Symplectic classification of semi-toric integrable systems: recent advances and examples

Jaume Alonso

Recent advances in Hamiltonian dynamics and symplectic topology

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Classifications

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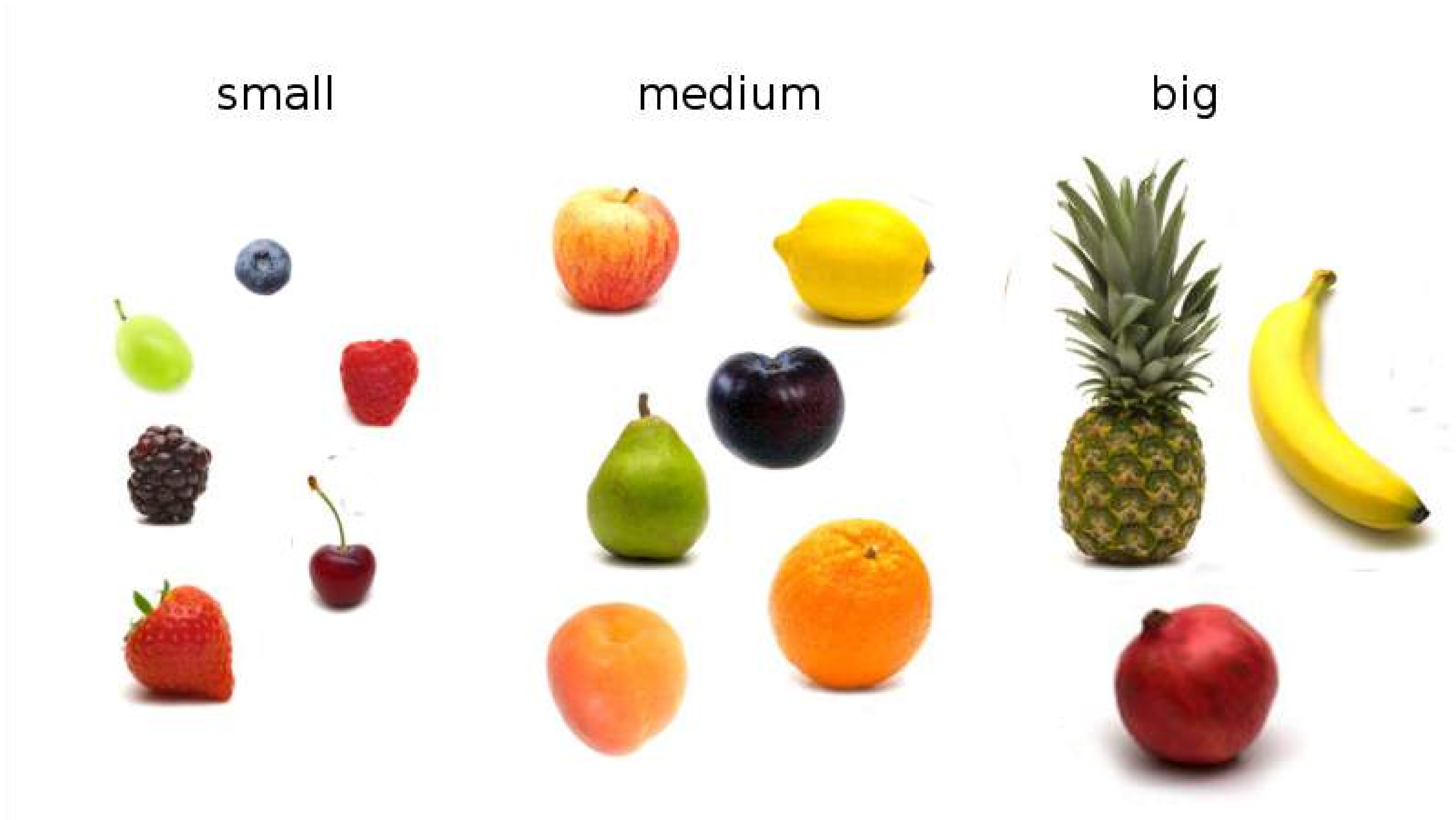

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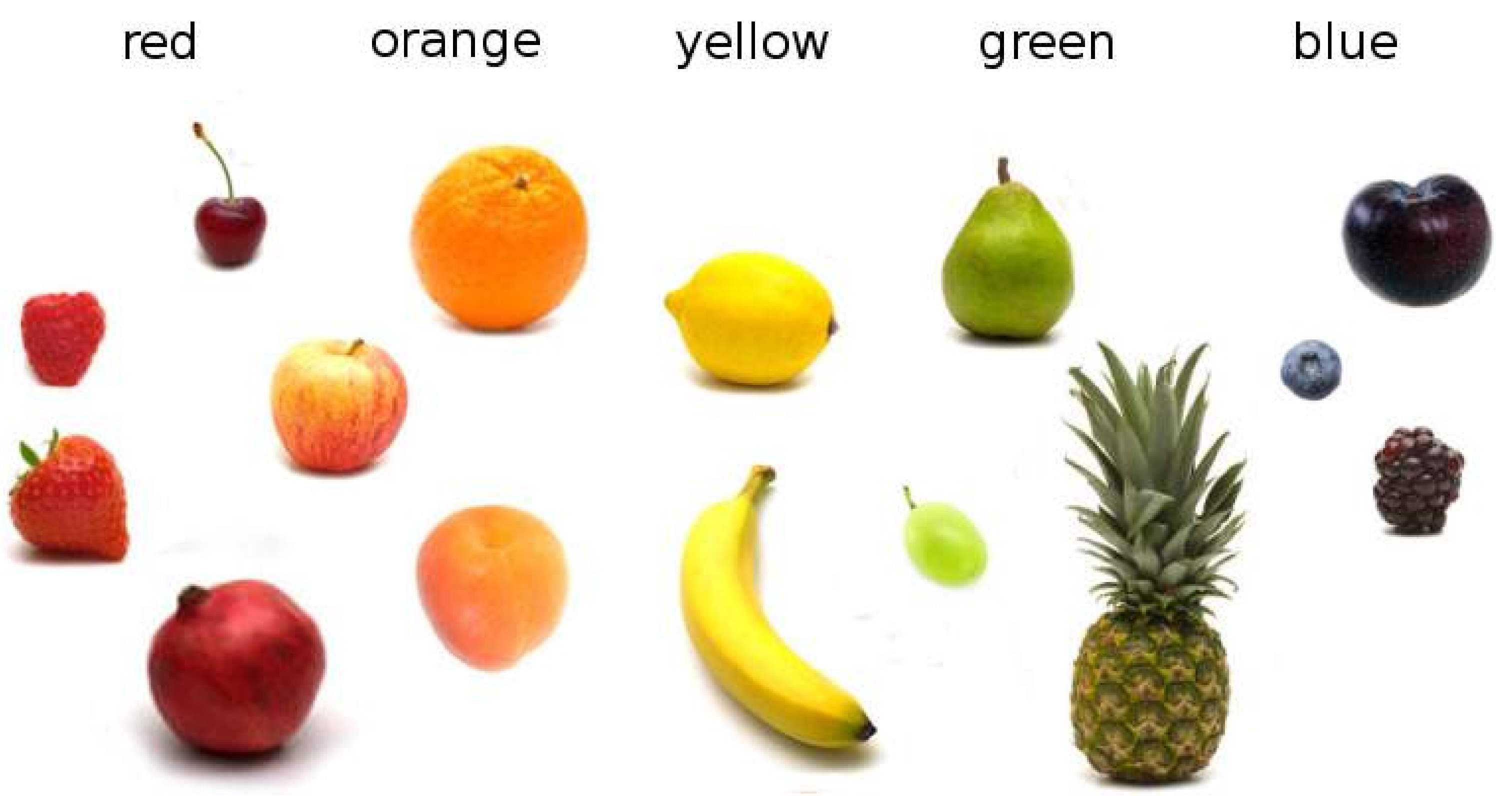

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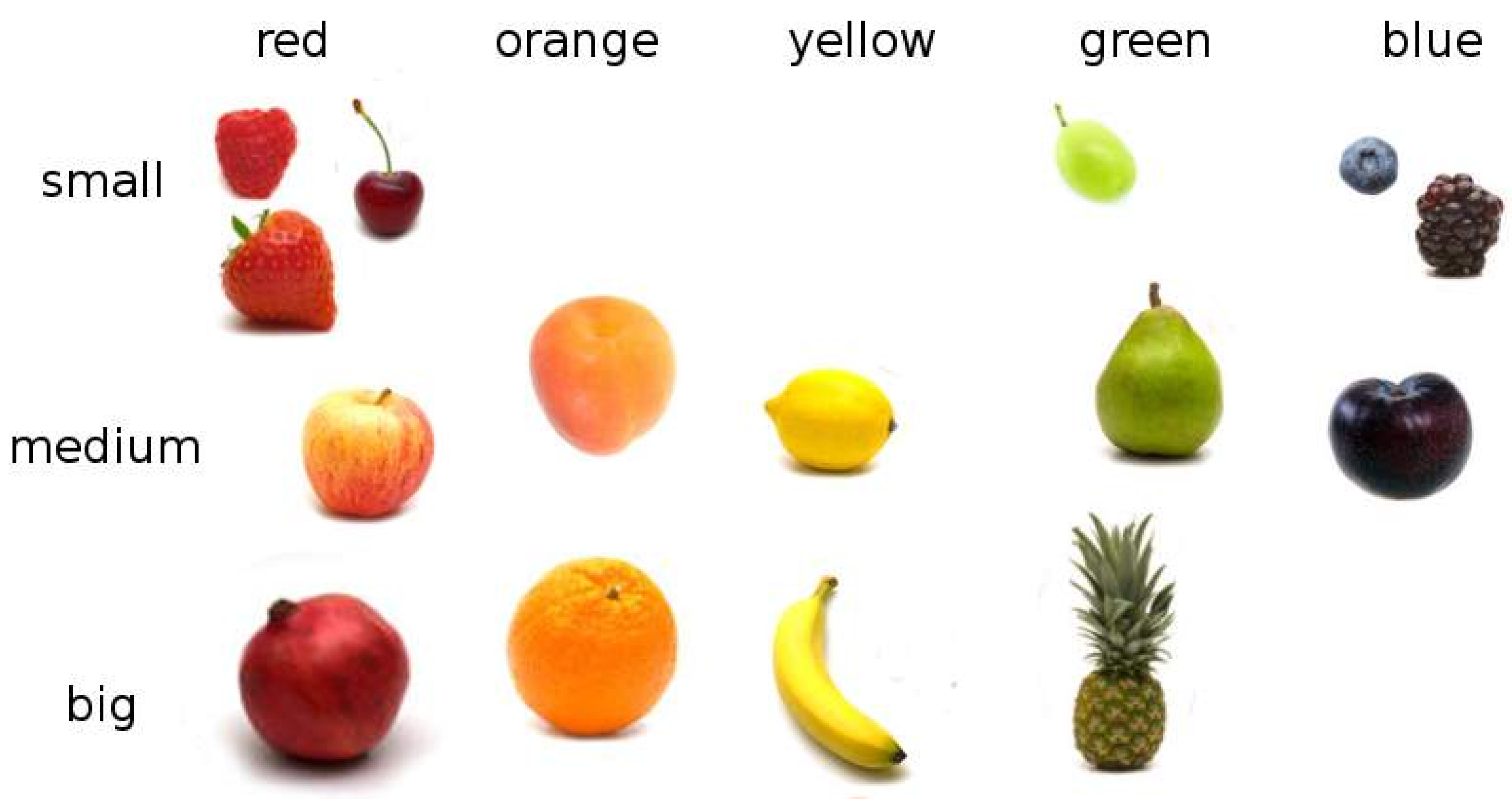

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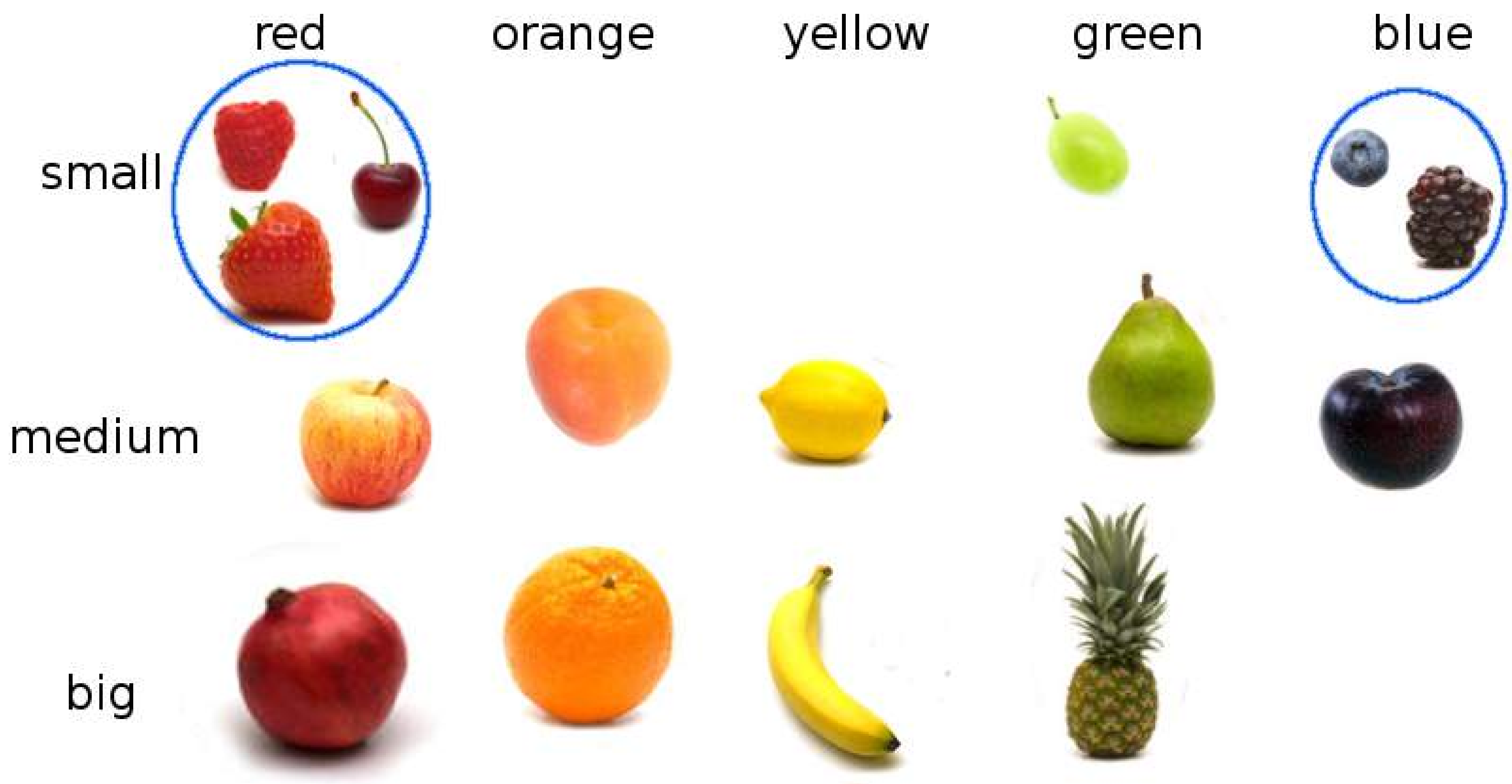
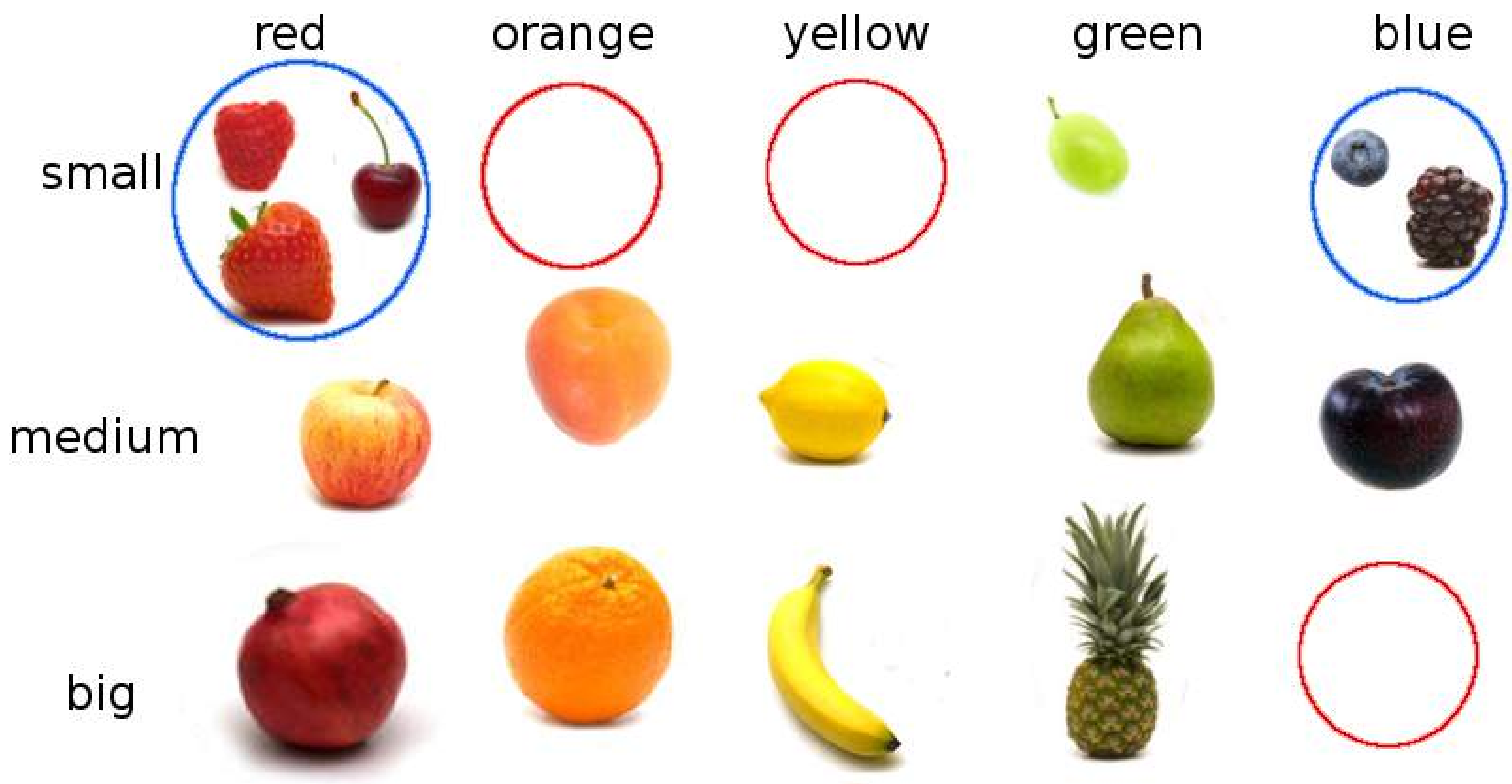


Classifications



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Classifications

Classification:

- We associate some data to each element of the collection (of topological nature, symplectic nature, diffeomorphic nature...)

Desirable characteristics:

- Two systems have the same data $\leftrightarrow$ they are equivalent (in some sense)
- Given some admissible data, we can construct the corresponding system

Classifications

- A symplectic classification of all completely integrable Hamiltonian systems is desirable but difficult
- Toric systems can be classified using certain polygons (Delzant polytopes)
- Recently, semi-toric systems have been classified in terms of five symplectic invariants by Pelayo and Vũ Ngọc

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Completely integrable systems

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Setting

- (M, ω) is a $2n$-dimensional symplectic manifold
- To each smooth function $H : M \to \mathbb{R}$ (autonomous Hamiltonian) we associate a Hamiltonian vector field X^H by

$$\omega(X^H, \cdot) = dH(\cdot)$$

  and a corresponding Hamiltonian flow φ^H
- The algebra of Hamiltonians $C^\infty(M)$ is endowed with the Poisson bracket: $\{J, F\} := \omega(X^J, X^F)$
- The time-evolution of a function F under the flow of X^H is $\dot{F} = \{H, F\}$
- If F satisfies $\{H, F\} = 0$, we call it a constant of motion / first integral

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Completely integrable systems

- A completely integrable system is a Hamiltonian on (M, ω) system where:
  - $H, F_2, \ldots, F_n : M \to \mathbb{R}$ constants of motion
  - $dH, dF_2, \ldots, dF_n$ linearly independent almost everywhere
  - $\{F_i, F_j\} = 0$ for $1 \leq i, j \leq n$, where $F_1 := H$
- The different constants of motion correspond to continuous symmetries of the system (Noether's theorem)
- The momentum map $\Phi = (F_1, F_2, ..., F_n) : M \to \mathbb{R}^n$. The flows $\varphi^{F_1}, ..., \varphi^{F_n}$ define an $\mathbb{R}^n$-action on M that fibrates the manifold

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Fibration

Understanding the fibration is key to understand the system:

- A point $p \in M$ is called regular if $D\Phi|_p$ has rank n. Otherwise it is a critical point
- If $\Phi^{-1}(c)$ has only regular points, it is a regular fibre and $c \in \mathbb{R}^n$ is a regular value
- If $\Phi^{-1}(c)$ contains critical points, it is a singular fibre and $c \in \mathbb{R}^n$ is a critical value

Regular and singular fibres

Regular fibres are “boring”:

- Assume the vector fields $X^{F_1}, ..., X^{F_n}$ are complete
- Then each connected component of a regular fibre is diffeomorphic to $\mathbb{R}^{n-k} \times \mathbb{T}^k$. If compact, then $\mathbb{T}^n$ (*Liouville tori*).

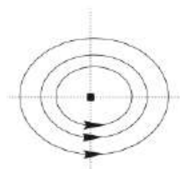
Singular fibres might be more “interesting”:

- If we have n integrals, we will (almost) always have singularities.
- They correspond to fixed points and relative equilibria subsets, so they describe the dynamics of the system.

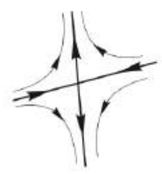
Singularities

Local normal form (Eliasson[1], Miranda & Zung[2]):
Integrable systems are linearisable around (non-degenerate) critical points $p \in M$, i.e. $\exists(x_1, ..., x_n, \xi_1, ..., \xi_n)$ local symplectic coordinates and functions $q_1, ..., q_n$ with $\{q_i, F_j\} = 0 \; \forall i, j$ where the q_i can be:

- Elliptic component: $q_j = (x_j^2 + \xi_j^2)/2$

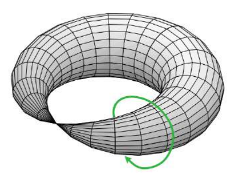


- Hyperbolic component: $q_j = x_j \xi_j$



- Focus-focus component: they come in pairs

$$\begin{cases} q_{j-1} = x_{j-1}\xi_j - x_j \xi_{j-1} \\ q_j = x_{j-1}\xi_{j-1} + x_j \xi_j \end{cases}$$



- Non-singular component: $q_j = \xi_j$

Moreover, if no hyperbolic components, then the equations $\{F_i, q_j\} = 0$ can be replaced by

$$(\Phi - \Phi(p)) \circ \varphi = g \circ (q_1, ..., q_n)$$

where $\varphi = (x_1, ..., x_n, \xi_1, ..., \xi_n)^{-1}$ and g is a diffeomorphism.

[1] L. Eliasson, Hamiltonian systems with Poisson commuting integrals, PhD Thesis, (1984)

[2] E. Miranda, N. T. Zung, *A note on equivariant normal forms of Poisson structures*, Math. Res. Lett., (2006)

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Semi-toric systems

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Singularities

Setting:

- (M, ω) 4-dimensional connected symplectic manifold
- $\Phi = (L, H) : M \to \mathbb{R}^2$ momentum map of a completely integrable system
- $p \in M$ critical point, with $D\Phi|_p < 2$

Definition:

- p is a non-degenerate critical point if the Hessians $D^2L|_p$, $D^2H|_p$ are linearly independent and if there is a linear combination of them having 4 distinct eigenvalues.

Semi-toric systems

Definition (Pelayo & Vũ Ngọc)[3]:

A semi-toric system is a 4-dimensional completely integrable system $(L, H) : (M, \omega) \to \mathbb{R}^2$ such that:

- L is proper
- L induces an effective Hamiltonian $\mathbb{S}^1$-action
- All singularities are non-degenerate
- There are no hyperbolic components

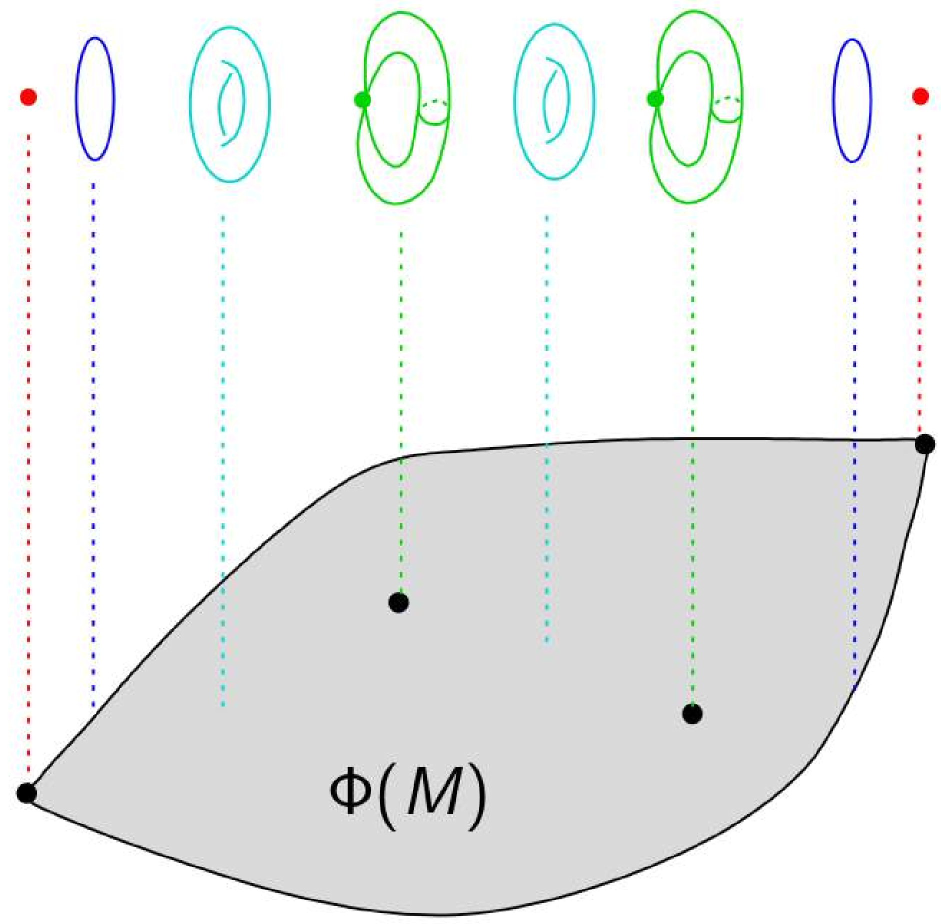
Conclusion: Possible singularities in dim=4:

- focus-focus (rank=0)
- elliptic-elliptic (rank=0)
- elliptic-regular (rank=1)

[3] Á. Pelayo, S. Vũ Ngọc, *Semitoric integrable systems on symplectic 4-manifolds*, Invent. Math., (2009)

Singularities

$$
\begin{array}{ccccc}
(M,\omega) & \supset & \Phi^{-1}(z) & & \\
\Big\downarrow \; \Phi = (J,H) & & \vdots & & \\
\mathbb{R}^2 \supset \Phi(M) & \ni & z & &
\end{array}
$$



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Example: the coupled spin-oscillator

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Coupled spin-oscillator

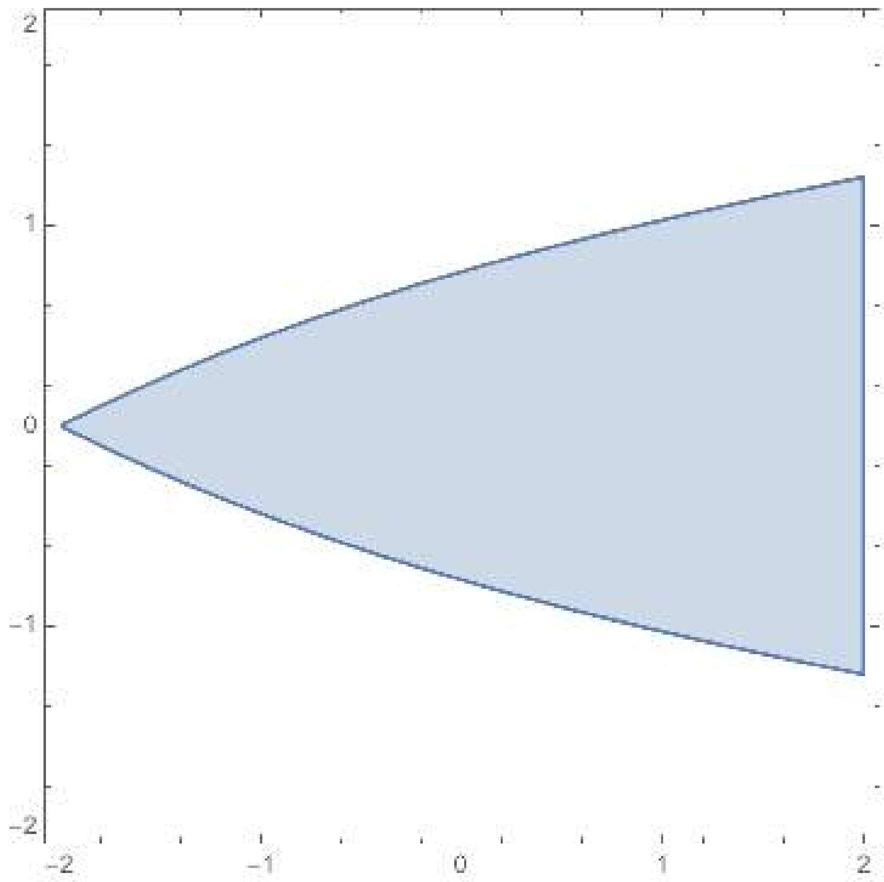
- The coupled spin-oscillator is a 4-dimensional integrable Hamiltonian system with 2 degrees of freedom
- It consists of the "coupling" of a classical spin on $\mathbb{S}^2$ and a classical harmonic oscillator on $\mathbb{R}^2$
- It appears in quantum mechanics (cold atoms in non-equilibrium / Jaynes-Cummings model) and there are also classical and semi-classical studies[4]
- Let $\lambda, \mu > 0$ be positive constants. Let $(x, y, z) \in \mathbb{S}^2$ and $(u, v) \in \mathbb{R}^2$ be coordinates of the space $\mathbb{S}^2 \times \mathbb{R}^2$. Then we define

$$L := \mu\frac{u^2+v^2}{2} + \lambda(z-1) \qquad H := \frac{1}{2}(ux+vy)$$

[4] O. Babelon, L. Cantini, B. Douçot, *A semi-classical study of the Jaynes-Cummings model*, J. Stat. Mech. Theory Exp., (2009)

Coupled spin-oscillator

L induces an $\mathbb{S}^1$-action. None of the functions are bounded, so the image of the energy-momentum map looks like:



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Example: the coupled angular momenta

Coupled angular momenta

- The coupled angular momenta is a 4-dimensional integrable Hamiltonian system with 2 degrees of freedom
- It consists of a non-trivial ”coupling” of two spin angular momenta, $S = (S_x, S_y, S_z)$ and $N = (N_x, N_y, N_z)$.
- It appears in quantum mechanics[5] and there are also classical and semi-classical studies
- The classical phase space is thus $\mathbb{S}^2 \times \mathbb{S}^2$ with symplectic form $\omega = -(R_1 \omega_{\mathbb{S}^2} \oplus R_2 \omega_{\mathbb{S}^2})$. We define

$$L := R_1 z_1 + R_2 z_2 \qquad H := (1-t)\frac{S_z}{|S|} + t\frac{S \cdot N}{|S||N|}$$

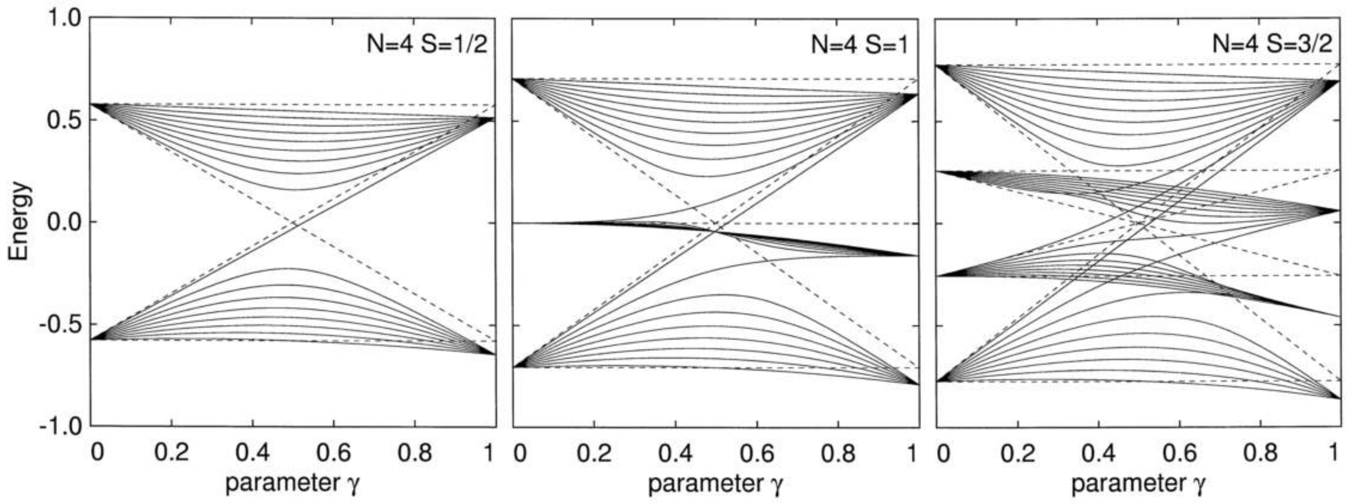
[5] D. A. Sadovskií, B. I. Zhilinskií, *Monodromy, diabolic points, and angular momentum coupling*, Phys. Lett. A, (1999)

Coupled angular momenta

- We have

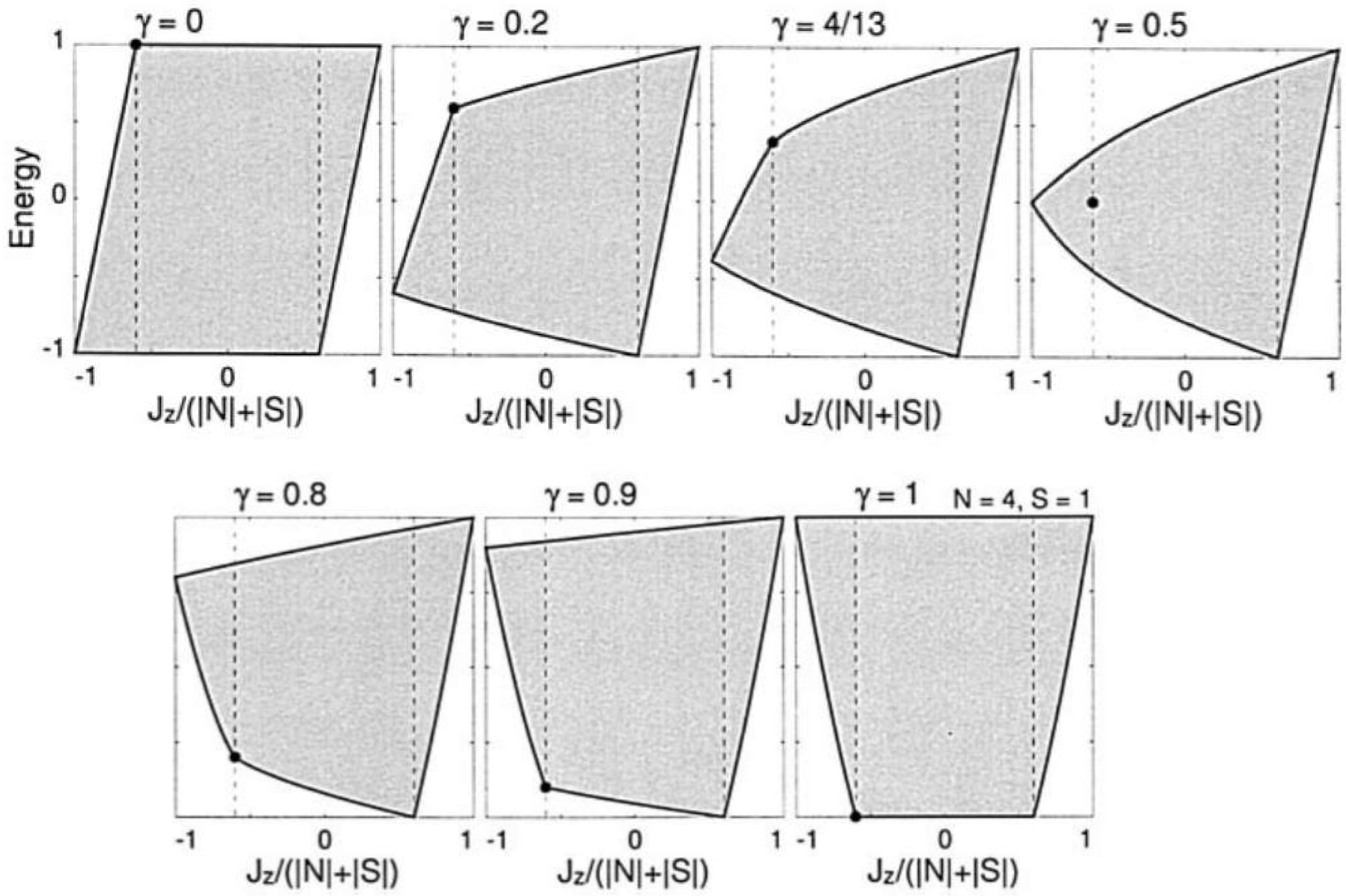
$$L := R_1 z_1 + R_2 z_2 \qquad H := (1-t)\frac{S_z}{|S|} + t\frac{S \cdot N}{|S||N|}$$

- For $t = 0$ we have only one angular spin momentum. N is free so it ”generates” the degeneracy
- For $t = 1$ we have a simple angular momenta addition. We get a ”multiplet” structure
- In between we have a ”diabolic point”



Coupled angular momenta

L induces an $\mathbb{S}^1$-action. We can see the evolution of the image of the momentum map as we move the parameter t:



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Coupled angular momenta

More mathematically:

- We have $M = \mathbb{S}^2 \times \mathbb{S}^2$. In Cartesian coordinates $(x_1, y_1, z_1, x_2, y_2, z_2)$:

$$\begin{cases} L \ := R_1(z_1 - 1) + R_2(z_2 + 1) \\ H := (1-t)z_1 + t(x_1x_2 + y_1y_2 + z_1z_2) + 2t - 1. \end{cases}$$

- The system has four critical points: $N \times N$, $N \times S$, $S \times N$ and $S \times S$.
- 3 of them are always elliptic-elliptic. But $N \times S$ is of focus-focus type for $t \in (t^-, t^+)$, where

$$t^{\pm} = \frac{R_2}{2R_2 + R_1 \mp 2\sqrt{R_1 R_2}}$$

- In particular, $t^- < 1/2 < t^+$.

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The symplectic classification of semi-toric systems

Classification

Pelayo and Vũ Ngọc have achieved[6][7] a complete 'generic' classification of semi-toric systems in terms of five symplectic invariants:

1. The number of focus-focus critical points invariant
2. The polygon invariant
3. The height invariant
4. The twisting-index invariant
5. The Taylor-series invariant

These invariants completely determine the integrable system up to global isomorphism of semi-toric systems.

[6] Á. Pelayo, S. Vũ Ngọc, *Semitoric integrable systems on symplectic 4-manifolds*, Invent. Math., (2009)
[7] Á. Pelayo, S. Vũ Ngọc, *Constructing integrable systems of semitoric type*, Acta Math., (2011)

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Symplectic invariants

The number of focus-focus critical points invariant

The number $m_f \in \mathbb{N} \cup \{0\}$ of singularities of focus-focus type that the system has.

- The coupled spin-oscillator has $m_f = 1$, corresponding to the point $N \times \{0\}$.
- The singularity on $S \times \{0\}$ is of elliptic-elliptic type.

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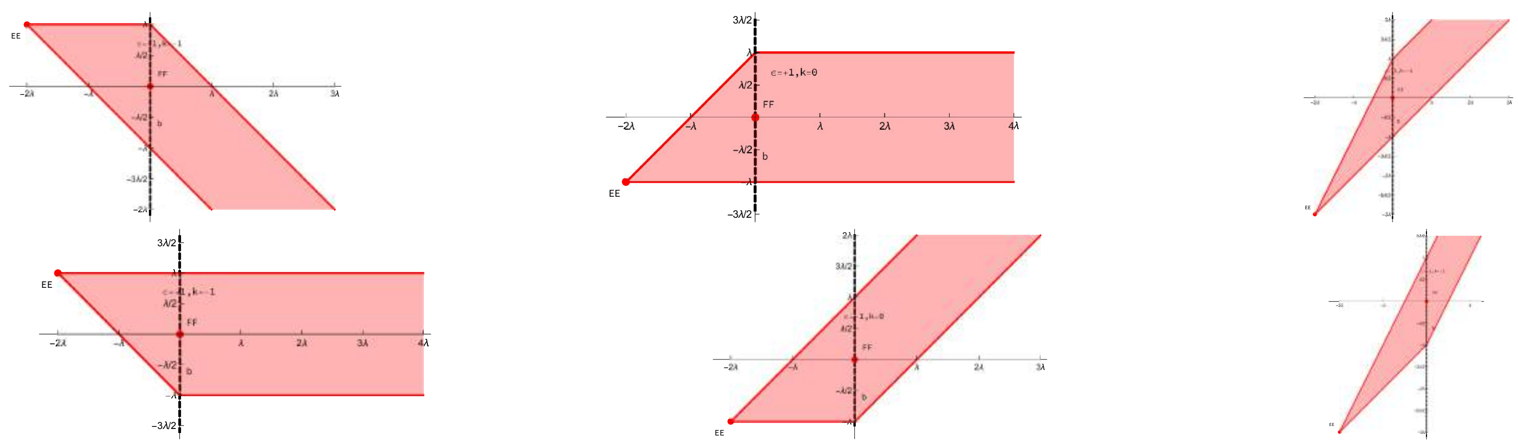
Symplectic invariants

The polygon invariant

We associate a family of simple, rational, convex polygons to a semi-toric system. There is a way to go from one polygon to the other (group action), so the invariant is the orbit of this action.

- They are a family of polygons (so more than one)
- Here we can see some of the polygons of the coupled spin-oscillator



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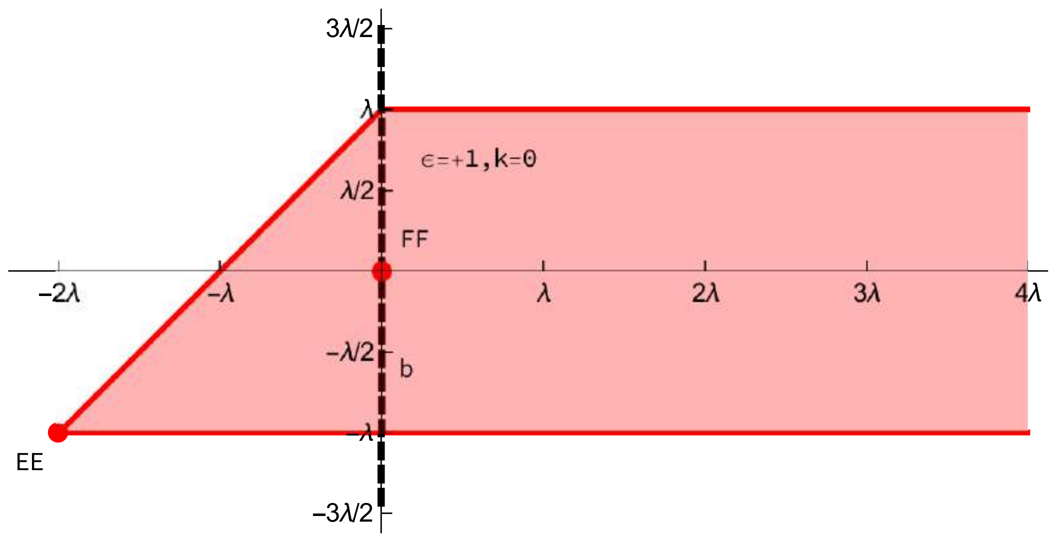
Symplectic invariants

The height invariant

Also called volume invariant, it measures the height of each focus-focus point in the polygons of the polygon invariant.

- The height invariant of the coupled spin-oscillator is 1.



Symplectic invariants

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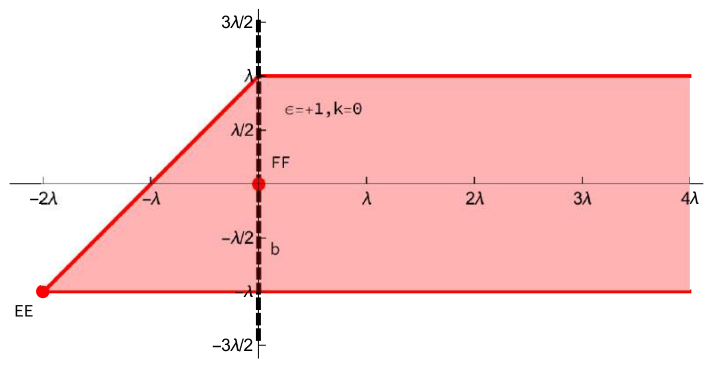
The twisting-index invariant

It is an equivalence class of integer m_f-tuples. It quantifies the dynamical complexity of the system at a global level, involving all focus-focus points at the same time.

- For systems with one focus-focus singularity it does not carry a lot of meaning.
- For the couple spin-oscillator, this polygon carries the value 0.



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Symplectic invariants

Taylor series invariant

It consists of m_f Taylor-series $(S_i)^\infty$, one per focus-focus singularity. It is a semi-global invariant of the fibration induced by (J, H). We associate an analytic function S_i that measures how the behaviour along the critical fibre differs from "perfect logarithmic" dynamics. Its Taylor-series is the invariant.

- Non-trivial to calculate. Until recently only linear terms were known.
- For the spin-oscillator is:

$$S(j,l) = (5\log 2 + \log\lambda)j + \frac{\pi}{2}l + \frac{1}{4\lambda}jl - \frac{1}{768\lambda^2}j(34j^2+39l^2)$$
$$+ \frac{1}{1536\lambda^3}lj(34j^2+23l^2) - \frac{1}{2621440\lambda^4}j(10727j^4+30620j^2l^2+13505l^4) + \ldots$$

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The Taylor-series invariant

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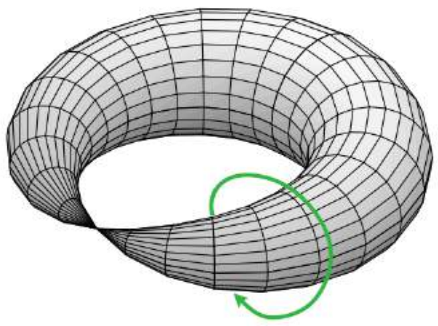
Taylor-series invariant

- (M, ω, Φ) semi-toric system, $\Phi = (L, H)$ that generates foliation $\mathcal{F}$.
- m focus-focus critical point, $\Phi(m) = 0$, $\Phi^{-1}(0)$ critical fibre.
- Thm. Eliasson/Miranda-Zung: $\exists(x_1, y_1, x_2, y_2)$ local symplectic coordinates around m:
  - The foliation $\mathcal{F}$ is given by connected components of $q := (q_1, q_2)$.

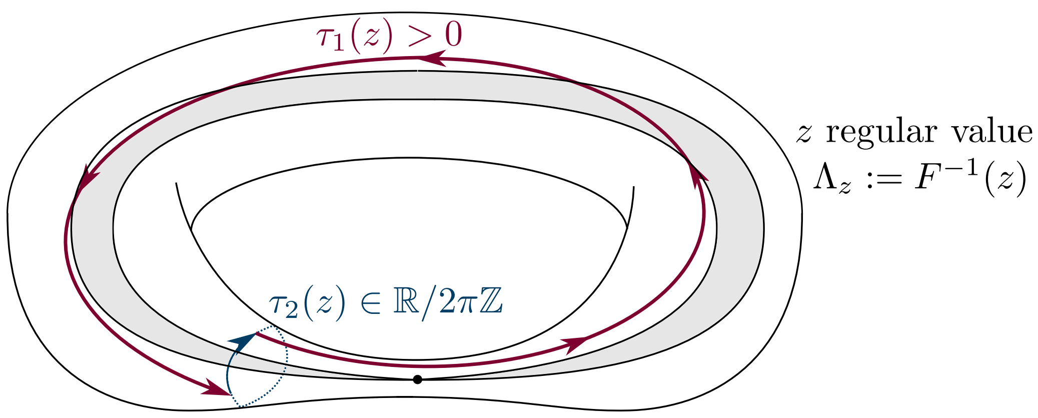
$$\begin{cases} q_1(x_1, y_1, x_2, y_2) = x_1x_2 + y_1y_2 \\ q_2(x_1, y_1, x_2, y_2) = x_1y_2 - y_1x_2 \end{cases}$$

  - Near m, q_2-orbits are 2π-periodic.
  - q_1 is hyperbolic, local stable manifold is (x_2, y_2)-plane, local unstable manifold is (x_1, y_1)-plane.
  - q_1 is radial, i.e. it tends towards 0 without spiraling in the unstable manifold.

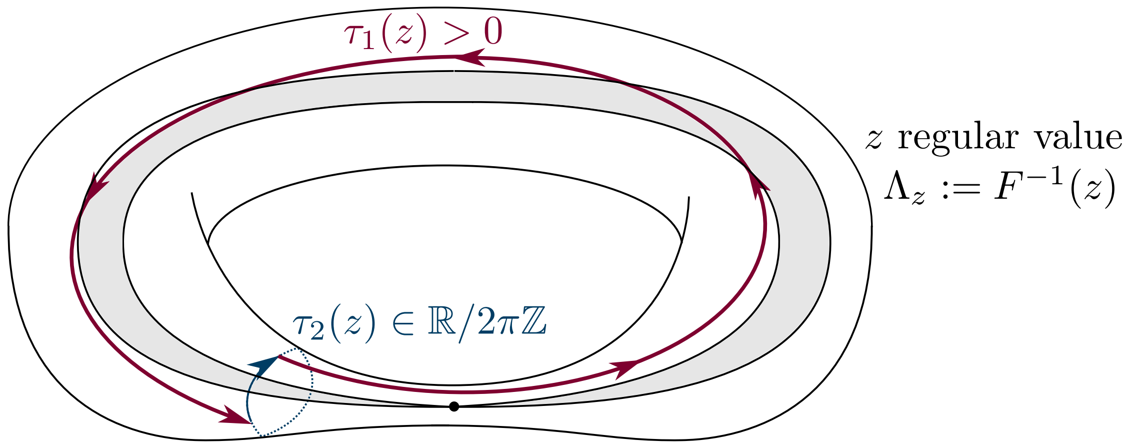


Taylor-series invariant

- $\exists \phi$ local diffeomorphism of $\mathbb{R}^2$ such that $q = \phi \circ \Phi$.
- We can extend to a global momentum map: $F := \phi \circ \Phi$. Locally agrees with q. We write $F := (F_1, F_2)$.
- Close to the critical point, the F_2-orbits must be periodic, because $F_2 = q_2$.
- Define fibres $\Lambda_z := F^{-1}(z)$ with $z := z_1 + iz_2$ close to 0. In particular, $F^{-1}(0)$ singular fibre.
- Take $a \in \Lambda_z$. Follow the flows on the regular fibre: $\tau_1(z)$ along φ^{F_1} and $\tau_2(z) \in \mathbb{R}/2\pi\mathbb{Z}$ along φ^{F_2}



Taylor-series invariant



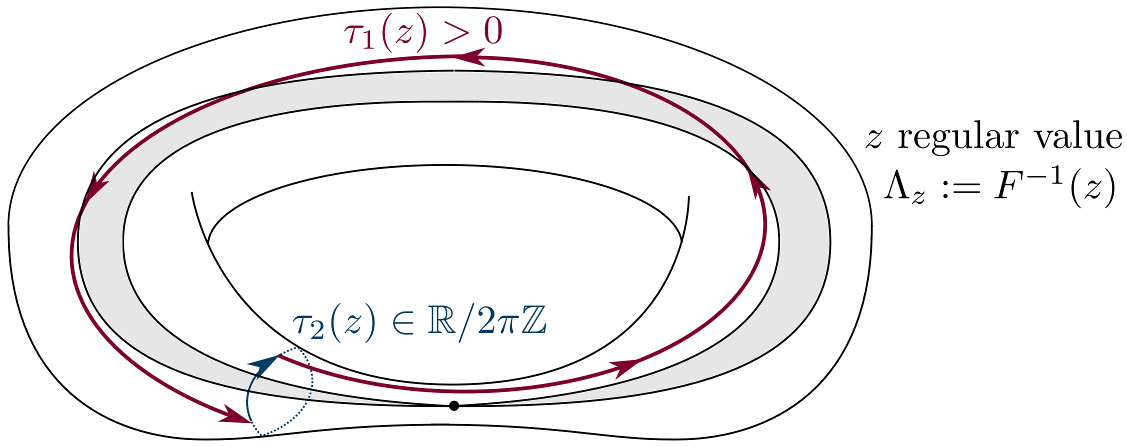
- Define

$$\begin{cases} \sigma_1(z) := \tau_1(z) + \mathrm{Re}(\ln z) \\ \sigma_2(z) := \tau_2(z) - \mathrm{Im}(\ln z) \end{cases}$$

- Then σ_1, σ_2 extend to smooth (single-valued) functions around 0
- The 1-form $\sigma := \sigma_1 dz_1 + \sigma_2 dz_2$ is closed[8] (Vũ Ngọc)
- Let S be the only function satisfying $dS = \sigma$, $S(0) = 0$
- Taylor series invariant = Taylor expansion of S around 0

[8] S. Vũ Ngọc, *On semi-global invariants for focus-focus singularities*, Topology, (2003)

Taylor-series invariant



The function S is related to the reduced action $I(z)$ of the system

- Take ϖ semi-global primitive of ω and γ_z our closed trajectory on the regular fibre
- Then the reduced action is $I(z) := \frac{1}{2\pi} \oint_{\gamma_z} \varpi$
- It satisfies $2\pi dI(z) = \tau_1(z)dz_1 + \tau_2(z)dz_2$
- Therefore the invariant is

$$S(z) = 2\pi I(z) - 2\pi I(0) + \text{Re}(z \ln z - z)$$

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Recent advances

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Recent advances

Recent advances:

- Calculation of the Taylor series invariant and the twisting index invariant of the coupled spin-oscillator[9] (A., Dullin, Hohloch)
- Calculation of the Taylor series invariant and the twisting index invariant of the coupled angular momenta depending on the different parameters[10] (A., Dullin, Hohloch)
- Description of a system with two focus-focus points[11] (Hohloch, Palmer)

[9] J. A., H. R. Dullin, S. Hohloch, *Taylor series and twisting-index invariants of coupled spin-oscillators*, arXiv:1712.06402 (2017).
[10] J. A., H. R. Dullin, S. Hohloch, *Taylor-series symplectic invariant of the coupled angular momenta*, (in progress).
[11] S. Hohloch, J. Palmer, *A family of compact semitoric systems with two focus-focus singularities*, arXiv:1710.05746 (2017).

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Coupled spin-oscillator

- We had $M = \mathbb{S}^2 \times \mathbb{R}^2$ with symplectic form $\omega = \lambda\,\omega_{\mathbb{S}^2} \oplus \mu\,\omega_{\mathbb{R}^2}$
- Functions:

$$L := \mu\frac{u^2+v^2}{2} + \lambda(z-1) \qquad H := \frac{1}{2}(ux+vy)$$

- After some transformations, the system can be rewritten as

$$L = p_1 \qquad H = \sqrt{\frac{-p_2(p_2-p_1)(p_2-p_1-2\lambda)}{2\lambda^2\mu}}\cos q_2$$

with symplectic coordinates (q_1, p_1, q_2, p_2) and symplectic form

$$\omega = dq_1 \wedge dp_1 \oplus dq_2 \wedge dp_2$$

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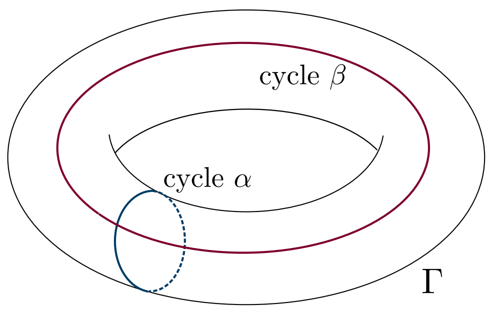
Coupled spin-oscillator

- Typical way to define action integral: $I(z) = \frac{1}{2\pi}\oint p\,dq$
- But requires solving a cubic equation. Better $I(z) = \frac{1}{2\pi}\oint q\,dp$
- If we integrate by parts, we get an elliptic integral

$$I(l,h) = \frac{1}{2\pi}\oint_\beta -h\left(3 + \frac{l}{p_2 - l} + \frac{l+2\lambda}{p_2 - l - 2\lambda}\right)\frac{dp_2}{w}$$

defined on

$$\Gamma_{l,h} := \{(p_2, w) : w^2 = P(p_2)\}, \quad P(p_2) = -\frac{2}{\lambda^2\mu}p_2(p_2-l)(p_2-l-2\lambda)-4h^2$$



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Coupled spin-oscillator

- When expanding elliptic integrals, the integral along the vanishing α cycle appears in front of the logarithm. We call it imaginary action:

$$J(l,h) := \frac{1}{2\pi i}\oint_{\alpha} -h\left(3+\frac{l}{p_2-l}+\frac{l+2\lambda}{p_2-l-2\lambda}\right)\frac{dp_2}{w}$$

- Since the α cycle vanishes as we approach the singularity m, we can use the residue theorem to expand it around m:

$$\frac{1}{\sqrt{\lambda\mu}}J(l,h) = -2h+\frac{1}{4\lambda}lh-\frac{1}{128\lambda^2}h(9l^2+20\lambda\mu h^2)+\ldots$$

- We can invert this series to obtain a “sort of” Birkhoff normal form $h=B(j,l)$ and get everything in terms of (j,l) instead of (l,h). The (j,l) actually correspond to the (z_1,z_2) where $z:=z_1+iz_2$.

Coupled spin oscillator

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And now we have all the ingredients we need:

- We expand the action integral (we get rational functions)

$$2\pi I(l,h) = 2\lambda\pi + \frac{\pi}{2} + l\arctan\left(\frac{l}{2\sqrt{\lambda\mu}h}\right) + J(l,h)\log\left(\frac{32\lambda}{\sqrt{l^2+4\lambda\mu h^2}}\right) - 2\sqrt{\lambda\mu}h$$
$$- \frac{\sqrt{\lambda\mu}}{2\lambda}lh + \frac{\sqrt{\lambda\mu}}{384\lambda^2(l^2+4\lambda\mu h^2)}h(63l^4 + 412\lambda\mu l^2h^2 + 544\lambda^2\mu^2h^4) + \ldots$$

- And now we substitute $h = B(j,l)$

$$2\pi I(j,l) = 2\lambda\pi - j\log|z| + j + l\arg(z)$$
$$+ (5\log 2 + \log\lambda)j + \frac{\pi}{2}l + \frac{1}{4\lambda}jl - \frac{1}{768\lambda^2}j(34j^2+39l^2) + \ldots$$

so all rational functions transform into a well-defined Taylor-series, which is what “theory predicted”.

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Coupled angular momenta

- A similar argument works for the coupled angular momenta but calculations are too complex
- Another approach: reduced period and rotation number

$$T := 2\pi \frac{\partial I}{\partial h}, \qquad W = -\frac{\partial I}{\partial l}$$

- We define imaginary reduced period and imaginary rotation number

$$T^\alpha := 2\pi \frac{\partial J}{\partial h}, \qquad W^\alpha = -\frac{\partial J}{\partial l}$$

- Then the derivatives of the invariant are (A., Dullin, Hohloch):

$$\frac{\partial S}{\partial j} = 2\pi \frac{T}{T^\alpha} + \ln|z|, \quad \frac{\partial S}{\partial l} = 2\pi \left(W^\alpha \frac{T}{T^\alpha} - W \right) - \arg(z)$$

System with two FF

System with two focus-focus singularities:

- Generalisation of the coupled angular momenta
- Defined on $M = \mathbb{S}^2 \times \mathbb{S}^2$ with $\omega = -(R_1 \omega_{\mathbb{S}^2} \oplus R_2 \omega_{\mathbb{S}^2})$, where $0 < R_1 < R_2$
- The system depends on four parameters $t := (t_1, t_2, t_3, t_4)$

$$\begin{cases} L = R_1 z_1 + R_2 z_2 \\ H = t_1 z_1 + t_2 z_2 + t_3 (x_1 x_2 + y_1 y_2) + t_4 z_1 z_2 \end{cases}$$

and it is semi-toric with two focus-focus singularities for some values of t.

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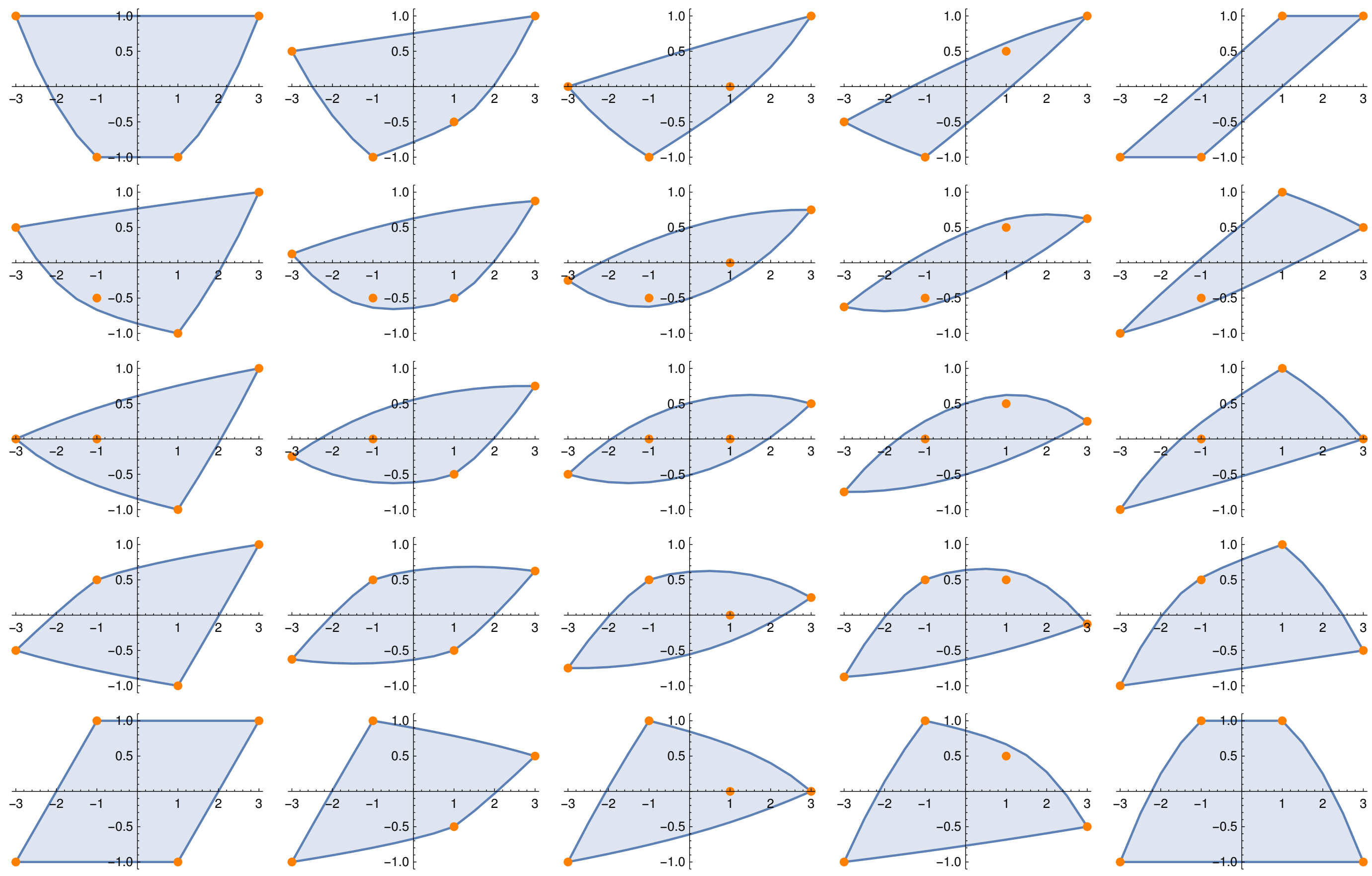
System with two FF

Special parameter choices with $s_1, s_2 \in [0,1]$:

$$t_1 = s_1(1-s_2) \qquad t_2 = s_2(1-s_1)$$
$$t_3 = (1-s_1)(1-s_2) + s_1 s_2 \qquad t_4 = (1-s_1)(1-s_2) - s_1 s_2$$



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Grazie per l’attenzione!