Lyapounov Functions of closed Cone Fields: from Conley Theory to Time Functions.

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Abstract. We propose a theory "à la Conley" for cone fields using a notion of relaxed orbits based on cone enlargements, in the spirit of space time geometry. We work in the setting of closed (or equivalently semi-continuous) cone fields with singularities. This setting contains (for questions which are parametrization independent such as the existence of Lyapounov functions) the case of continuous vector-fields on manifolds, of differential inclusions, of Lorentzian metrics, and of continuous cone fields. We generalize to this setting the equivalence between stable causality and the existence of temporal functions. We also generalize the equivalence between global hyperbolicity and the existence of a steep temporal functions.

Résumé. On développe une théorie à la Conley pour les champs de cones, qui utilise une notion d'orbites relaxées basée sur les élargissements de cones dans l'esprit de la géométrie des espaces temps. On travaille dans le contexte des champs de cones fermés (ou, ce qui est équivalent, semi-continus), avec des singularités. Ce contexte contient (pour les questions indépendantes de la paramétrisation, comme l'existence de fonctions de Lyapounov) le cas des champs de vecteurs continus, celui des inclusions différentielles, des métriques Lorentziennes, et des champs de cones continus. On généralise à ce contexte l'équivalence entre la causalité stable et l'existence d'une fonction temporale. On généralise aussi l'équivalence entre l'hyperbolicité globale et l'existence d'une fonction temporale uniforme.

The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement 307062.

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Contents

0.1 Introduction

Lyapounov functions play an important role in dynamical systems. Their existence is related to basic dynamical behaviors such as stability and recurrence. The second aspect was made precise by Conley, who showed an equivalence between the existence of Lyapounov functions and the absence of chain recurrence. This result was extended by Hurley, see [18, 19], to non compact spaces. See also [26] for a different point of view based on Mather-Fathi theory.

On the other hand the causality theory of space times studies (among other things) time functions on Lorentzian manifold, see [22] for example. The existence of continuous time functions for smooth stably causal space times was proved in [16] and [17]. The condition of stable causality of space time is analogous to the absence of chain recurrence in Conley's theory. Still in the context of smooth space times. the equivalence between stable causality and the existence of a smooth temporal function (a regular Lyapounov functions in the terminology of the present paper) was proved in [3]. Motivated by solutions to the Einstein equations with low regularity the problem has been revisited in [8], [7] and [27] where continuous metrics are studied. The existence of smooth time functions for continuous stably causal cone fields (hence in particular for continuous, stably causal, Lorentzian metrics) was proved in [14] and [15] by methods inspired from weak-KAM theory.

In the present paper, we propose a theory "à la Conley" for cone fields. Such a program was already carried out in [23] in the case of Lorentzian metrics, but our approach is different. We use a notion of relaxed orbits based on cone enlargements, in the spirit of space time geometry. This notion has the advantage of not resting on the choice of an auxiliary metric and its bypasses some technical difficulties related to the non continuity of the length. It allows us to work without difficulty in the very general setting of closed (or equivalently semi-continuous) cone fields with singularities. This setting contains (for questions which are parametrization independent such as the existence of Lyapounov functions) the case of continuous vector-fields on manifolds, of differential inclusions, of Lorentzian metrics, and of continuous cone fields. One drawback of our approach is that it requires a manifold structure on the phase space, but the associated advantage is that we directly deal with smooth Lyapounov (or time) functions. We generalize to this setting the equivalence between stable causality and the existence of temporal functions. We also prove that every globally hyperbolic cone field admits a steep Lyapounov function. The term *steep temporal function* was introduced in [24], see section 0.2 for the definition and a discussion. We finally recover classical statements on the relation between Lyapounov functions and asymptotic stability in their most general setting, as obtained in [11, 28, 29]. Since our original motivation was to prove the existence of steep temporal functions in a generalized setting, we work with the usual convention of space time geometry and consider Lyapounov functions which are non decreasing along orbits.

We work on a complete Riemannian manifold M.

A convex cone of the vector space E is a convex subset $C \subset E$ such that $tx \in C$ for each $t > 0$ and $x \in C$. It is convenient to exclude the case $C = \{0\}$, we however include the case $C = \emptyset$. We will also call this cone degenerate. All the other cones are called non degenerate. Cones may or not contain the origin. We say that the cone C' is wider than the cone C, noted $C \prec C'$ if $C \cup \{0\} \subset C' \cup \{0\}$. A cone is called singular if it contains a straight line. In the case of an open cone, this implies that $C = E$. Cones which are not singular are called regular. The cone C is regular if and only there exists a linear form p on E such that $p \cdot v > 0$ for each non zero vector $v \in C$.

A closed cone field on M is a closed subset $\mathcal{C}$ of TM such that the set $\mathcal{C}(x) := \{v \in T_xM, (x, v) \in \mathcal{C}\}$ is a convex closed cone for each x. It is easy to see that continuous cone fields as considered for example in [14] are closed, hence our setting is more general. We say that x is a regular (resp. singular) point of $\mathcal{C}$ if $\mathcal{C}(x)$ is a regular (resp. singular) cone. The set of singular points of a closed cone field is closed.

As a first example, we can associate to each continuous vector-field V on M a closed cone field $\mathcal{C}_V$ defined by $\mathcal{C}_V(x) = \mathbb{R}^+ \cdot V(x)$ if $V(x) \neq 0$ and $\mathcal{C}_V(x) = T_xM$ if $V(x) = 0$. The point x is singular for $\mathcal{C}_V$ if and only if it is singular for V (*i.e.* if $V(x) = 0$). Already this simple example shows the usefulness of allowing non continuous and singular closed cone fields. This example also motivate our terminology for singular points.

An open cone field on M is an open subset $\mathcal{E}$ of TM such that the set $\mathcal{E}(x) := \{v \in T_xM, (x, v) \in \mathcal{E}\}$ is a convex open cone of T_xM for each $x \in M$. The set of singular points of an open cone field (the points x such that $\mathcal{E}(x) = T_xM$) is open.

We denote by $\mathcal{D}(\mathcal{C})$ the domain of the cone field $\mathcal{C}$, which is the set of non degenerate points of $\mathcal{C}$. Open cone fields have open domains, and closed cone fields have closed domains. A cone field is said non degenerate if is has no degenerate point *i.e.* if $\mathcal{D}(\mathcal{C}) = M$.

The cone field $\mathcal{C}'$ is said to be wider than the cone field $\mathcal{C}$ if $\mathcal{C}'(x)$ is wider than $\mathcal{C}(x)$ for each x. We use the notation $\mathcal{C} \prec \mathcal{C}'$. We say that $\mathcal{C}'$ is an enlargement of $\mathcal{C}$ if there exists an open cone field $\mathcal{E}$ such that $\mathcal{C} \prec \mathcal{E} \prec \mathcal{C}'$. An open enlargement of $\mathcal{C}$ is just an open cone field wider than $\mathcal{C}$.

The closure (as a subset of TM) of a cone field $\mathcal{E}$ is noted $\bar{\mathcal{E}}$, it is a closed cone field. Note that $\overline{\mathcal{E}(x)} \subset \bar{\mathcal{E}}(x)$, but with equality only at continuity points of $\mathcal{E}$.

Given an open cone field $\mathcal{E}$, we say that the curve $\gamma : I \longrightarrow M$ is $\mathcal{E}$-timelike (or just timelike) if it is piecewise smooth (we shall see later that this regularity can be relaxed) and if $\dot{\gamma}(t) \in \mathcal{E}(\gamma(t))$ for all t in I (at nonsmooth points, the inclusion is required to hold for left and right differentials). The chronological future $\mathcal{I}^+_{\mathcal{E}}(x)$ of x is the set of points $y \in M$ such that there exists a non constant timelike curve $\gamma : [0, T] \longrightarrow M$ satisfying $\gamma(0) = x$ and $\gamma(T) = y$. The chronological past $\mathcal{I}^-_{\mathcal{E}}(x)$ is the set of points $x' \in M$ such that $x \in \mathcal{I}^+_{\mathcal{E}}(x')$. Note that

$\mathcal{I}^-_{\mathcal{E}}(x) = \mathcal{I}^+_{-\mathcal{E}}(x)$. More generally, for each subset $A \subset M$, we denote by $\mathcal{I}^{\pm}_{\mathcal{E}}(A) := \cup_{x\in A}\mathcal{I}^{\pm}_{\mathcal{E}}(x)$ the chronological future and past of A. They are open subsets of M. We have the inclusion $\mathcal{I}^+_{\mathcal{E}}(y) \subset \mathcal{I}^+_{\mathcal{E}}(x)$ if $y \in \mathcal{I}^+_{\mathcal{E}}(x)$.

Given a closed cone field $\mathcal{C}$, we say that the curve $\gamma : I \longrightarrow M$ is $\mathcal{C}$-causal (or just causal) if it is Lipschitz and if the inclusion $\dot\gamma(t) \in \mathcal{C}(\gamma(t))$ holds for almost all $t \in I$. The causal future $\mathcal{J}^+_{\mathcal{C}}(x)$ of x is the set of points $y \in M$ such that there exists a (possibly constant) causal curve $\gamma : [0,T] \longrightarrow M$ satisfying $\gamma(0) = x$ and $\gamma(T) = y$. The causal past $\mathcal{J}^-_{\mathcal{C}}(x)$ is the set of points $x' \in M$ such that $x \in \mathcal{J}^+_{\mathcal{C}}(x')$. More generally, for each subset $A \subset M$, we denote by $\mathcal{J}^{\pm}_{\mathcal{C}}(A) := \cup_{x\in A}\mathcal{J}^{\pm}_{\mathcal{C}}(x)$ the causal future and past of A. We have the inclusion $\mathcal{J}^+_{\mathcal{C}}(y) \subset \mathcal{J}^+_{\mathcal{C}}(x)$ if $y \in \mathcal{J}^+_{\mathcal{C}}(x)$.

The smooth function $\tau : M \longrightarrow \mathbb{R}$ is called a Lyapounov function for the cone field $\mathcal{C}$ if $d\tau_x \cdot v \geqslant 0$ for each $(x,v) \in \mathcal{C}$ and if, at each regular point x of τ (which means that $d\tau_x \neq 0$), we have $d\tau_x \cdot v > 0$ for each $v \in \mathcal{C}(x) - \{0\}$. When $\mathcal{C}$ is the cone field associated to a vector-field V, a Lyapounov function for $\mathcal{C}$ is the same as a Lyapounov function for V.

Note that if the cone field is induced by a time orientable Lorentzian metric a smooth Lyapounov function without critical points is a temporal function for the Lorentzian metric. In the same vein time/temporal function were considered in [14] for continuous cone fields.

Given a closed cone field $\mathcal{C}$, we define

$$\mathcal{F}^+_{\mathcal{C}}(x) := \{x\} \cup \bigcap_{\mathcal{E}\succ\mathcal{C}} \mathcal{I}^+_{\mathcal{E}}(x)$$

where the intersection is taken on all open enlargements $\mathcal{E}$ of $\mathcal{C}$. We call it the stable future of x. A point x is said to be *stably recurrent* (for $\mathcal{C}$) if, for each open enlargement $\mathcal{E}$ of $\mathcal{C}$, there exists a closed $\mathcal{E}$-timelike curve passing through x. We denote by $\mathcal{R}_{\mathcal{C}}$ the set of stably recurrent points. Let us state our first result, which will be proved in Section 4.

Theorem 1. *Let $\mathcal{C}$ be a closed cone field.*

The set $\mathcal{F}^+_{\mathcal{C}}(x)$ is the set of point $x' \in M$ such that $\tau(x') \geqslant \tau(x)$ for each smooth Lyapounov function τ (it is thus a closed set).

The point x is stably recurrent if and only if all smooth Lyapounov functions τ satisfy $d\tau_x = 0$ (hence $\mathcal{R}_{\mathcal{C}}$ is closed).

Two points x and x' of $\mathcal{R}_{\mathcal{C}}$ are called stably equivalent if $x' \in \mathcal{F}^+_{\mathcal{C}}(x)$ and $x \in \mathcal{F}^+_{\mathcal{C}}(x')$; this is an equivalence relation on $\mathcal{R}_{\mathcal{C}}$. The classes of this equivalence relation are called stable classes.

Following the terminology of general relativity, we call a closed cone field *stably causal* if it is without stably recurrent points. The following statement is also proved in Section 4.

Theorem 2. *Let $\mathcal{C}$ be a closed cone field. There exists a smooth Lyapounov function τ with the following properties:*

- *The function τ is regular at each point of $\mathcal{D}(\mathcal{C}) - \mathcal{R}_{\mathcal{C}}$.*
- *Two points x and x' of $\mathcal{R}_{\mathcal{C}}$ belong to the same stable class if and only if $\tau(x') = \tau(x)$.*
- *If x and x' are two points of M such that $x' \in \mathcal{F}^+_{\mathcal{C}}(x)$ and $x \notin \mathcal{F}^+_{\mathcal{C}}(x')$, then $\tau(x') > \tau(x)$.*

This implies that $\mathcal{R}_{\mathcal{C}}$ is a closed set, as well as the stable components.

We recover the classical fact that a closed cone field is stably causal if and only if it admits a smooth temporal function (in our terminology, a smooth Lyapounov function without critical points). This result has a long history and several variants. To our knowledge, the most general known variant is due to Fathi and Siconolfi in [14], in the context of continuous cone fields. Our variant is more general, since we allow closed (equivalently : semi-continuous) cone fields with singularities, and our proof is completely different.

We now present some more specific applications of our methods:

0.2 Hyperbolic cone fields

Following the terminology of space time geometry, we say that the cone field $\mathcal{C}$ on M is *globally hyperbolic* if

(GH0) $\mathcal{C}$ is closed and non degenerate.

(GH1) $\mathcal{C}$ is causal, which means that that all closed Lipschitz $\mathcal{C}$-causal curves are constant and all points are regular.

(GH2) The set $\mathcal{J}_{\mathcal{C}}(K,K') := \mathcal{J}^+_{\mathcal{C}}(K)\cap \mathcal{J}^-_{\mathcal{C}}(K')$ is compact for each compact sets K and K'.

We stress that stable causality is not assumed here, as it is in [14] (it will be indirectly proved to be a consequence of hyperbolicity). In the classical context of Lorentzian metrics, the definition was given in a weaker form where (GH2) is replaced by

(GH3) The set $\mathcal{J}_{\mathcal{C}}(x,y) = \mathcal{J}^+_{\mathcal{C}}(x)\cap \mathcal{J}^-_{\mathcal{C}}(y)$ is compact for each x and y in M.

Our definition is equivalent in the Lorentzian case, as follows from:

Proposition 1. *If the closed cone field $\mathcal{C}$ is wider than a non degenerate open cone field and satisfies (GH3), then it satisfies (GH2).*

Proof. Our assumption is that there exist a non degenerate open cone field $\mathcal{E}\prec\mathcal{C}$. It follows from Lemma 8 below that $\mathcal{E}$ contains a smooth vector-field $V(x)$. This vector-field can be assumed complete by reparametrization, we denote by ϕ^t its flow. Let K and K' be two compact sets. We consider a sequence $z_n \in \mathcal{J}_{\mathcal{C}}(K,K')$, $z_n\in \mathcal{J}_{\mathcal{C}}(x_n,y_n)$ with $x_n\in K$ and $y_n \in K'$. We can assume that the sequences x_n and y_n have limits x and y in K and K', respectively. For each $t>0$, $x\in \mathcal{I}^+_{\mathcal{E}}(\phi^{-t}(x))\subset \mathcal{J}^+_{\mathcal{C}}(\phi^{-t}(x))$ and $y\in \mathcal{I}^-_{\mathcal{E}}(\phi^{t}(y))\subset \mathcal{J}^-_{\mathcal{C}}(\phi^{t}(y))$. Since $\mathcal{I}^+_{\mathcal{E}}(\phi^{-t}(x))$ and $\mathcal{I}^-_{\mathcal{C}}(\phi^{t}(y))$ are open, $x_n\in \mathcal{I}^+_{\mathcal{E}}(\phi^{-t}(x))\subset \mathcal{J}^+_{\mathcal{C}}(\phi^{-t}(x))$ and $y_n\in \mathcal{I}^-_{\mathcal{E}}(\phi^{t}(y))\subset \mathcal{J}^-_{\mathcal{C}}(\phi^{t}(y))$ when n is large enough, hence $z_n\in \mathcal{J}_{\mathcal{C}}(\phi^{-t}(x),\phi^t(y))$, which is a compact set by (GH3). We can thus assume by taking a subsequence that z_n has a limit z which is contained in $\mathcal{J}_{\mathcal{C}}(\phi^{-t}(x),\phi^t(y))$ for each $t>0$. By (GH3), the set $\mathcal{J}_{\mathcal{C}}(\phi^{-1}(x),z)$ is compact and it contains $\phi^{-t}(x)$ for each $t\in]0,1[$, hence it contains x. This implies that $z\in \mathcal{J}^+_{\mathcal{C}}(x)$. We prove similarly that $z\in \mathcal{J}^-_{\mathcal{C}}(y)$. $\square$

The Lyapounov function τ is said to be *steep* if the inequality

$$d\tau_x\cdot v \geqslant |v|_x$$

holds for each $(x,v)\in\mathcal{C}$. Recall that we work with a complete Riemannian metric. The following statement extends a classical result (see [24], [21]) to our more general setting:

Theorem 3. *The non degenerate closed cone field $\mathcal{C}$ is globally hyperbolic if and only if it admits a smooth steep Lyapounov function. Then, the relations $\mathcal{J}_{\mathcal{C}}$ and $\mathcal{F}_{\mathcal{C}}$ are identical.*

Note that the definition of global hyperbolicity does not involve the metric. We deduce that, if $\mathcal{C}$ is globally hyperbolic and if $\tilde{g}$ is a (not necessarily complete) metric, then there exists a Lyapounov function which is steep with respect to $\tilde{g}$. This follows from the theorem applied to the complete metric $g+\tilde{g}$ (where g is a complete metric on M).

At this point a comment on the definition of steep Lyapounov functions is in order. A similar notion appears in [24] as the sharp criterion for the isometric embeddability of space times into Minkowski space. There a function τ on the space time (M,g_L) is steep if $d\tau\cdot v\geq \sqrt{|g_L(v,v)|}$ for all future pointing vectors $(x,v)\in TM$. The existence of steep smooth temporal functions for globally hyperbolic space times is proved in [24] and [21]. Theorem 3 implies the existence of steep temporal functions on space times as the Riemannian metric g can be chosen to satisfy $g(v,v) \geq |g_L(v,v)|$ on all tangent vectors, especially the future pointing ones.

The conclusion of Theorem 3 is false if (GH2) is replaced by (GH3) without assuming that $\mathcal{C}$ has non empty interior. Any vector-field admitting non trivial recurrence provides a counter-example. We have the following corollaries:

Corollary 2. *Each globally hyperbolic cone field admits a globally hyperbolic enlargement.*

In particular, hyperbolicity implies stable causality. The splitting theorem, see [2, 3], also holds in our setting:

Corollary 3. *Let $(M, \mathcal{C})$ be globally hyperbolic. Then there exists a manifold N and a diffeomorphism $\psi : M \longrightarrow \mathbb{R} \times N$ whose first component is a steep time function on M.*

Proof. Let τ be a steep time function. We consider the vector-field $V(x) = \nabla\tau/|\nabla\tau|^2$, it has the property that $d\tau_x \cdot V(x) = 1$. Note that $|d\tau_x| \geqslant 1$ hence $|\nabla\tau_x| \geqslant 1$ hence $|V(x)| \leqslant 1$. As a consequence, the flow φ^t of V is complete. Setting $N = \tau^{-1}(0)$, the map $(t,x) \longmapsto \varphi^t(x)$ is a diffeomorphism from $\mathbb{R} \times N$ to M. The inverse diffeomorphism ψ is as desired. □

As was noticed in [6], if M is moreover assumed contractible, it is then diffeomorphic to a Euclidean space.

0.3 A Lemma of Sullivan

We start with the definition of complete causal curves, which are the analogs in our setting of maximal solutions of vector fields.

Definition 4. *The causal curve γ is called complete if it is defined on an open (possibly unbounded) interval $]a,b[$ and if the two following conditions hold:*

- *Either $\gamma_{|[s,b[}$ has infinite length for each $s \in]a,b[$ or $\lim_{t \longrightarrow b} \gamma(t)$ is a singular point of $\mathcal{C}$ (we say that γ is forward complete).*
- *Either $\gamma_{|]a,s]}$ has infinite length for each $s \in]a,b[$ or $\lim_{t \longrightarrow a} \gamma(t)$ is a singular point of $\mathcal{C}$ (we say that γ is backward complete).*

We have:

Proposition 5. *Let $(M, \mathcal{C})$ be a closed cone field and let $F \subset M$ be a closed set. Let $Z \subset F$ be the union of all complete causal curves contained in F. Then, there exists a Lyapounov function τ for $\mathcal{C}$ on M which is regular on $F - Z$.*

Proof. We consider the closed cone field $\mathcal{C}_F$ which is equal to $\mathcal{C}$ on F and degenerate outside of F. Each curve which is causal and complete for $\mathcal{C}_F$ is causal and complete for $\mathcal{C}$. The proposition follows from Theorem 2 and the observation that $\mathcal{R}(\mathcal{C}_F) \subset Z$, which follows from Corollary 19 below, applied to $\mathcal{C}_F$. □

In the case where $\mathcal{C}$ is the cone field generated by a continuous vector field X, where F is compact, and where Z is empty, we recover a famous Lemma of Sullivan, [31]:

If X is a continuous vector field on M, and if K is a compact set which does not contain any full orbit of X, then there exists a Lyapounov function for X which is regular on K, i.e. $d\tau_x \cdot X(x) > 0$ for each $x \in K$.

The proof of Sullivan in [31] was based on Hahn-Banach Theorem, a more elementary proof was given in [20]. Proposition 5 extends this result to the non compact case, and also to the case where some full orbits exist.

0.4 Asymptotic stability

We consider a closed cone field $\mathcal{C}$. A compact set $Y \subset M$ is called *asymptotically stable* if, for each neighborhood U of Y, there exists a neighborhood $V \subset U$ of Y such that $\mathcal{J}^+_{\mathcal{C}}(V) \subset U$ and if each forward complete causal curve starting in V converges to Y (which means that the distance to Y converges to zero). If $Y = \{y\}$ is a point, then this requires that y be singular (or degenerate).

We can recover in our setting the following restatement of several known results on converse Lyapounov theory for differential inclusions, see [11] for the case where Y is a singular point, and [28, 29] for the general case. Our setting in terms of cone fields is parametrization-invariant, in contrast to the formulation in terms of differential inclusions used in these papers. Since both properties of being asymptotically stable and of admitting a Lyapounov function are parametrization invariant, these settings are equivalent. Note that our sign convention for Lyapounov functions is non standard: they increase along orbits.

Proposition 6. *Let $Y \subset M$ be a compact set and let $\mathcal{C}$ be a closed cone field which is non degenerate in a neighborhood of Y. The following properties are equivalent:*

1. *Y is asymptotically stable.*
2. *$\mathcal{J}^+_{\mathcal{C}}(Y) = Y$ and there exists a neighborhood U of Y such that each backward complete causal curve γ contained in U is contained in Y.*
3. *$\mathcal{F}^+_{\mathcal{C}}(Y) = Y$ and there exists a neighborhood U of Y such that $\mathcal{R}_{\mathcal{C}} \cap U \subset Y$.*
4. *There exists a smooth Lyapounov function τ which is non positive, null on Y, and regular on $U - Y$, where U is a neighborhood of Y.*

Proof. 1 $\Rightarrow$ 2. The asymptotic stability implies that $\mathcal{J}^+_{\mathcal{C}}(Y) \subset U$ for each neighborhood U of Y, hence $\mathcal{J}^+_{\mathcal{C}}(Y) \subset Y$. Let U_0 be a compact neighborhood of Y which has the property that all forward complete curves contained in U_0 converge to Y. Let us suppose that there exists a backward complete causal curve $\gamma :]-T, 0] \longrightarrow U_0$ such that $\gamma(0)$ does not belong to Y. Let U_1 be a compact neighborhood of Y which does not contain $\gamma(0)$. There exists an open neighborhood V_1 of Y such that $\mathcal{J}^+_{\mathcal{C}}(V_1) \subset U_1$, which implies that γ does not enter V_1 on $]-T, 0]$. Since $U_0 - V_1$ does not contain singular points of $\mathcal{C}$, the curve γ has infinite length, we parametrize it by arclength, $\gamma : (-\infty, 0] \longrightarrow M$. By Ascoli Theorem, there exists a sequence $t_n \longrightarrow -\infty$ such that the curves $t \longmapsto \gamma(t - t_n)$ converge, uniformly on compact intervals, to a Lipschitz curve $\eta : \mathbb{R} \longrightarrow U_0 - V_1$. By Lemma 18, the curve η is causal and forward complete. This implies that η converges to Y, which is a contradiction since $\eta(\mathbb{R}) \subset U_0 - V_1$.

2 $\Rightarrow$ 3. Let U be the neighborhood with property 2, and W be a compact neighborhood of Y contained in U. If $\mathcal{F}^+_{\mathcal{C}}(Y)$ was not contained in W, then there would exists a backward complete causal curve contained in W but not in Y, by Corollary 20. This contradiction implies that $\mathcal{F}^+_{\mathcal{C}}(Y) \subset W$, and, since this holds for each compact neighborhood W of Y contained in U, that $\mathcal{F}^+_{\mathcal{C}}(Y) \subset Y$. The part of the statement concerning $\mathcal{R}_{\mathcal{C}}$ follows immediately from Corollary 19.

3 $\Rightarrow$ 4. It is a direct consequence of Proposition 35.

4 $\Rightarrow$ 1 Let U be a compact neighborhood of Y such that τ is regular on $U - Y$. For each neighborhood W of Y contained in U, we set $a = \max_{\partial W} \tau$ (by compactness, $a < 0$) and $V := \{x \in W, \tau(x) \geqslant a/2\}$. We have $\mathcal{J}^+_{\mathcal{C}}(V) \subset V \subset U$. Let $\gamma : [0, T[\longrightarrow V$ be a complete causal curve parametrized by arclength. The function $\tau \circ \gamma$ is increasing, hence is converges to $b \in]a/2, 0]$. We have to prove that $b = 0$. The set $V^b := \{x \in V, \tau(x) \leqslant b\}$ is compact. If $b < 0$, then τ is regular on V^b, hence there exists $\delta > 0$ such that $d\tau_x \cdot v \geqslant \delta |v|$ for each $(x, v) \in \mathcal{C}, x \in V^b$. This implies that $\tau \circ \gamma(t) \geqslant \tau \circ \gamma(0) + \delta t$, hence that $T \leqslant (b - a)/\delta$ is finite.

The complete causal curve γ has finite length, hence it converges to a limit $x \in V^b$ which is a singular point of $\mathcal{C}$ hence a critical point of τ, a contradiction. □

1 Preliminaries

1.1 On cone fields

We state here useful results on cone fields.

Lemma 7. *If $\mathcal{C}$ is a closed cone field and $\mathcal{E}$ an open cone field, then the set of point $x \in M$ such that $\mathcal{C}(x) \prec \mathcal{E}(x)$ is open.*

PROOF. It is the projection on M of the open set $\mathcal{E} - \mathcal{C}$. □

A standard partition of the unity argument implies:

Lemma 8. *Let $\mathcal{E}$ be a non degenerate open cone field ($\mathcal{E}(x) \neq \emptyset$ for each x). Then there exists a smooth vector-field V such that $V(x) \in \mathcal{E}(x)$ for each x. Moreover, given $(x, v) \in \mathcal{E}$, the vector field V can be chosen such that $V(x) = v$. In particular, there exists a smooth curve $\gamma(t) : \mathbb{R} \longrightarrow U$ which is $\mathcal{E}$-timelike and such that $(\gamma(0), \dot{\gamma}(0)) = (x, v)$.*

Note that $V(x) \neq 0$ if x is a regular point of $\mathcal{E}$. This implies that a non degenerate open cone field on a manifold must admit singular points if the Euler characteristic is not zero.

A smooth function τ defined near x is called a *local time function* at x if $d\tau_x \neq 0$ and $d\tau_x \cdot v > 0$ for each non zero vector $v \in \mathcal{C}(x)$. This property then holds in a neighborhood of x. Local time functions at x exist if and only if x is a regular point of $\mathcal{C}$. The cone $\mathcal{C}(x)$ is the set of vectors $v \in T_xM$ such that $d\tau_x \cdot v \geqslant 0$ for each local time function τ at x.

Let C be a closed cone and $\Omega \succ C$ be an open cone. Then there exists an open cone Ω' such that $C \prec \Omega' \prec \bar{\Omega}' \prec \Omega$. Given a diffeomorphism onto its image $\phi : N \longrightarrow U \subset M$ and a cone field $\mathcal{C}$ on M, we denote by $\phi^*\mathcal{C} := (T\phi)^{-1}(\mathcal{C})$ the preimage of the cone field $\mathcal{C}$, where $T\phi$ is the tangent map $(x, v) \longmapsto (\phi(x), d\phi_x \cdot v)$. We define similarly the forward image $\phi_*\mathcal{C} := T\phi(\mathcal{C})$ of a cone field on N, this is a cone field on $U = \phi(N)$. We denote by $Q_s, s \geqslant 0$ the standard open cone

$$Q_s := \{(y, z) \in \mathbb{R}^{d-1} \times \mathbb{R} : z > s|y|\} \subset \mathbb{R}^d.$$

The following Lemma on the local structure of open cone fields is obvious, but very useful:

Lemma 9. *Let $\mathcal{E}$ be open cone field and let x_0 be a point which is non degenerate for $\mathcal{E}$ and regular for $\bar{\mathcal{E}}$. There exists a chart $\phi : B^{d-1} \times]-1, 1[\longrightarrow M$ at x_0 such that*

$$Q_1 \prec \phi^*\mathcal{E}(y, z) \prec \phi^*\bar{\mathcal{E}}(y, z) \prec Q_0$$

for each $(x, y) \in B^{d-1} \times]-1, 1[$.

PROOF. Let τ be a local regular Lyapounov function for $\bar{\mathcal{E}}$ such that $\tau(x_0) = 0$. Let V be a vector contained in $\mathcal{E}(x_0)$ and $\psi : M \longrightarrow \mathbb{R}^{d-1}$ be a smooth local map sending x_0 to 0 and such that the kernel of $d\psi_{x_0}$ is $\mathbb{R}V$. For each $a > 0$, the map $\Psi := (a\tau, \psi)$ is a local diffeomorphism, such that $d\Psi_{x_0} \cdot V = (ad\tau_{x_0} \cdot V, 0)$ and $\Psi_*\bar{\mathcal{E}}(0, 0) \prec Q_0$. If $a > 0$ is small enough, we have $Q_{1/2} \prec \Psi_*\bar{\mathcal{E}}(0, 0)$. As a consequence, there exists $s > 0$ such that

$$Q_1 \prec \Psi_*\mathcal{E} \prec \Psi_*\bar{\mathcal{E}} \prec Q_0$$

on $B^{d-1}(s) \times]-s, s[$. The inverse map ϕ of Ψ/s then satisfies the conclusions of the statement.
□

Lemma 10. *Let $\mathcal{E}_1$ and $\mathcal{E}$ be two open cone fields such that $\mathcal{E} \prec \bar{\mathcal{E}} \prec \mathcal{E}_1$. For each $A \subset M$, we have*

$$\mathcal{I}^+_{\mathcal{E}}(\bar{A}) \subset \mathcal{I}^+_{\mathcal{E}_1}(A).$$

Proof. It is enough to prove that $\mathcal{I}^+_{\mathcal{E}}(x) \subset \mathcal{I}^+_{\mathcal{E}_1}(A)$ for each $x \in \bar{A}$. Considering an $\mathcal{E}$-timelike curve $\gamma(t)$ satisfying $\gamma(0) = x$, it is enough to prove that $\gamma(t) \in \mathcal{I}^+_{\mathcal{E}_1}(A)$ for $t > 0$ small enough. Working in a chart at x, we can assume that $M = \mathbb{R}^d$ and $x = 0$. We consider an open cone Ω such that $\overline{\mathcal{E}(0)} \prec \Omega \prec \bar{\Omega} \prec \mathcal{E}_1(0)$. The inclusions $\mathcal{E}(y) \prec \Omega \prec \bar{\Omega} \prec \mathcal{E}_1(y)$ then also hold for all y in an open ball U centered at 0. Since 0 is in the closure of A, we deduce that $\Omega \cap U \subset \mathcal{I}^+_{\mathcal{E}_1}(A)$. On the other hand, we have $\gamma(t) \in \Omega$ for $t > 0$ small enough, hence $\gamma(t) \in \mathcal{I}^+_{\mathcal{E}_1}(A)$. $\square$

In the sequel we will need the notion of sums of convex cones or cone fields. The sum of a family of convex cones is defined as the convex envelop of their union. The sum of cone fields is defined pointwise.

Lemma 11. *The sum $\mathcal{E} = \sum_\alpha \mathcal{E}_\alpha$ of an arbitrary family of open cone fields is an open cone field.*

Proof. Let (x, v) be a point of $\mathcal{E}$. We can assume that $M = \mathbb{R}^d$ by working in a chart at x. The vector v belongs to the convex closure of the union $\cup_\alpha \mathcal{E}_\alpha(x)$, hence it is a finite sum of elements of this union: There exists a finite set J of indices such that $v = \sum_{i\in J} v_i$ with $v_i \in \mathcal{E}_i(x)$. Let $B_i \subset \mathcal{E}_i(x)$ be a compact neighborhood of v_i in $\mathbb{R}^d$. For each $i \in J$, there exists a neighborhood of x on which $B_i \subset \mathcal{E}_i(y)$. As a consequence, there exists a neighborhood U of x such that $B_i \subset \mathcal{E}_i(y)$ for each $y \in U$ and each $i \in J$. We conclude that $(\sum_i B_i) \times U \subset \mathcal{E}$. $\square$

The sum of two nonempty closed cones C and C' is equal to $\{v + v', v \in \Omega, v' \in \Omega'\}$.

It is not always true that the sum of two closed cones is closed. This is however true in the case where there exists an open half plane Q containing both.

Lemma 12. *Let Ω be an open cone and let C_i be finitely many closed cones such that $C_i \prec \Omega$. Then there exists an open cone Ω' such that $\sum C_i \prec \Omega' \prec \bar{\Omega}' \prec \Omega$.*

Proof. In the case where $\Omega = \mathbb{R}^d$, we can take $\Omega' = \Omega$. Otherwise we can assume that $\Omega \prec Q_0$ (the open upper half space). Each of the closed cones C_i then satisfies $C_i \prec Q_{s_i}$ for some $s_i > 0$. We can take $\Omega' = Q_s$ with $s = \min s_i$. $\square$

Lemma 13. *If $\mathcal{E}$ is an open enlargement of the closed cone field $\mathcal{C}$, then there exists an open cone field $\mathcal{E}'$ such that*

$$\mathcal{C} \prec \mathcal{E}' \prec \bar{\mathcal{E}}' \prec \mathcal{E}.$$

Proof. For each $x_0 \in M$, there exists a chart $\phi : B^{d-1} \times]-1, 1[\longrightarrow M$ at x_0 and an open cone $\Omega \subset \mathbb{R}^d$ such that $\phi^*\mathcal{C}(y, z) \prec \Omega \prec \bar{\Omega} \prec \phi^*\mathcal{E}(y, z)$ for each $(y, z) \in B^{d-1} \times]-1, 1[\longrightarrow M$. We take a locally finite covering of M by open sets U_i which are of the form $\phi_i(B^{d-1}(1/2) \times] - 1/2, 1/2[)$ for such charts, and denote by Ω_i the corresponding open cones. We consider the open cone fields $\mathcal{E}_i$ which are equal to $\phi_*\Omega_i$ on U_i and which are empty outside of U_i. The closure $\bar{\mathcal{E}}_i$ is the cone field equal to $\phi_*\bar{\Omega}_i$ on $\bar{U}_i$ and empty outside of $\bar{U}_i$. Then we consider the open cone field $\mathcal{E}' = \sum_i \mathcal{E}_i$. For each $x \in M$, there exists i such that $x \in U_i$, hence $\mathcal{C}(x) \prec \mathcal{E}_i(x) \prec \mathcal{E}'(x)$.

Let us now prove that $\bar{\mathcal{E}}'(x) \prec \mathcal{E}(x)$ for each $x \in M$. Let $J(x)$ be the finite set of indices such that x belongs to the closure of U_i. For each $i \in J(x)$, $\bar{\mathcal{E}}_i(x) \prec \mathcal{E}(x)$. Lemma 12 implies the existence of a convex open cone $\Omega' \subset T_xM$ such that

$$\bar{\mathcal{E}}_i(x) \prec \Omega' \prec \bar{\Omega}' \prec \mathcal{E}(x)$$

for each $i \in J(x)$. Extending locally Ω' to a continuous open cone field , we obtain that

$$\bar{\mathcal{E}}_i(y) \prec \Omega'(y) \prec \bar{\Omega}'(y) \prec \mathcal{E}(y)$$

for each $i \in J(x)$ and for y close to x. This implies that $\bar{\mathcal{E}}'(x) \prec \bar{\Omega}' \prec \mathcal{E}(x)$. □

Lemma 14. *There exists a sequence $\mathcal{E}_n$ of open cone fields which is strictly decreasing to $\mathcal{C}$, which means that $\bar{\mathcal{E}}_{n+1} \prec \mathcal{E}_n$ for each n and that $\mathcal{C} = \cap \mathcal{E}_n$. Such a sequence has the property that, for each open enlargement $\mathcal{E}$ of $\mathcal{C}$ and each compact set $K \subset M$, there exists n such that $\mathcal{E}_n(x) \prec \mathcal{E}(x)$ for each $x \in K$.*

PROOF. For each point $(x, v) \in TM - \mathcal{C}$, there exists an open enlargement $\mathcal{E}$ of $\mathcal{C}$ which is disjoint from a neighborhood U of (x, v). We can cover the complement of $\mathcal{C}$ in TM by a sequence U_i of open sets such that, for each i, there exists an open enlargement $\mathcal{E}'_i$ of $\mathcal{C}$ disjoint from U_i. We define inductively the open cone field $\mathcal{E}_n$ as an enlargement of $\mathcal{C}$ satisfying

$$\bar{\mathcal{E}}_n \prec \mathcal{E}'_n \cap \mathcal{E}_{n-1}.$$

It is obvious from the construction that $\mathcal{C} = \cap \mathcal{E}_n$. Finally, let $K \subset M$ be compact and $\mathcal{E}$ be an open enlargement of $\mathcal{C}$. For each $x \in K$, there exists n_x such that $\bar{\mathcal{E}}_{n_x}(x) \prec \mathcal{E}_{n_x - 1}(x) \prec \mathcal{E}(x)$. Then the inclusion $\bar{\mathcal{E}}_{n_x}(y) \prec \mathcal{E}(y)$ holds on an open neighborhood of x. We can cover K by finitely many such open sets, hence $\mathcal{E}_n(y) \prec \mathcal{E}(y)$ for each $y \in K$ when n is large enough. □

1.2 Clarke differential, causal and timelike curves

We will use the notion of Clarke differential of curves and functions, see [10] for example.

The Clarke differential of a Lipschitz function $f : \mathbb{R} \longrightarrow \mathbb{R}$ at a given point x is the compact interval

$$\partial f(x) = \left[\liminf_{y_2 \to x, y_1 \to x, y_2 > y_1} \frac{f(y_2) - f(y_1)}{y_2 - y_1}, \limsup_{y_2 \to x, y_1 \to x, y_2 > y_1} \frac{f(y_2) - f(y_1)}{y_2 - y_1} \right].$$

The interval $\partial f(x) = [p_-, p_+]$ can be characterized in the following way: for $p < p_-$, the function $t \longmapsto f(t) - pt$ is increasing near $t = x$, it is decreasing for $p > p_+$, and it is not monotone in any neighborhood of x for $p \in]p_-, p_+[$.

The Clarke differential of a Lipschitz curve $\gamma : \mathbb{R} \longrightarrow M$ at a given time t is the compact convex subset $\partial \gamma(t) \subset T_{\gamma(t)} M$ defined as the convex hull of limit points of sequences of the form $(\gamma(t_n), \dot{\gamma}(t_n))$ in TM, where t_n is a sequence of differentiability points of γ, see [10]. It satisfies the equality

$$df_{\gamma(t)} \cdot \partial \gamma(t) = \partial (f \circ \gamma)(t)$$

for each smooth function f, and this characterizes $\partial \gamma(t)$. In other words, $v \in \partial \gamma(t)$ if and only if $df_{\gamma(t)} \cdot v \in \partial (f \circ \gamma)(t)$ for each smooth function f.

Lemma 15. *Given a closed cone field $\mathcal{C}$ on M, the following statements are equivalent for a locally Lipschitz curve $\gamma : I \longrightarrow M$:*

- *$\dot{\gamma}(t) \in \mathcal{C}(\gamma(t))$ for almost every $t \in I$.*
- *$\partial \gamma(t) \subset \mathcal{C}(\gamma(t))$ for each $t \in I$.*
- *For each $t \in I$ and each local time function τ at $\gamma(t)$, the function $\tau \circ \gamma$ is non decreasing in a neighborhood of t.*

We call the corresponding Lipschitz curves $\mathcal{C}$-causal.

PROOF. The second point implies the first one since $\gamma'(s)$ exists almost everywhere, and then is contained in $\partial\gamma(s)$.

Assume the first point. For each $t \in I$ and each local time function τ at $\gamma(t)$, we consider a neighborhood of $\gamma(t)$ such that τ is a regular Lyapounov function on U. We have $\gamma(s) \in U$ for s close to t. Then, for almost every point s in a neighborhood of t, the derivative $(\tau \circ \gamma)'(s) = d\tau_{\gamma(s)} \cdot \gamma'(s)$ exists and it is non negative. This implies that the Lipschitz function $\tau \circ \gamma$ is non decreasing near t.

If the second point does not hold at some time t, then there exists $w \in \partial\gamma(t)$ and a local time function τ at $\gamma(t)$ such that $d\tau_{\gamma(t)} \cdot w < 0$. This implies that $\partial(\tau \circ \gamma)(t)$ contains a negative value, hence that $\tau \circ \gamma$ is not non decreasing near t. $\square$

If γ is a Lipschitz curve and g a Lipschitz function, then we have the chain rule (see [10], Theorem 2.3.9)

$$\partial(g \circ \gamma)(t_0) \subset [\inf_{p,v} p \cdot v, \sup_{p,v} p \cdot v]$$

where the sup and inf are taken on $p \in \partial g(\gamma(t_0)), v \in \partial\gamma(t_0)$. This inclusion is an equality if γ or g are smooth, but may be strict in general.

1.3 Limit curve Lemma

Let us consider a sequence $\mathcal{E}_n$ of open cone fields strictly decreasing to $\mathcal{C}$.

Lemma 16. *Let $\gamma_n : I \longrightarrow M$ be an equi-Lipschitz sequence of $\mathcal{E}_n$-timelike curves converging to $\gamma : I \longrightarrow M$ uniformly on compact subintervals of I, then γ is $\mathcal{C}$-causal.*

PROOF. Note first that γ is Lipschitz. Let $t \in I$ be given, and let τ be a local time function at $\gamma(t)$. In view of Lemma 15, it is enough to prove that $\tau \circ \gamma$ is non decreasing near t.

Let U be a compact neighborhood of $\gamma(t)$ such that τ is a non degenerate Lyapounov function on U. Then τ is still a non degenerate Lyapounov function on U for the closed cone field $\bar{\mathcal{E}}_n$ for $n \geqslant n_0$. There exists a neighborhood J of t and $n_1 \geqslant n_0$ such that $\gamma_n(s) \in U$ for each $s \in J$, $n \geqslant n_1$. These properties imply that $\tau \circ \gamma_n$ is non decreasing on J provided $n \geqslant n_1$. At the limit, we deduce that $\tau \circ \gamma$ is non decreasing on J. $\square$

It is useful to control the length of the limit curve:

Lemma 17. *Let $\mathcal{C}$ be a closed cone field and $\gamma : [0, 1] \longrightarrow M$ be a $\mathcal{C}$-causal curve which does not contain any singular point of $\mathcal{C}$. There exists $L > 0$ such that:*

For each $T \in]0, 1[$, there exists $\epsilon > 0$ and an open enlargement $\mathcal{E}$ of $\mathcal{C}$ such that each $\mathcal{E}$-timelike curve $\eta : [0, T] \longrightarrow M$ satisfying $d(\gamma(t), \eta(t)) \leqslant \epsilon$ for each $t \in [0, T]$ has a length less than L.

PROOF. Let $\mathcal{E}_n$ be a sequence of open cone fields strictly decreasing to $\mathcal{C}$, and let γ be as in the statement. We denote by $\ell(\gamma)$ the length of a curve.

We cover the image of γ by finitely many bounded open sets $U_1, \ldots, U_k$ each of which has the property that there exists a time function τ_i on an open neighborhood V_i of $\bar{U}_i$, which satisfies $|v|_x/2\delta_i \geqslant d(\tau_i)_x \cdot v \geqslant 2\delta_i|v|_x$ for some $\delta_i > 0$, and for each $v \in \mathcal{C}(x), x \in U_i$. We set $\delta := \min \delta_i$ and prove the statement with $L = (1 + \ell(\gamma)/\delta)/\delta$. We consider a sequence $\eta_n : [0, 1[\longrightarrow M$ of $\mathcal{E}_n$-timelike curves converging, uniformly on compact subsets of $[0, 1[$, to γ. We have to prove that $\ell(\eta_{n|[0,T]}) \leqslant L$ for n large enough.

Given $T \in]0,1[$, there exists a finite increasing sequence of times $0 = t_0 < t_1 < \cdots < t_N = T$ such that $\gamma_{[t_j,t_{j+1}]}$ is contained in one of the open sets U_i for each j. Then for n large enough, this is also true for $\eta_{n|[t_j,t_{j+1}]}$ and $|\tau_i(\eta(t)) - \tau_i(\gamma(t))| \leq \frac{1}{2N}$. We obtain, for n large enough :

$$\begin{aligned}\delta\ell(\eta_{n|[t_j,t_{j+1}]}) &\leqslant \tau_i(\eta_n(t_{j+1})) - \tau_i(\eta_n(t_j)) \leqslant \tau_i(\gamma(t_{j+1})) - \tau_i(\gamma(t_j)) + 1/N \\ &\leqslant 1/N + \ell(\gamma_{|[t_j,t_{j+1}]})/\delta.\end{aligned}$$

Taking the sum, we obtain that the inequality

$$\delta\ell(\eta_{n|[0,T]}) \leqslant 1 + \ell(\gamma)/\delta$$

holds for n large enough, which ends the proof. □

Proposition 18. *Let $\gamma_n : [0,a_n[\longrightarrow M$ be a sequence of $\mathcal{E}_n$-timelike curves parametrized by arclength, such that $\gamma_n(0)$ is bounded and $a_n \longrightarrow \infty$. Then along a subsequence, the sequence γ_n converges, uniformly on compact intervals of $[0,\infty)$, to a limit $\gamma : [0,\infty) \longrightarrow M$ which is $\mathcal{C}$-causal and complete.*

Proof. Since the curves γ_n are 1-Lipschitz, Ascoli Arzela's Theorem gives, for each $T > 0$, the existence of a subsequence along which γ_n converge uniformly on $[0,T]$. By a diagonal extraction, we get a subsequence along which γ_n converge uniformly on compact intervals. By Lemma 16, the limit γ is $\mathcal{C}$-causal. Let us prove that this limit is complete. If it was not complete, it would have finite length and a regular limit y at infinity. Since the set of regular points is open, there would exit $T > 0$ such that $\gamma([T,\infty])$ contains only regular points. We could reparameterize γ on $[T,\infty)$ to a curve $\tilde{\gamma} = \gamma \circ \lambda : [0,1[\longrightarrow M$, and extend $\tilde{\gamma}$ to a causal curve $\tilde{\gamma} : [0,1] \longrightarrow M$. Lemma 17, applied to the causal curve $\tilde{\gamma}$ and the sequence $\tilde{\gamma}_n = \gamma_n \circ \lambda$, gives $L > 0$ such that, for each $S \in]0,1[$, the curve $\tilde{\gamma}_{n|[0,S]}$ has length less than L for n large enough. Observing that $\ell(\tilde{\gamma}_{n|[0,S]}) = \lambda(S) - T$, this would imply that $\lambda(S) \leqslant T + L$ for each $S \in]0,1[$. This is a contradiction since λ maps $[0,1[$ onto $[T,\infty)$. □

Corollary 19. *For each $x \in \mathcal{R}_{\mathcal{C}}$, there exists a complete causal curve γ passing through x.*

Proof. Let $\mathcal{E}_n$ be a sequence of open cone fields strictly decreasing to $\mathcal{C}$. For each n, there exists a closed $\mathcal{E}_n$-timelike curve passing through x, that we see as a periodic $\mathcal{E}_n$-timelike curve $\gamma_n : \mathbb{R} \longrightarrow M$ satisfying $\gamma_n(0) = x$. The curve γ_n is periodic and not constant, hence it has infinite length. At the limit, we obtain a complete causal curve passing through x. □

The same method also yields:

Corollary 20. *Let $Y \subset K$ be two compact sets. If $\mathcal{J}^+_{\mathcal{C}}(Y)$ is contained in the interior of K, and $\mathcal{F}^+_{\mathcal{C}}(Y)$ is not contained in K, then there exists a backward complete causal curve $\gamma :]-T,0] \longrightarrow M$ contained in K and such that $\gamma(0) \in \partial K$.*

Proof. For each n, there exists an $\mathcal{E}_n$-timelike curve $\gamma_n :]-T_n, 0] \longrightarrow K$ such that $\gamma_n(0) \in \partial K$ and $\gamma_n(-T_n) \in Y$, parametrized by arclength. If the sequence T_n was bounded, then at the limit we would obtain a $\mathcal{C}$-causal curve joining a point of Y to a point of ∂K, which contradicts the hypothesis that $\mathcal{J}^+_{\mathcal{C}}(Y)$ is contained in the interior of K. We deduce that T_n is unbounded, and at the limit we obtain the desired backward complete causal curve. □

2 Direct Lyapounov theory

We consider a closed cone field $\mathcal{C}$ and explain how to deduce information about stable causality from the existence of appropriate smooth Lyapounov functions. More precisely we prove the following parts of Theorem 1, and some variations:

- If there exists a smooth Lyapounov function τ such that $\tau(x') < \tau(x)$, then $x' \notin \mathcal{F}^+_{\mathcal{C}}(x)$.
- If there exists a smooth Lyapounov function τ such that $d\tau_x \neq 0$, then $x \notin \mathcal{R}_{\mathcal{C}}$.

We say that the open set A is a *trapping domain* for the open cone field $\mathcal{E}$ if $\mathcal{I}^+_{\mathcal{E}}(A) \subset A$. We say that A is a trapping domain for the closed cone field $\mathcal{C}$ if it is a trapping domain for some open enlargement $\mathcal{E}$ of $\mathcal{C}$.

Lemma 21. *If A is a trapping domain for $\mathcal{C}$, then there exists an enlargement $\mathcal{E}$ of $\mathcal{C}$ such that $\mathcal{I}^+_{\mathcal{E}}(\bar{A}) \subset A$, in particular, $\mathcal{F}^+_{\mathcal{C}}(\bar{A}) \subset A$.*

PROOF. Let $\mathcal{E}$ be an open enlargement of $\mathcal{C}$ such that $\mathcal{I}^+_{\mathcal{E}}(A) \subset A$ and let $\mathcal{E}_1$ be an open cone field such that $\mathcal{C} \prec \mathcal{E}_1 \prec \bar{\mathcal{E}}_1 \prec \mathcal{E}$. By Lemma 10, $\mathcal{I}^+_{\mathcal{E}_1}(\bar{A}) \subset \mathcal{I}^+_{\mathcal{E}}(A) \subset A$. $\square$

Lemma 22. *Let f be a C^1 function, and $a \in \mathbb{R}$. If the inequality $df_x \cdot v > 0$ holds for each $x \in f^{-1}(a)$ and $v \in \mathcal{C}(x) - \{0\}$, then $\{f > a\}$ is a trapping domain.*

In particular, if a is a regular value of the smooth Lyapounov function τ, then $\{\tau > a\}$ is a trapping domain.

PROOF. Let us consider the open cone field $\mathcal{E}$ defined by $\mathcal{E}(x) = T_xM$ if $f(x) \neq a$ and $\mathcal{E}(x) = \{v \in T_xM : df_x \cdot v > 0\}$ if $f(x) = a$. Our hypothesis on f is that $\mathcal{E}$ in an enlargement of $\mathcal{C}$. If $\gamma(t)$ is an $\mathcal{E}$-timelike curve, then $f \circ \gamma$ is increasing near each time t such that $\gamma(t) = a$. As a consequence, if $f \circ \gamma(t) > a$, then $f \circ \gamma(s) > a$ for each $s > t$. This implies that $\{f > a\}$ is a trapping domain. $\square$

Corollary 23. *Let τ be a Lyapounov function. If $\tau(x') < \tau(x)$, then $x' \notin \mathcal{F}^+_{\mathcal{C}}(x)$.*

PROOF. Let $a \in]\tau(x'), \tau(x)[$ be a regular value (there exists one by Sard's theorem). We have $\mathcal{F}^+_{\mathcal{C}}(x) \subset \mathcal{F}^+_{\mathcal{C}}(\{\tau > a\}) \subset \{\tau > a\}$. $\square$

Lemma 24. *Let τ be a smooth Lyapounov function and x a regular point of τ. Then there exists a smooth Lyapounov function $\tilde{\tau}$ which has the same critical set as τ, and such that $\tilde{\tau}(x)$ is a regular value of $\tilde{\tau}$. This implies that x is not stably recurrent.*

PROOF. Given a neighborhood U of x on which τ is regular, let f be a smooth function supported in U and such that $f(x) = 1$. For $\delta > 0$ small enough, the function $\tau + sf$ is a smooth Lyapounov function, which is regular on U for each $s \in] - \delta, \delta[$. The interval $]\tau(x) - \delta, \tau(x) + \delta[$ contains a regular value a of τ. The function $\tilde{\tau} := \tau + (a - \tau(x))f$ is a smooth Lyapounov function which is regular on U. The number $a := \tilde{\tau}(x)$ is a regular value of $\tilde{\tau}$: If $\tilde{\tau}(y) = a$, then either $y \in U$ and then $d\tilde{\tau}_y \neq 0$ or y does not belong to the support of f, and then $d\tilde{\tau}_y = d\tau_y \neq 0$ since a is a regular value of τ. $\square$

3 Smoothing

The goal of the present section is to prove the following regularization statement, which is one of our main technical tools to prove the existence of Lyapounov functions. We work with a fixed cone field $\mathcal{C}$ on the manifold M.

Proposition 25. *Let A_0 be a trapping domain, let F_i be a closed set contained in A_0, let F_e be a closed set disjoint from $\bar{A}_0$, and let θ_0 be a point in the boundary of A_0.*

Then there exists a smooth (near $\mathcal{D}(\mathcal{C})$) trapping domain A_0' which contains F_i, whose boundary contains θ_0, and whose closure is disjoint from F_e.

3.1 Local properties of trapping domains

We say that the closed cone field $\mathcal{C}$ is *strictly entering* A at $x \in \partial A$ if there exists an open cone field $\mathcal{E}$ which contains $\{x\} \times \mathcal{C}(x)$ and such that $\mathcal{I}^+_{\mathcal{E}}(A) \subset A$. The cone field $\mathcal{E}$ may have degenerate points. Given an open neighborhood U of x, then $\mathcal{C}$ is strictly entering A at x if and only if it is strictly entering $U \cap A$ at x.

Lemma 26. *The open set A is a trapping domain for $\mathcal{C}$ if and only if $\mathcal{C}$ is strictly entering A at each point $x \in \partial A$.*

Proof. If A is a trapping domain, then there exists an open enlargement $\mathcal{E}$ of $\mathcal{C}$ such that $\mathcal{I}^+_{\mathcal{E}}(A) \subset A$. This implies that $\mathcal{C}$ is strictly entering A at each point of ∂A.

Let us now prove the converse. For each point $x \in \partial A$, there exists an open cone field $\mathcal{E}_x$ such that $\mathcal{C}(x) \subset \mathcal{E}_x(x)$ and $\mathcal{I}^+_{\mathcal{E}_x}(A) \subset A$. The inclusion $\mathcal{C}(y) \subset \mathcal{E}_x(y)$ then holds for all y in an open neighborhood U_x of x in ∂A. We consider a sequence x_i such that the open sets U_{x_i} form a locally finite covering of ∂A. For each $x \in \partial A$, we denote by $J(x)$ the finite set of indices such that $x \in \bar{U}_{x_i}$. Since the covering is locally finite, there exists a neighborhood V of x in ∂A which is disjoint from U_{x_i} for each $i \notin J(x)$. We define, for each $x \in \partial A$, the open cone $\mathcal{E}(x) := \bigcap_{i \in J(x)} \mathcal{E}_{x_i}(x)$. For $x \notin \partial A$, we set $\mathcal{E}(x) = T_xM$. We claim that $\mathcal{E} := \bigcup_{x \in M}\{x\} \times \mathcal{E}(x)$ is an open cone field. Indeed, for each $x \in A$, the intersection $\bigcap_{i \in J(x)} \mathcal{E}_{x_i}$ is an open cone field which is contained in $\mathcal{E}$ in a neighborhood of x, and equal to $\mathcal{E}$ at x.

By construction, $\mathcal{E}$ is an enlargement of $\mathcal{C}$. Let us verify that $\mathcal{I}^+_{\mathcal{E}}(A) \subset A$. If not, there exists a $\mathcal{E}$-timelike curve γ such that $\gamma(t) \in A$ on $[0, T[$ and $\gamma(T) \in \partial A$. We have $\dot{\gamma}(T) \in \mathcal{E}(\gamma(T)) \subset \mathcal{E}_{x_i}(T))$ for some i (any i such that $\gamma(T) \in U_{x_i}$). For this fixed i, the curve $t \mapsto \gamma(t)$ is then $\mathcal{E}_{x_i}$-timelike on $[S, T[$ for some $S < T$. This contradicts the inclusion $\mathcal{I}^+_{\mathcal{E}_{x_i}}(A) \subset A$. □

Let $\mathcal{E}$ be an open cone field, and A be a trapping domain for $\bar{\mathcal{E}}$. Then $\bar{\mathcal{E}}_x$ is regular for all $x \in \partial A$. Thus at each point $x \in \partial A$, there exists a chart $\phi : B^{d-1}(2)\times]-2, 2[\longrightarrow M$ which sends $(0, 0)$ to x and has the property that

$$Q_1 \prec \phi^*\bar{\mathcal{E}}(y, z) \prec Q_0$$

for all $(y, z) \in B^{d-1}(2)\times]-2, 2[$. We recall that $B^d(r)$ is the open ball of radius r centered at 0 in $\mathbb{R}^d$ and that $Q_s, s \geqslant 0$ is the open cone $Q_s = \{(y, z) \in \mathbb{R}^{d-1} \times \mathbb{R} : z > s|y|\} \subset \mathbb{R}^d$.

Lemma 27. *There exists a 1-Lipschitz function $g : B^{d-1}(2) \longrightarrow]-2, 2[$ such that $\phi^{-1}(A)$ is the open epigraph $\{z > g(y)\}$, hence $\phi^{-1}(\partial A)$ is the graph of g. Note that $g(0) = 0$.*

Proof. Let us define the function $g(y) = \inf\{z \in]-2, 2[: \phi_i(y, z) \in A\}$ (we decide to set $g(y) = 2$ if the infimum is taken on the empty set). Then $\phi(y, g(y)) \in \bar{A}$ for each $y \in B^{d-1}(2)$. The curve $t \longmapsto \phi(y, z+t)$ is $\bar{\mathcal{E}}$-causal hence the set $\{(y, z) : z < g(y) < 2\}$ is contained in A. Furthermore, since $Q_1 \prec \phi^*\bar{\mathcal{E}}$, the curve $\phi_i(y + tv, g(y) + t)$ is $\bar{\mathcal{E}}$-causal for each $y \in B^{d-1}(1)$ and

$v \in \bar{B}^{d-1}(1)$. This implies that g is 1-Lipschitz. $\square$

Let $\mathcal{C}$ be a closed cone field on $\mathbb{R}^{d-1} \times \mathbb{R}$, and let $A \subset \mathbb{R}^{d-1} \times \mathbb{R}$ be a trapping domain which is the open epigraph of the Lipschitz function $g : \mathbb{R}^{d-1} \longrightarrow \mathbb{R}$. For each point $x = (y, g(y))$ in the boundary of A, we denote by $\mathcal{C}^\circ(x)$ the open subset of $(\mathbb{R}^{d-1})^*$ formed by linear forms p such that $p \cdot v_y < v_z$ for each $(v_y, v_z) \in \mathcal{C}(x) - \{0\}$. The set $\mathcal{C}^\circ := \{(y, p) : p \in \mathcal{C}^\circ(y, g(y))\}$ is open in $\mathbb{R}^{d-1} \times (\mathbb{R}^{d-1})^*$.

Lemma 28. *For each point $x = (y, g(y))$ of ∂A, the following statements are equivalent:*

- *The cone field $\mathcal{C}$ is strictly entering A at x,*
- *The Clarke differential $\partial g(y)$ is contained in $\mathcal{C}^\circ(x)$.*

Proof. If $\partial g(y)$ is not contained in $\mathcal{C}^\circ(x)$, then there exists $p \in \partial g(y)$ and $(v_y, v_z) \in \mathcal{C}(x) - \{0\}$ such that $p \cdot v_y \geqslant v_z$. Then for each open cone field $\mathcal{E}$ containing $\{x\} \times \mathcal{C}(x)$, there exists a smooth timelike curve $\gamma = (\gamma_y, \gamma_z)$ such that $\dot{\gamma}_z(0) < p \cdot \dot{\gamma}_y(0)$. This implies that the function $g \circ \gamma_y(t) - \gamma_z(t)$ is increasing near $t = 0$, hence $\mathcal{I}^+_{\mathcal{E}}(A)$ is not contained in A. We conclude that the cone $\mathcal{C}$ is not strictly entering at x.

Conversely, if $\partial g(y) \subset \mathcal{C}^\circ(x)$, then we consider the cone $\Omega = \{(v_y, v_z) : v_z > p \cdot v_y, \forall p \in \partial g(y)\}$. Since $\partial g(y)$ is compact, this is an open cone. We consider an open cone Ω_1 such that

$$\mathcal{C}(x) \prec \Omega_1 \prec \bar{\Omega}_1 \prec \Omega.$$

In view of the semi-continuity of the Clarke differential, there exists an open neighborhood U of y in $\mathbb{R}^{d-1}$ such that the inequality $v_z > \sup_{p \in \partial g(y')} p \cdot v_y$ holds for each $(v_y, v_z) \in \Omega_1$ and each $y' \in U$. We consider the open cone field $\mathcal{E}$ which is equal to Ω_1 on $U \times \mathbb{R}$ and empty outside, and prove that $\mathcal{I}^+_{\mathcal{E}}(A) \subset A$. Otherwise, there exists a smooth curve $\gamma = (\gamma_y, \gamma_z)$, which is timelike for $\mathcal{E}$, and such that $\gamma_z(T) = g(\gamma_y(T))$ and $\gamma_z(t) > g(\gamma_y(t))$ for each $t \in [0, T[$. Then, we have $\gamma_y(T) \in U$ and $(\dot{\gamma}_y(T), \dot{\gamma}_z(T)) \in \Omega_1$ hence $\dot{\gamma}_z(T) > \sup_{p \in \partial g(\gamma_y(T))} p \cdot \dot{\gamma}_y(T)$. This implies that the function $\gamma_z(t) - g(\gamma_y(t))$ is increasing near $t = T$, a contradiction. $\square$

3.2 De Rham Smoothing

Proposition 29. *For each Lipschitz function $g : \mathbb{R}^d \longrightarrow \mathbb{R}$, there exists a family $g_s, s > 0$ of Lipschitz functions on $\mathbb{R}^d$ which converge uniformly to g as $s \longrightarrow 0$ and such that :*

- *g_s is smooth on $B^d(1)$ for each $s > 0$, and moreover g_s is smooth on any open subset $O \subset \mathbb{R}^d$ where g is already smooth.*
- $\limsup_{s \longrightarrow 0}(\mathrm{Lip}\ g_s) \leqslant \mathrm{Lip}\ g$.
- *If $V \subset \mathbb{R}^d \times (\mathbb{R}^d)^*$ is an open set containing the graph ∂g of the Clarke differential of g, then V contains the graph of ∂g_s for s small enough.*

If $y_1, \ldots, y_N$ are finitely many points in $B^d(1)$, then we can assume in addition that $g_s(y_i) = g(y_i)$ for each $i = 1, \ldots, N$ and each $s > 0$.

Proof. We use De Rham smoothing procedure. We follow the notations of [5], Lemma A.1. There exists a smooth action $a(y, x)$ of $\mathbb{R}^d$ on itself (meaning that $a(y, a(y', x)) = a(y + y', x)$) such that :

- $a(y, x) = x$ for each $y \in \mathbb{R}^d$ and $x \in \mathbb{R}^d - B^d(1)$

- The action of $\mathbb{R}^d$ on $B^d(1)$ is conjugated to the standard action of $\mathbb{R}^d$ on itself by translations (there exists a diffeomorphism $\varphi : B^d(1) \longrightarrow \mathbb{R}^d$ such that $a(y, \varphi(x)) = y + \varphi(x)$).
- The diffeomorphisms a_y converge to the identity C^1-uniformly for $y \longrightarrow 0$.

Given a Lipschitz function $g : \mathbb{R}^d \longrightarrow \mathbb{R}$, we define

$$g_s(x) := \int_{\mathbb{R}^d} s^{-d} g(a(y,x))\rho(-y/s)dy$$

where ρ is a mollification kernel supported in $B^d(1)$. The two first points of the Proposition are proved, for example, in [5], Lemma A.1. Let us now prove the last point.

We cover the compact set $\bar{B}^d(1)$ by finitely many balls B_i each of which has the following property: There exists convex open sets W_i and V_i in $(\mathbb{R}^d)^*$ such that $\partial g(x) \subset W_i \subset \bar{W}_i \subset V_i \subset V(x)$ for each $x \in 2B_i$ (the ball of same center and double radius).

For each i, we define $n_i(v) := \sup_{p \in W_i} p \cdot v$ and $m_i(v) = \sup_{p \in V_i} p \cdot v$ which are convex and positively one-homogeneous (hence subadditive) functions. There exists $\delta > 0$ such that $m_i(v) \geqslant n_i(v) + \delta|v|$. Note that V_i (resp. W_i) is precisely the set of linear forms p satisfying $p \cdot v \leqslant m_i(v)$ (resp. $n_i(v)$) for each v . The function g is n_i-Lipschitz on $2B_i$, which means that

$$g(x') - g(x) \leqslant n_i(x' - x)$$

for each x and x' in $2B_i$. Since the diffeomorphisms a_y converge to the identity C^1-uniformly as $y \longrightarrow 0$, we have

$$\left|a(y,x') - a(y,x) - x' + x\right| = \left|\int_0^1 (\partial_x a(y, x + t(x'-x)) - Id) \cdot (x'-x)dt\right| \leqslant \delta(|y|)|x' - x|$$

with a function δ converging to 0 at 0. For s small enough, we have $a(y,x) \in 2B_i$ for each $x \in B_i$ and $|y| \leqslant s$, and $\delta(s) < \delta$. We then obtain, for x and x' in B_i,

$$\begin{aligned}
|g_s(x') - g_s(x)| &\leqslant \int s^{-d} \big|g \circ a_y(x') - g \circ a_y(x)\big| \rho(-y/s)dy \\
&\leqslant \int s^{-d} n_i\big(a_y(x') - a_y(x)\big)\rho(-y/s)dy \\
&\leqslant \int s^{-d} m_i(x' - x)\rho(-y/s)dy = m_i(x' - x).
\end{aligned}$$

This implies that $dg_s(x) \cdot v \leqslant m_i(v)$ for each v, hence that $\partial g_s(x) \in V_i \subset V(x)$ for each $x \in B_i$. Since the covering B_i is finite, this inclusion holds for all $x \in \mathbb{R}^d$ provided s is small enough.

The function g_s constructed so far does not necessarily satisfy the additional conditions $g_s(y_i) = g(y_i)$. We thus consider the modified function

$$\tilde{g}_s(x) = g_s(x) + \sum_{i=1}^{N} (g(y_i) - g_s(y_i))h_i(x),$$

where $h_i, 1 \leqslant i \leqslant N$ are non negative smooth function supported on $B^d(1)$ and satisfying $h_i(y_i) = 1$ and $h_i(y_j) = 0$ for $j \neq i$. This modified family of functions satisfies the three points of the statements since $g_s(y_i) \longrightarrow g(y_i)$ for each i, and

$$\partial \tilde{g}_s(x) = \partial g_s(x) + \sum_i (g(y_i) - g_s(y_i))dh_x$$

for each x. □

3.3 Proof of Proposition 25

We first give the proof under the assumption that $\mathcal{D}(\mathcal{C}) = M$. Since A_0 is also a trapping domain for some open enlargement $\mathcal{E}$ of $\mathcal{C}$, we can assume without loss of generality that $\mathcal{C}$ is the closure of a non degenerate open cone field.

We consider a locally finite covering of ∂A_0 by domains

$$U_k(1) = \phi_k(B^{d-1}(1)\times]-1,1[)$$

associated to charts $\phi_k : B^{d-1}(2)\times]-2,2[\longrightarrow M, k \geqslant 1$ which have the property that

$$Q_1 \prec \phi_k^*\mathcal{C}(y,z) \prec Q_0$$

for all $(y,z) \in B^{d-1}(2)\times]-2,2[$. Locally finite means that each point $x \in M$ has a neighborhood which intersects only finitely many of the sets $U_k(1)$. We denote by x_k the points $\phi_k(0)$, $k \geqslant 1$ and set $x_0 = \theta_0$. We moreover assume that the open sets $U_k(2) := \phi_k(B^{d-1}(2)\times]-2,2[)$ are all disjoint from F_i and F_e.

By Lemma 27, the open set $\phi_1^{-1}(A)$ is the epigraph of a 1-Lipschitz function $f_1 : B^{d-1}(2) \longrightarrow]-2,2[$ such that $f_1(0) = 0$. The bounded set $U_1(1)$ contains finitely many of the points x_i. We denote by $y_1,\ldots,y_N$ the first component of the preimages of these points. So those of the points x_i which are contained in $U_1(1)$ are $\phi_1(y_1,f_1(y_1)),\ldots,\phi_1(y_N,f_1(y_N))$.

As in Lemma 28, we define $\mathcal{C}^\circ \subset B^{d-1}(2)\times(\mathbb{R}^{d-1})^*$ by

$$\mathcal{C}^\circ := \{(y,p) : p \in \mathcal{C}^\circ(y,f_1(y))\}.$$

Since A_0 is a trapping domain for $\mathcal{C}$, the set $\mathcal{C}^\circ$ contains the graph ∂f_1 of the Clarke differential of f_1, by Lemma 28.

By Proposition 29, there exists a function $g_1 : B^{d-1}(2) \longrightarrow \mathbb{R}$ which is 1-Lipschitz, smooth on $B^{d-1}(1)$, and satisfies $\partial g_1(y) \subset \mathcal{C}^0(y)$ for each $y \in B^{d-1}(2)$, and $g_1(y_j) = f_1(y_j)$ for $j = 1,\ldots,N$. In particular, $g_1(0) = 0$, hence g_1 takes vales in $]-2,2[$.

Let A_1 be the open set such that $A_1 \cap (M - U_1(1)) = A_0 \cap (M - U_1(1))$ and such that $\phi_1^{-1}(A_1)$ is the open epigraph of g_1. The cone field $\mathcal{C}$ is strictly entering A_1 at each point x of the boundary ∂A_1. Indeed, such a point x either belongs to $\partial A_0 \cap (M - \bar{U}_1(1))$, and then $A_1 = A_0$ near x, or it is of the form $\phi_1(y,g_1(y))$ for some $y \in B^{d-1}(2)$. In this second case, the conclusion follows from the inclusion $\partial g_1(y) \subset \mathcal{C}^0(y)$, by Lemma 28. We deduce by Lemma 26 that A_1 is a trapping domain for $\mathcal{C}$.

By the same method, we build inductively a sequence $A_m, m \geqslant 0$ of trapping domains which have the following properties:

- ∂A_m contains all the points x_k (hence the point θ_0), F_i is contained in A_m and F_e is disjoint from $\bar{A}_m$.
- The boundary ∂A_m is contained in $\bigcup_{k\geqslant 1} U_k(1)$, and its intersection with $\bigcup_{m\geqslant k\geqslant 1} U_k(1)$ is a smooth hypersurface.
- The symmetric difference between A_m and A_{m-1} is contained in $U_m(1)$.

We denote by $A_0' := \liminf A_m$ the set of points x which belong to all but finitely many of the sets A_m. We claim that A_0' satisfies the conclusions of Proposition 25.

Since the covering $U_k(1)$ is locally finite, the intersection $A_m \cap K$ stabilizes to $A_0' \cap K$ for each compact K, *i. e.* $K \cap A_m = K \cap A_0'$ for all m large enough.

This implies that A_0' is open, and that $\partial(A_0') = \liminf \partial(A_m)$. This boundary is smooth, contains all the points x_k, and is contained in U.

To prove that A_0' is a trapping domain, it is enough to observe that the cone field $\mathcal{C}$ is strictly entering A_0' at each point $x \in \partial A_0'$. Since the sequence A_k stabilizes in a neighborhood of x, this follows from the fact that each of the open sets A_k is attracting.

This ends the proof of Proposition 25 under the assumption that $\mathcal{D}(\mathcal{C}) = M$.

If we do not make this assumption we consider an enlargement $\mathcal{E}$ of $\mathcal{C}$ such that A_0 is a trapping domain for $\bar{\mathcal{E}}$. We can apply the result just proved on the manifold $\mathcal{D}(\mathcal{E})$, to the cone field $\bar{\mathcal{E}}$. We deduce the existence of a smooth trapping region A_0' for $\bar{\mathcal{E}}$ in $\mathcal{D}(\mathcal{E})$ which contains $F_i \cap \mathcal{D}(\mathcal{E})$, is disjoint from $F_e \cap \mathcal{D}(\mathcal{E})$, and whose boundary contains θ_0. Let O be an open subset of M which contains F_i and whose closure is disjoint from F_e, and let $Z \subset \mathcal{D}(\mathcal{E})$ be a closed neighborhood of $\mathcal{D}(\mathcal{C})$ in M. The open set $A_0' \cup ((M - Z) \cap O)$ then satisfies the conclusions of Proposition 25. □

4 Existence of Lyapounov functions

We consider in this section a closed cone field $\mathcal{C}$ and prove several existence results for Lyapounov functions, in particular Theorems 1, 2 and 3.

4.1 Smooth trapping domains and Lyapounov functions

We associate smooth Lyapounov functions to smooth trapping domains:

Proposition 30. *Let A be smooth trapping domain, then there exists a smooth Lyapounov function τ such that $A = \{\tau > 0\}$ and 0 is a regular value of A (hence $\partial A = \{\tau = 0\}$).*

If F_i and F_e are closed sets contained in A and disjoint from $\bar{A}$, respectively, we can moreover impose that $\tau = 1$ on F_i and $\tau = -1$ on F_e.

Proof. We consider a collar of ∂A in the manifold $M - (F_e \cup F_i)$, that is a smooth embedding $\psi : H \times \mathbb{R} \longrightarrow M - (F_e \cup F_i)$ such that $\psi(H \times \{0\}) = \partial A$ and $\psi^{-1}(A) = H \times]0, \infty)$. We will prove the existence of a Lyapounov function $\tilde{\tau}$ on $H \times \mathbb{R}$ for the cone field $\psi^* \mathcal{C}$, which has the following properties:

- $\tilde{\tau} = 0$ on $H \times \{0\}$ and 0 is a regular value of $\tilde{\tau}$.
- $\tilde{\tau} = 1$ on $H \times [1, \infty)$ and $\tilde{\tau} = -1$ on $H \times (-\infty, -1]$.

Assuming the existence of the function $\tilde{\tau}$, we obtain the Lyapounov function τ on M as follows: $\tau = \tilde{\tau} \circ \psi^{-1}$ on $U = \psi(H \times \mathbb{R})$, $\tau = 1$ on $A - U$, and $\tau = -1$ on $M - (A \cup U)$.

Let us now prove the existence of the Lyapounov function $\tilde{\tau}$ on $H \times \mathbb{R}$. We denote by (y, z) the points of $H \times \mathbb{R}$. The cone field

$$\tilde{\mathcal{C}}(y, z) = \psi^* \mathcal{C}(y, z) = (d\psi^{-1}_{(y,z)} \cdot \mathcal{C}(\psi(y, z)))$$

is a closed cone field on $H \times \mathbb{R}$. The cones $\tilde{\mathcal{C}}(y, 0)$ satisfy $v_z > 0$ for each $(v_y, v_z) \subset \tilde{\mathcal{C}}(y, 0) - \{0\}$. Fixing a Riemannian metric on H, there exists a smooth positive function $\delta(y)$ on M such that

$$\tilde{\mathcal{C}}(y, 0) \subset \{(v_y, v_z) : v_z \geqslant 3\delta(y)\|v_y\|\}$$

for each $y \in H$. Then, there exists a smooth positive function $\epsilon(y)$ such that

$$\tilde{\mathcal{C}}(y, z) \subset \{(v_y, v_z) : v_y \geqslant 2\delta(y)\|v_y\|\}$$

provided $|z| \leqslant \epsilon(y)$. Let $f : H \longrightarrow \mathbb{R}$ be a smooth positive function such that $\|df_y\| \leqslant \delta(y)$ and $f(y) \leqslant \epsilon(y)$ for all $y \in M$. Finally, let us set

$$\tilde{\tau}(y, z) = \phi(z/f(y)),$$

where $\phi : \mathbb{R} \longrightarrow [-1, 1]$ is a smooth nondecreasing function which has positive derivative on $]-1, 1[$ and is equal to 1 on $[1, \infty)$ and to -1 on $(-\infty, -1]$. The set of regular points of the function $\tilde{\tau}$ is $H\times]-1, 1[$. At such a point (y, z), we compute

$$d\tilde{\tau}_{(y,z)} \cdot (v_x, v_z) = \frac{\phi'(z/f(y))}{f(y)}\big(v_z - \frac{z}{f(y)} df_y \cdot v_y\big) \geqslant \frac{\phi'(z/f(y))}{2f(y)} v_z$$

for $(v_y, v_z) \in \tilde{\mathcal{C}}(y, z)$ since

$$|(z/f(y))df_y \cdot v_y| \leqslant \delta(y)\|v_y\| \leqslant v_z/2.$$

□

We will also need a variant of the above result. We say that the open set A is smooth near the set X if there exists an open set U containing X such that $U \cap \partial A$ is a smooth hypersurface.

Corollary 31. *Let A be a trapping domain which is smooth near $\mathcal{D}(\mathcal{C})$. Then there exists a smooth Lyapounov function τ such that $A = \{\tau > 0\}$ and such that τ is regular at each point of $\tau^{-1}(0) \cap \mathcal{D}(\mathcal{C})$.*

If F_i and F_e are closed sets contained in A and disjoint from $\bar{A}$, respectively, we can moreover impose that $\tau = 1$ on F_i and $\tau = -1$ on F_e.

Proof. Let U be an open neighborhood of $\mathcal{D}(\mathcal{C})$ such that $\partial A \cap U$ is smooth. Let V be the complement of $\mathcal{D}(\mathcal{C})$. Let T be a smooth Lyapounov function of $\mathcal{C}$ on U such that 0 is a regular value of T and $A \cap U = \{T > 0\}$. We obtain such a function by applying the Proposition to U. Let f be a smooth function on M such that $f = 1$ on F_i, $f > 0$ on A, $f < 0$ outside of $\bar{A}$, and $f = -1$ on F_e. Let g, h be a partition of the unity associated to the open covering (U, V) of M. We set $\tau = gT + hf$. □

4.2 Conley Theory for closed cone fields

We prove Theorem 1 and 2.

Proposition 32. *If x is not chain recurrent, then there exists a Lyapounov function τ such that $\tau(x)$ is a regular value of τ (in particular, τ is regular at x).*

Proof. There are two cases. Either $\mathcal{C}(x)$ is degenerate, or there exists an enlargement $\mathcal{E}$ of $\mathcal{C}$ such that $x \notin \mathcal{I}^+_{\mathcal{E}}(x)$ and such that $\mathcal{E}(x) \neq \emptyset$. In each of these cases, we will prove the existence of a Lyapounov function regular at x, Lemma 24 then implies the proposition.

In the first case each function τ which is supported in the open set $M - \mathcal{D}(\mathcal{C})$ is a Lyapouov function for $\mathcal{C}$. Since $x \in M - \mathcal{D}(\mathcal{C})$ there exists such a function satisfying $d\tau_x \neq 0$.

In the second case, the set $A_0 := \mathcal{I}^+_{\mathcal{E}}(x)$ is a trapping domain for $\mathcal{C}$ whose boundary contains x. Proposition 25 gives the existence of a smooth trapping domain whose boundary contains x. Corollary 31 then implies the existence of a smooth Lyapounov function τ such that $d\tau_x \neq 0$. □

Proposition 33. *Let x and x' be two points such that x' does not belong to $\mathcal{F}^+_{\mathcal{C}}(x)$. Then there exists a Lyapounov function τ such that $\tau(x') < \tau(x)$. If the point x does not belong to $\mathcal{R}_{\mathcal{C}}$ then the function τ can be chosen such that $\tau(x)$ is a regular value of τ, similarly for x'.*

Proof. To prove the first statement, we consider two cases. Either $\mathcal{C}(x) = \emptyset$ or there exists an an enlargement $\mathcal{E}$ of $\mathcal{C}$ such that $x' \notin \mathcal{I}^+_{\mathcal{E}}(x) \cup \{x\}$ and $\mathcal{E}(x) \neq \emptyset$.

In the first case, we take a smooth function τ supported in a small neighborhood of x and satisfying $\tau(x) > 0 = \tau(x')$ and $d\tau_x \neq 0$.

In the second case, the set $A_0 := \mathcal{I}^+_{\mathcal{E}}(x)$ is a trapping domain containing x in its closure and not containing x'. Proposition 25 then implies the existence of a smooth (near $\mathcal{D}(C)$) trapping domain containing x in its closure and not containing x'. Proposition 30 implies the existence of a smooth Lyapounov function τ such that $\tau(x') \leqslant 0$ and $\tau(x) \geqslant 0$ and 0 is a regular value of τ. If $\tau(x) = 0$, then $d\tau_x \neq 0$ hence the Lyapounov function τ can be slightly modified near x to a Lyapounov function $\tilde{\tau}$ such that $\tilde{\tau}(x) > 0$ and $\tilde{\tau}(x') = \tau(x') \leqslant 0$.

We have proved, in each case, the existence of a smooth Lyapounov function τ such that $\tau(x') < \tau(x)$. If x is not stably recurrent and is a critical point of τ, we consider a smooth Lyapounov function f regular at x. By composing on the left with a non decreasing function, the function f can be made arbitrarily C^0-small. Then the Lyapounov function $\tau_1 = \tau + f$ is regular at x and satisfies $\tau_1(x') < \tau_1(x)$. If, in addition, x' is not stably recurrent and is a singular point of $\tau + f$ then we consider a Lyapounov function g regular at x', C^0-small, and such that $dg(x)$ is small. The Lyapounov function $\tau_2 = \tau + f + g$ then satisfies $\tau_2(x') < \tau_2(x)$, $d\tau_2(x) \neq 0$ and $d\tau_2(x') \neq 0$.

Finally, in the case where $d\tau_2(x) \neq 0$, we use Lemma 24, to obtain a Lyapounov function τ_3, which has the same critical set as τ_2, and such that $\tau_3(x)$ is a regular value of τ_3. The proof of Lemma 24 shows that the function τ_3 can be chosen arbitrarily C^0 close to τ_2, hence $\tau_3(x') < \tau_3(x)$. A last application of Lemma 24 gives a Lyapounov function τ_4 such that $\tau_4(x')$ and $\tau_4(x)$ are regular values of τ_4, and $\tau_4(x') < \tau_4(x)$. □

Theorem 1 obviously follows from the two propositions above. Let us prove Theorem 2.

Proof of Theorem 2. Let us consider the set $\mathcal{L}$ of smooth Lyapounov functions which have the property that they take values in $[0,1]$ and have only two singular values 0 and 1. We endow $\mathcal{L}$ with the topology of C^1 convergence on compact sets. It is a separable metric space. We consider a dense sequence τ_i in $\mathcal{L}$. There exists a positive sequence a_i such that $\tau = \sum a_i\tau_i$ converges in C^k for each k on each compact set. We can moreover assume that $a_{i+1} \leqslant a_i/5$. We claim that the sum τ then satisfies all the conclusions of Theorem 2.

For each point x which is not chain recurrent, there exists a Lyapounov function $f \in \mathcal{L}$ such that $df_x \neq 0$ (just take any Lyapounov function regular at x and compose it on the left with an appropriate non decreasing function). As a consequence, there exists i such that $d\tau_i(x) \neq 0$. If $\mathcal{C}(x)$ contains a nonzero vector v, then all terms of the sum $d\tau_x \cdot v = \sum_i a_i d\tau_i(x) \cdot v$ are non negative, and one of them is positive, hence the sum is positive. We deduce that x is a regular point of τ.

If x and x' are two chain recurrent points which do not belong to the same stable class, there exists a Lyapounov function $f \in \mathcal{L}$ such that $f(x) \neq f(x')$ (once again, we just consider any Lyapounov function g such that $g(x') \neq g(x)$, and compose it on the left by a non decreasing function). Then, there exists i such that $\tau_i(x) \neq \tau_i(x')$, and we consider the first index j with this property. Since x and x' are chain recurrent, the values of τ_j on x and x' are 0 and 1, and we assume (by possibly renaming x and x') that $\tau_j(x) = 0$ and $\tau_j(x') = 1$. Then

$$\tau(x') - \tau(x) = \sum_i a_i(\tau_i(x') - \tau_i(x)) \geqslant a_j - \sum_{i>j} a_i \geqslant 3a_j/4 > 0$$

since $a_i \leqslant a_j 5^{i-j}$ for each $i \geqslant j$. We conclude that $\tau(x') \neq \tau(x)$.

Finally, let us consider two points $x \neq x'$ in M such that $x' \in \mathcal{F}^+(x)$ and $x \notin \mathcal{F}^+(x')$. The first point implies that $\tau(x') \geqslant \tau(x)$ for each Lyapounov function τ. The second point implies that there exists a Lyapounov function τ such that $\tau(x') > \tau(x)$. By composition with a non increasing function, we can assume that $\tau \in \mathcal{L}$. Then, by density, there exists j such that $\tau_j(x') > \tau_j(x)$. The difference $\tau(x') - \tau(x)$ is thus the sum of non negative terms one of which is positive. ∎

∎

4.3 More existence results of Lyapounov functions

We will use the following easy Lemma in our next result:

Lemma 34. *Let $\tau_i, 1 \leqslant i \leqslant k$, be finitely many smooth non negative Lyapounov functions, then the product $\tau = \tau_1\tau_2\cdots\tau_k$ is a non negative smooth Lyapounov function. If all the τ_i are regular at some point x_0, then so is τ.*

PROOF. By recurrence, it is enough to prove the statement for $k = 2$. The expression

$$d\tau(x) = \tau_1(x)d\tau_2(x) + \tau_2(x)d\tau_1(x)$$

implies that $d\tau_x \cdot v \geqslant 0$ for each $(x, v) \in \mathcal{C}$. Assume now that there exists $(x, v) \in \mathcal{C}$, $v \neq 0$, such that $d\tau_x \cdot v = 0$. Then each of the terms $\tau_1(x)d\tau_2(x) \cdot v$ and $\tau_2(x)d\tau_1(x) \cdot v$ vanish, which implies that each of the linear forms $\tau_1(x)d\tau_2(x)$ and $\tau_2(x)d\tau_1(x)$ vanish, hence that $d\tau(x) = 0$. We have proved that τ is a smooth Lyapounov function. If the τ_1 and τ_2 are regular at x_0, then $\tau_i(x_0) > 0$ and we see that $d\tau(x_0) \neq 0$. ∎

Proposition 35. *Let $K \subset M$ be a compact set. Then there exists a smooth non negative Lyapounov function τ_+ such that $\tau_+ = 0$ on K (hence on $\mathcal{F}^-_{\mathcal{C}}(K)$) and $\tau_+ > 0$ outside of $\mathcal{F}^-_{\mathcal{C}}(K)$. This implies in particular that $\mathcal{F}^-_{\mathcal{C}}(K)$ is closed. The function τ_+ can be chosen regular on $\mathcal{D}(\mathcal{C}) - \big(\mathcal{F}^-_{\mathcal{C}}(K) \cup \mathcal{R}_{\mathcal{C}}\big)$.*

There also exists a non positive smooth Lyapounov function τ_- such that $\tau_- = 0$ on K (hence on $\mathcal{F}^+_{\mathcal{C}}(K)$) and $\tau_- < 0$ outside of $\mathcal{F}^+_{\mathcal{C}}(K)$. This implies that $\mathcal{F}^+_{\mathcal{C}}(K)$ is closed. The function τ_- can be chosen regular on $\mathcal{D}(\mathcal{C}) - \big(\mathcal{F}^+_{\mathcal{C}}(K) \cup \mathcal{R}_{\mathcal{C}}\big)$.

PROOF. The second part of the statement is a consequence of the first part applied to the reversed cone $-\mathcal{C}$. More precisely, we have $\tau_-(\mathcal{C}) = -\tau_+(-\mathcal{C})$.

To prove the first part, we fix a point $x_0 \in M - \mathcal{F}^-_{\mathcal{C}}(K)$. For each $y \in K$, there exists a smooth Lyapounov function f such that $f(y) < f(x_0)$. If moreover $x_0 \notin \mathcal{R}_{\mathcal{C}}$, then the function f can be chosen regular at x_0. By composing f on the left with a non decreasing function, we deduce the existence of a Lyapounov function τ_y such that $\tau_y \geqslant 0$, $\tau_y = 0$ in a neighborhood U_y of y, and $\tau_y(x_0) > 0$. If $x_0 \notin \mathcal{R}_{\mathcal{C}}$, then in addition τ_y is regular at x_0.

Since K is compact, there exist finitely many points $y_1, \ldots, y_k$ such that the corresponding open sets U_{y_i} cover K. The product $\tau_0 := \tau_{y_1}\tau_{y_2}\cdots\tau_{y_k}$ is a smooth non negative Lyapounov function such that $\tau_0(x_0) > 0$, and, if $x_0 \notin \mathcal{R}_{\mathcal{C}}$, $d\tau_0(x_0) \neq 0$.

For each $x_0 \in M - \mathcal{F}^-_{\mathcal{C}}(K)$, we have proved the existence of an open neighborhood V_0 of x_0 and of a smooth non negative Lyapounov function τ which is null on K and positive on V_0. We can cover the separable metric space $M - \mathcal{F}^-_{\mathcal{C}}(K)$ by a sequence V_i of open sets such that, for each i, there exists a smooth non negative Lyapounov function τ_i which is null on K and positive on V_i. Then there exists a positive sequence a_i such that $\tau := \sum_i a_i\tau_i$ is a smooth non negative function which is positive on $M - \mathcal{F}^-_{\mathcal{C}}(K)$.

By exactly the same method we can also obtain a smooth non negative Lyapounov function τ which is null on K and which has the property that $d\tau_x \cdot v > 0$ for each $x \in M - (\mathcal{F}^-_{\mathcal{C}}(K) \cup \mathcal{R}_{\mathcal{C}})$ and $v \in \mathcal{C}(x) - \{0\}$. ∎

By adding the functions τ_+ and τ_-, we obtain:

Corollary 36. *Given a compact $K \subset M$, there exists a Lyapounov function which is null on K and regular on $\mathcal{D}(\mathcal{C}) - \big(\mathcal{F}_{\mathcal{C}}(K,K) \cup \mathcal{R}_{\mathcal{C}}\big)$.*

Let us also state the following :

Proposition 37. *Let $A \subset M$ be a trapping domain. There exists a non negative Lyapounov function τ such that $\tau > 0$ on A and $\tau < 0$ outside of $\bar{A}$. The function τ can be chosen regular on $\mathcal{D}(\mathcal{C}) - (\mathcal{R}_{\mathcal{C}} \cup \partial A)$.*

Proof. We consider an enlargement $\mathcal{E}$ of $\mathcal{C}$ such that A is a trapping domain for $\bar{\mathcal{E}}$.

We first fix a point $x_0 \in A$ and prove the existence of a smooth Lyapounov function which is non negative, null outside of A, positive at x_0 and, if x_0 is not stably recurrent, regular at x_0.

We consider a point $x_1 \in A \cap \mathcal{I}^-_{\mathcal{E}}(x_0)$. Then the set $A_1 := \mathcal{I}^+_{\mathcal{E}}(x_1)$ is open, it contains $F_i := \mathcal{F}^+_{\mathcal{C}}(x_0)$, and its closure is contained in $\mathcal{F}^+_{\bar{\mathcal{E}}}(x_1)$, hence in A. In other words, the closure of A_1 is disjoint from the $F_e := M - A$. By Proposition 25, there exists a smooth (near $\mathcal{D}(\mathcal{C})$) trapping domain A'_1 which contains F_i and whose closure is disjoint from F_e. By Proposition 30, there exists a smooth Lyapounov function $\tau : M \longrightarrow [-1,1]$ (for $\mathcal{C}$) which is equal to 1 on F_i and to -1 on F_e. The non negative Lyapounov function $1+\tau$ is then null outside of A and positive at x_0.

In the case where x_0 is not stably recurrent and non degenerate, we can take $\mathcal{E}$ in such a way that $x_0 \notin A_2 := \mathcal{I}^+_{\mathcal{E}}(x_0)$, hence x_0 belongs to the boundary of this trapping domain. The closure of A_2 is disjoint from the complement F_e of A. By Propositions 25 and 30, we find a non negative Lyapounov function τ which is regular (hence positive) at x_0 and null outside of A.

By considering a convex combination of countably many of the Lyapounov functions we just built, we obtain a non negative Lyapounov function τ_i which is positive on A and regular on $(A \cap \mathcal{D}(\mathcal{C})) - \mathcal{R}_{\mathcal{C}}$.

We can apply the same result to the cone $-\mathcal{C}$ and the trapping domain $M - \bar{A}$, and get a Lyapounov function τ_e (for $\mathcal{C}$) which is non positive, negative outside of $\bar{A}$, and regular on $(\mathcal{D}(\mathcal{C}) - \bar{A}) - \mathcal{R}_{\mathcal{C}}$.

The sum $\tau := \tau_i + \tau_e$ then satisfies the conclusions of the proposition. □

4.4 Globally hyperbolic cone fields

We prove Theorem 3. We start with an easy observation:

Lemma 38. *If the closed cone field $\mathcal{C}$ satisfies (GH2), then $\mathcal{J}^{\pm}_{\mathcal{C}}(x)$ is closed for all $x \in M$.*

Proof. Let $y_n \in \mathcal{J}^+_{\mathcal{C}}(x)$ be a convergent sequence with limit $y \in M$. Let Y be the compact set $Y := \{y, y_1, y_2, \ldots\}$. The set $\mathcal{J}_{\mathcal{C}}(x, Y)$ is compact and it contains y_n for each n, hence it contains the limit y. □

Let us denote by $\mathcal{C}_K$ the cone field which is equal to $\mathcal{C}$ on K and degenerate outside of K. If $\mathcal{C}$ is a closed cone field and K is a closed set, then $\mathcal{C}_K$ is a closed cone field. If $\mathcal{C}$ is causal, then so is $\mathcal{C}_K$.

Lemma 39. *Let $\mathcal{C}$ be a causal closed cone field and K be a compact set. Then there exists an open enlargement $\mathcal{E}$ of $\mathcal{C}_K$ and a real number $L > 0$ such that all $\mathcal{E}$-timelike curves have length less than L.*

Proof. Let $\mathcal{E}_n$ be a decreasing sequence of open cone fields converging to $\mathcal{C}_K$. We can assume that $U_n := \mathcal{D}(\mathcal{E}_n)$ is bounded for each n. If the conclusion of the Lemma does not hold, there exists a sequence $\gamma_n : [-l_n, l_n] \longrightarrow M$ of $\mathcal{E}_n$-timelike curves parametrized by arclength with l_n

unbounded. By Proposition 18, there exists a complete $\mathcal{C}_K$-causal curve $\gamma : \mathbb{R} \longrightarrow M$. Since $\mathcal{C}_K$ has no singular point, this curve has infinite length in the forward direction. Let ω be a limit point of γ at $+\infty$. For each $s > t \in \mathbb{R}$, we have $\gamma(s) \in \mathcal{J}^+_{\mathcal{C}}(\gamma(t))$. Since this set is closed (Lemma 38), we deduce that $\omega \in \mathcal{J}^+_{\mathcal{C}}(\gamma(t))$, or in other words that $\gamma(t) \in \mathcal{J}^-_{\mathcal{C}}(\omega)$, and this holds for all t. Since ω is not singular, there exists a local time function, and this implies that γ has another limit point ω'. Since $\mathcal{J}^-_{\mathcal{C}}(\omega)$ is closed, we obtain that $\omega' \in \mathcal{J}^-_{\mathcal{C}}(\omega)$, and similarly $\omega \in \mathcal{J}^-_{\mathcal{C}}(\omega')$. This is in contradiction with $\mathcal{C}$ being causal. □

Corollary 40. *Let $\mathcal{C}$ be a globally hyperbolic closed cone field and K be a compact set. The stably recurrent set $\mathcal{R}(\mathcal{C}_K)$ is empty.*

Proof. If $\mathcal{R}(\mathcal{C}_K)$ is not empty, then it contains a complete causal curve γ (Definition 4). Since $\mathcal{C}_K$ has no singular point, this curve has infinite length, which contradicts Lemma 39. □

Corollary 41. *Let $\mathcal{C}$ be a globally hyperbolic closed cone field, and K_1, K_2 be two compact sets. Let K be a compact set containing $\mathcal{J}_{\mathcal{C}}(K_1, K_2)$. Then*

$$\mathcal{F}_{\mathcal{C}_K}(K_1, K_2) = \mathcal{J}_{\mathcal{C}_K}(K_1, K_2) = \mathcal{J}_{\mathcal{C}}(K_1, K_2).$$

Proof. The second equality is clear. To prove the first equality, we consider a sequence $\mathcal{E}_n$ of open enlargements of $\mathcal{C}_K$ decreasing to $\mathcal{C}_K$. By Lemma 39, we can assume that each $\mathcal{E}_1$-timelike curve has length less than $L > 0$. This is then true for all $\mathcal{E}_n$. Given $x \in \mathcal{F}_{\mathcal{C}_K}(K_1, K_2)$, there exists a sequence $\gamma_n : [0, 1] \longrightarrow M$ of $\mathcal{E}_n$-timelike curves connecting K_1 to K_2, parametrized proportionally to arclength, and passing through x. Since the curves γ_n have bounded length, they are equi-Lipschitz. Up to a subsequence, they converge uniformly to a Lipschitz curve $\gamma : [0, 1] \longrightarrow M$ which is $\mathcal{C}_K$-causal by Lemma 17, passes through x, and connects K_1 to K_2. This implies that $x \in \mathcal{J}_{\mathcal{C}_K}(K_1, K_2)$. □

We are now ready to prove the existence of a steep Lyapounov function. Let $K_i, i \geqslant 1$ be a sequence of compact sets such that $\mathcal{J}_{\mathcal{C}}(K_i, K_i)$ is contained in the interior of K_{i+1} and such that $M = \cup_i K_i$.

For each $i \geqslant 1$, we apply Corollary 36 to the cone field $\mathcal{C}_{K_{i+2}}$ and the compact set K_i. Since $\mathcal{F}_{\mathcal{C}_{K_{i+2}}}(K_i, K_i) = \mathcal{J}_{\mathcal{C}}(K_i, K_i) \subset \mathring{K}_{i+1}$ and since $\mathcal{R}(\mathcal{C}_{K_{i+2}})$ is empty, we obtain a smooth function $\tau_i : M \longrightarrow \mathbb{R}$ with the following properties:

- τ_i is a Lyapounov function on K_{i+2}, which means that $d\tau_i(x) \cdot v > 0$ for each $x \in K_{i+2}$ such that $d\tau_i(x) \neq 0$ and each $v \in \mathcal{C}(x) - \{0\}$.
- τ_i is regular on $K_{i+2} - \mathring{K}_{i+1}$, which means that $d\tau_i(x) \neq 0$ for each $x \in K_{i+2} - \mathring{K}_{i+1}$.
- τ_i is null on K_i.

We also let τ_0 be a smooth function on M which is a Lyapounov function on K_2.

We now prove the existence of a sequence a_i of positive numbers such that the sum $\tau := \sum_{i \geqslant 0} a_i \tau_i$ is a steep Lyapounov function. Note that this sum is locally finite.

We build the sequence a_i by induction, in such a way that the partial sum $\sum_{i=0}^{k} a_i \tau_i$ is a steep Lyapounov function on K_{k+2} for each k.

The function τ_0 is a Lyapounvov function on the compact set K_2, hence there exists $a_0 > 0$ such that $a_0 \tau_0$ is steep on K_2. The function τ_1 is Lyapounov on K_3 and regular on $K_3 - \mathring{K}_2$. Then there exists $a_1 > 0$ such that $a_0 \tau_0 + a_1 \tau_1$ is a steep Lyapounvov function on $K_3 - \mathring{K}_2$, hence on K_3. Assuming that $a_0, \ldots, a_k$ have been constructed, observe that the function τ_{k+1} is

Lyapounov on K_{k+2} and non degenerate on $K_{k+2} - \mathring{K}_{k+1}$. On the other hand the partial sum $\sum_{i=0}^{k} a_i \tau_i$ is a smooth function on M which is a steep Lyapounvov function on K_{k+1}. There exists $a_{k+1} > 0$ such that $\sum_{i=0}^{k+1} a_i \tau_i$ is a steep Lyapounvov function on $K_{k+2} - \mathring{K}_{k+1}$, hence on K_{k+2}. This ends the proof of the existence of a steep Lyapounvov function.

Conversely, let us assume the existence of a steep Lyapounvov function τ. It is clear that $\mathcal{C}$ is causal. Let us prove that $\mathcal{F}^{\pm}_{\mathcal{C}}(x) = \mathcal{J}^{\pm}_{\mathcal{C}}(x)$ for each x. We consider a decreasing sequence $\mathcal{E}_n$ of enlargements of $\mathcal{C}$, which have the property that $d\tau_y \cdot v \geqslant |v|_y/2$ for each $(y, v) \in \mathcal{E}_n$. Given $z \in \mathcal{F}^{+}_{\mathcal{C}}(x)$, there exists a sequence $\gamma_n : [0, 1] \longrightarrow M$ of smooth $\mathcal{E}_n$-timelike curves such that $\gamma_n(0) = x$ and $\gamma_n(1) = z$. We can assume that γ_n is parametrized proportionally to arclength, hence is L_n-Lipschitz, where L_n is the length of γ_n. The hypothesis made on $\mathcal{E}_n$ implies that $L_n \leqslant 2(\tau(z) - \tau(x))$ is bounded. At the limit, we obtain a Lipschitz causal curve $\gamma : [0, 1] \longrightarrow M$ connecting x to z. We have proved that $\mathcal{F}^{+}_{\mathcal{C}}(x) \subset \mathcal{J}^{+}_{\mathcal{C}}(x)$, hence these sets are equal.

We finally prove (GH2). The set $\mathcal{J}_{\mathcal{C}}(K, K') = \mathcal{F}_{\mathcal{C}}(K, K')$ is closed. If γ is a causal curve joining K to K', then the length of γ is bounded by $\max_{K'} \tau - \min_K \tau$. This means that γ is contained in a bounded set, hence that $\mathcal{J}_{\mathcal{C}}(K, K')$ is bounded. Being closed and bounded in the complete Riemannian manifold M, the set $\mathcal{J}_{\mathcal{C}}(K, K')$ is compact. □

5 Semi-continuity of the chain recurrent set

Assume that the chain recurrent set $R_{\mathcal{C}}$ of the closed cone field $\mathcal{C}$ is compact.

Proposition 42. *For every neighborhood U of $R_{\mathcal{C}}$ there exists a closed enlargement $\mathcal{C}_U$ of $\mathcal{C}$ such that $\mathcal{R}_{\mathcal{C}_U} \subset U$.*

Proof. We assume, without loss of generality, that U is bounded, hence ∂U is compact. It is enough to prove the existence of an enlargement $\mathcal{E}$ of $\mathcal{C}$ such that $\mathcal{R}_{\bar{\mathcal{E}}}$ is disjoint from ∂U.

Let us fix a point $z \in \partial U$. By Lemma 24, there exists a Lyapounov function τ^z for $\mathcal{C}$ such that $a := \tau^z(z)$ is a regular value of τ^z. Then, there exists a closed enlargement $\mathcal{C}^z$ of $\mathcal{C}$ such that τ^z is a regular Lyapounov function for $\mathcal{C}^z$ in a neighborhood of $\{\tau^z = a\}$. This implies, by Lemma 22, that $\{\tau^z > a\}$ is a trapping region for $\mathcal{C}^z$, hence that $z \notin \mathcal{R}_{\mathcal{C}^z}$.

The open sets $M - \mathcal{R}_{\mathcal{C}^z}$, $z \in \partial U$ thus cover the compact set ∂U, hence finitely many of them cover ∂U. By taking the intersection of the corresponding cone fields $\mathcal{C}^z$, we obtain a closed enlargement of $\mathcal{C}$ whose stably recurrent set is disjoint from ∂U, as was claimed. □

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