for every $\sigma \in]0, \min(t, s)]$.

Proof of the local contractibilty result Theorem 1.

For every open subset $O \subset M \setminus \mathcal{I}(u)$, we first construct an open subset $\tilde{O} \subset O$, such that

$$\mathrm{Sing}(u) \cap \tilde{O} = \mathrm{Sing}(u) \cap O,$$

together with a homotopy $G_O : \tilde{O} \times [0,1] \to O$, which satisfies:

(i) $G_O(x,0) = x$ for every $x \in \tilde{O}$;

(ii) $G_O((\mathrm{Sing}(u) \cap O) \times [0,1]) \subset \mathrm{Sing}(u) \cap O = \mathrm{Sing}(u) \cap \tilde{O}$;

(iii) $G_O(\tilde{O} \times \{1\}) \subset \mathrm{Sing}(u) \cap O = \mathrm{Sing}(u) \cap \tilde{O}$.

To define $\tilde{O}$, we introduce the function $\eta_O : O \to [0, \infty]$ defined by

$$\eta_O(x) = \sup\{t \in [0, +\infty[: F(x,s) \in O, \text{ for all } s \in [0,t]\}.$$

Since O is open, F is continuous, with $F(x, 0) = x$, the function η_O defined by

$$\eta_O(x) = \sup\{t \in [0, +\infty[: F(x, s) \in O, \text{ for all } s \in [0, t]\},$$

is lower semi-continuous and everywhere > 0 on O.

Using that η_O is lower semi-continuous and the cut time function τ is upper semi-continuous, we conclude that the subset

$$\tilde{O} = \{x \in O : \tau(x) < \eta_O(x)\} \subset O$$

is indeed open.

Furthermore since τ is 0 on $\mathrm{Sing}(u)$ and $\eta > 0$ everywhere in O, we get

$$\mathrm{Sing}(u) \cap \tilde{O} = \mathrm{Sing}(u) \cap O.$$

It remains to construct the homotopy G_O.

To construct G_O, we observe that $\tau < \eta_O$ on $\tilde{O} \subset O$, with τ upper semi-continuous, and η_O lower semi-continuous.

Hence, Baire's interpolation theorem guarantees the existence of a continuous function $\alpha_O : \tilde{O} \to]0, +\infty[$ such that $\tau < \alpha_O < \eta_O$ everywhere on $\tilde{O}$.

It is not difficult to check that the map $G_O : \tilde{O} \times [0,1] \to O$, defined by

$$G_O(x,s) = F(x, s\alpha_O(x)),$$

satisfies the required conditions (i),(ii), and (iii).

Property (i) of G_O states that $G_O(x,0) = x$ for every $x \in \tilde{O}$. Therefore, from property (ii) of G_O

$$G_O\big((\mathrm{Sing}(u) \cap O) \times [0,1]\big) \subset \mathrm{Sing}(u) \cap O = \mathrm{Sing}(u) \cap \tilde{O},$$

we obtain that the identity on $\mathrm{Sing}(u) \cap O = \mathrm{Sing}(u) \cap \tilde{O}$ is homotopic to the restriction $G_{O,1} : \mathrm{Sing}(u) \cap O \to \mathrm{Sing}(u) \cap O$ of the time one map $G_{O,1}$ of G_O as maps with values in $\mathrm{Sing}(u) \cap O$.

Let B be an open subset included in $\tilde{O}$, which is homeomorphic to an Euclidean ball.

Since B is contractible and $G_{O,1}(B) \subset \text{Sing}(u) \cap O$, composing the contraction of B in itself with $G_{O,1}$, we see tahat the restriction of $G_{O,1}$ to B is homotopic to a constant as maps from B to $\text{Sing}(u) \cap O$.

Therefore the restriction of $G_{O,1}$ to $\text{Sing}(u) \cap B$ is homotopic to a constant as maps with values in $\text{Sing}(u) \cap O$.

Since $\text{Sing}(u) \cap B \subset \text{Sing}(u) \cap O$, and $G_{O,1} : \text{Sing}(u) \cap O \rightarrow \text{Sing}(u) \cap O$ is homotopic to the identity as maps with values in $\text{Sing}(u) \cap O$, it follows that the inclusion $\text{Sing}(u) \cap B \hookrightarrow \text{Sing}(u) \cap O$ is homotopic to a constant as maps with values in $\text{Sing}(u) \cap O$. $\qquad\square$