ON THE GRAPH THEOREM FOR LAGRANGIAN MINIMIZING TORI

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ABSTRACT. We study the graph property for Lagrangian minimizing submanifolds of the geodesic flow of a Riemannian metric in the torus (T^n, g), $n > 2$. It is well known that the transitivity of the geodesic flow in a minimizing Lagrangian submanifold implies the graph property. We replace the transitivity by three kind of assumptions: (1) r-density of the set of recurrent orbits for some $r > 0$ depending on g, (2) r-density of the limit set, (3) every point is nonwandering. Then we show that a Lagrangian, minimizing torus satisfying one of such assumptions is a graph.

INTRODUCTION

The theory of Lagrangian submanifolds that are invariant by a Hamiltonian flow has two well known statements called Birkhoff Theorems for Lagrangian invariant tori. These statements. which are actually open problems with many partial interesting answers, can be viewed as higher dimensional versions of the celebrated Birkhoff Theorems for invariant curves of measure preserving twist maps of the annulus. Let us recall briefly the context that these problems appear. Let $L : TM \longrightarrow \mathbb{R}$ be a convex, superlinear Lagrangian (or optic Lagrangian) defined in the tangent space TM of an n dimensional C^∞ manifold M. Let $\pi : TM \longrightarrow M$, $\pi(p, v) = p$, be the canonical projection.The first statement asserts that each continuous, Lagrangian torus $W \subset TT^n$ that is invariant by the Euler-Lagrange flow of L and that is minimizing, i.e., each orbit in W projects into a global minimizer of the Lagrangian action of L (see Section 1 for details), is in fact the graph of a section of the tangent bundle. Namely, the canonical projection restricted to W is a homeomorphism.

The second statement says that the same property is valid for a continuous, Lagrangian torus W, invariant by the Euler-Lagrange flow of L and that is homologous to the zero section. Notice that both problems seek to show that the set of points in the Lagrangian torus where the canonical projection is not injective is empty. If we assume in addition that the Lagrangian torus in the statements is smooth, this is equivalent to show that the set of singular points of the canonical projection restricted to the Lagrangian torus is empty.

The above statements are conjectures, and the best understood of them is the second one. Let us make a brief account of the results known up to date, to

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our best knowledge For two dimensional tori, Bialy [8] proved that a Lagrangian torus homologous to the zero section which is invariant by the geodesic flow of a reversible Finsler metric and has no periodic orbit is a graph of the canonical projection. Carneiro-Ruggiero [15] showed the same result for C^1 tori dropping the assumption on the periodic orbits. Thus, a C^1 Lagrangian torus homologous to the zero section invariant by the geodesic flow of a reversible Finsler metric in T^2 is a graph, our second statement is a true theorem in this case. In higher dimensions, Bialy-Polterovich [10], [11], Polterovich [37] showed that a C^3 Lagrangian torus of sufficiently high energy that is invariant by the Euler-Lagrange flow of an optic Lagrangian is a graph of the canonical projection provided that the torus is homologous to the zero section and that the flow in the torus is chain recurrent. The smoothness assumption on the Lagrangian torus is very important, it allows the application of powerful tools of symplectic topology to study the singular set of the restriction of the canonical projection to the Lagrangian torus. The proof in this case relies strongly on a result proved by Viterbo [39] which essentially implies that the Maslov cycle of a C^3, Lagrangian, invariant torus homologous to the zero section is trivial. Without any assumption on the dynamics of the flow in the Lagrangian torus, using graph selectors Marie Claude Arnaud [1] shows that if the torus is exact and Hamiltonianly isotopic to the zero section then it is a graph. See also a generalization of this result, by Bernard and dos Santos, for Lipschitz Lagrangian torus in [7]. For further generalizations using Floer homology by Amorim, Oh and dos Santos see [4].

For the case of the geodesic flow, In the book [35], the graph property is proven using a theorem due to Viterbo in [39], under the assumptions that the torus is homologous to the zero section and the set of nonwandering points of the restriction of the flow coincides with the whole of the torus.

The first of the above statements is much less explored in the literature. It was proven for C^1 Lagrangian invariant tori of Riemannian metrics in T^2 by Bialy and Polterovich [9] and Bangert [5] in the early 1980's; the statements holds even for continuous invariant tori without closed orbits. It was proved for C^1 Lagrangian, invariant, nonsingular tori in the Mañé critical level of an optic Lagrangian in T^2 by Carneiro and Ruggiero [14] (2004). By the way, in [15] it is given an example of a continuous, invariant minimizing torus with periodic orbits that is not a graph of the canonical projection. As far as we know, no further progress has been made in the subject (even assuming high smoothness of the Lagrangian invariant tori) in higher dimensions. Although the assumptions in both statements are related, the variational assumption in the first one is different in nature from the topological assumption in the second one.

The graph theorem for minimizing tori in dimension 2 follows from a combination of calculus of variations and the well understood dynamics of nonsingular flows in the 2-torus. Since the seminal works of Morse and Hedlund it is known that accumulation properties of globally minimizing geodesics determine their intersection properties: canonical projections of minimizing recurrent orbits are laminations in the two-torus.

On the other hand, in higher dimensions, the nonwandering set of a nonsingular flow in the torus might be extremely more complicated than in dimension two, giving no clue at all about the accumulation properties of minimizing orbits in

a Lagrangian invariant torus. It is not difficult to show that a C^1, Lagrangian, invariant torus that is a graph must be minimizing.

So the two categories, minimizing and homologous to the zero section, are closely connected. We might think that both graph problems could be solved by applying the same sort of techniques. However, the minimizing assumption does not give any hint about the triviality of the Maslov cycle.

The first main result of the paper relates the geometry to the dynamics, as a sufficient condition to have the graph property:

Theorem A: Let (T^n, g) be a C^∞ Riemannian metric in the torus, and let $W \subset T_1 T^n$ be a C^2 invariant, Lagrangian torus. There exists $r > 0$ depending on the supremum of sectional curvatures of g and the supremum of the seccional curvatures of W with respect to the Sasaki metric such if W is minimizing and the limit set of W is r-dense, then the canonical projection $\pi : T_1 T^n \longrightarrow T^n$ restricted to W is a diffeomorphism.

Given a metric space (X, d) we say that a subset Y is r-dense in X if the open ball of radius r around each point of X contains an element of Y. The next main contribution of the paper is another extension of a well known result in the context of Lagrangian tori homologous to the zero section.

Theorem B: Let (T^n, g) be a C^∞ Riemannian metric in the torus, and let $W \subset T_1 T^n$ be a C^1 minimizing Lagrangian torus such that every point is non-wandering for the restricted flow. Then W is a graph.

Theorems A and B extend to minimizing tori well known results that are known for invariant tori which are homologous to the zero section, after the work of Polterovich [37].

The proofs of Theorems A and B can be extended to Finsler metrics in the torus, therefore to Euler-Lagrange flows in sufficiently high energy levels of Tonelli Lagrangians. We would like to point out that the known partial results of Birkhoff's theorems do not imply either Theorem A or Theorem B.

Indeed, a Lagrangian invariant tori homologous to the zero section is known to be a graph if the geodesic flow in W is chain recurrent or if every point is nonwandering as we mentionned before. Neither the r-density of the recurrent set of W or the limit set for some $r > 0$ imply either of the above properties. Moreover, notice that items (1) and (2) in Theorem A are independent: the limit set might contain wandering orbits, so the assumption of item (1) does not imply item (2).

The strategy of the proof of Theorem B is different from the strategy of the proof of Theorem A of course, since the assumptions of both theorems are of different nature. Moreover, the theory of the Maslov cycle of invariant Lagrangian tori homologous to the zero section (see Section 2 for details) cannot be applied to the case of minimizing tori. The minimizing assumption does not imply a priori that the torus is homologous to the zero section.

In the proof of Theorem B we apply a general result that is interesting in itself, a topological description of the singular set of a C^1, Lagrangian invariant torus.

Theorem C Let (M, g) be a C^∞ compact Riemannian manifold, let $\phi_t : T_1 M \longrightarrow T_1 M$ be the geodesic flow, and let $W \subset T_1 M$ be a C^1 compact, invariant Lagrangian submanifold. Then the set of points $S(W)$ where the canonical projection is not

regular has the following property: for every $\theta \in S(W)$ there exists a connected open neighborhood $B(\theta) \subset W$ such that

(1) $B(\theta)$ is a flow box

$$B(\theta) = \cup_{|t|<\epsilon} \phi_t(\Sigma(\theta))$$

where $\Sigma(\theta) \subset S(W)$ contains θ and is a continuous cross section of the geodesic flow.
(2) $S(W) \cap B(\theta)$ is the union of a finite number of continuous cross sections of the geodesic flow.

For the definition of a topological cross section we refer to Section 4.

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1. Preliminaries

Let us introduce some notations. TM is the tangent bundle of a C^∞ manifold M, T_pM is the tangent space of M at a point p, the unit tangent bundle of a Riemannian manifold (M, g) is T_1M, and the canonical projection $\pi : TM \longrightarrow M$ is the map $\pi(p, v) = p$, where (p, v) is a point in TM in canonical coordinates. We shall always assume that M is complete. The universal covering of M will be denoted by $\tilde{M}$.

The geodesic flow of (M, g) in T_1M will be denoted by $\phi_t : T_1M \longrightarrow T_1M$. The canonical one form of the geodesic flow will be α, the pull back by the Legendre transform of the Liouville form on T^*M and the canonical two-form $\omega = d\alpha$. The two-form ω is symplectic (non-degenerate and closed) and preserved by ϕ_t. The tangent space $T_\theta TM$ is the orthogonal (with respect to the Sasaki metric) sum of the horizontal subspace H_θ, the kernel of the connection map, and the vertical subspace $V_\theta = Ker(D_\theta\pi)$. The subspace $N_\theta \in T_\theta T_1M$ of vectors which are orthogonal to the geodesic vector field $X(\theta)$ is preserved by $D\phi_t$ for every $t \in \mathbb{R}$, and we have $N_\theta = \mathcal{H}_\theta \bigoplus \mathcal{V}_\theta$, where $\mathcal{H}_\theta = H_\theta \cap N_\theta$ and $\mathcal{V}_\theta = V_\theta \cap N_\theta$.

Many forthcoming definitions and results have natural generalizations to Tonelli Lagrangians: a $C^k, k \geq 3$ Lagrangian $L : TM \to \mathbb{R}$ is a Tonelli Lagrangian if it is strictly convex and superlinear (i.e., $\lim_{\|v\|\to+\infty} \frac{L(p,v)}{\|v\|} = +\infty$ for every $v \in T_pM$, for every $p \in M$) when restricted to each tangent space T_pM, $p \in M$. A Lagrangian L defined in TM has a natural lift in $T\tilde{M}$ that we shall denote by $\tilde{L}$.

1.1. **Lagrangian submanifolds.** Suppose that the dimension of M is n. A subspace L_θ of $T_\theta TM$ is called Lagrangian if its dimension is n and the restriction of ω to L_θ is vanishes. In the notation of symplectic geometry, L_θ is Lagrangian for ω if it is an isotropic subspace (i.e., ω vanishes in $L_\theta \times L_\theta$) with maximal dimension (n). A continuous vector subbundle L of TM is called invariant if it is is invariant by the action of $d\phi_t$ for every $t \in \mathbb{R}$.

A smooth submanifold $W \subset TM$ is called Lagrangian if it is isotropic of dimension n, that is canonical symplectic the two-form ω in T_pW vanishes for every $p \in W$. A smooth Lagrangian submanifold W is called a **graph** if the canonical projection π restricted to W is a diffeomorphism. Let us denote by S, the singular set, the set of points $\theta \in W$ where the canonical projection is singular, namely, where $T_\theta W \cap V_\theta$ is a nontrivial subspace.

1.2. **Jacobi fields and conjugate points.** Invariant Lagrangian subspaces can be described in terms of Jacobi fields and symmetric operators with very special properties, let us recall briefly this description for geodesic flows.

Let γ_θ be a geodesic of (M, g). Taking a parallel frame $\{e_i(t)\}$, $i = 1, 2, .., n - 1$ of vector fields along the geodesic $\gamma_\theta(t) = \pi(\varphi_t(\theta))$, which are orthogonal to $\gamma_\theta'(t)$, the subspace of perpendicular Jacobi fields along γ_θ is obtained as the image of a one parameter family of linear operators $\mathcal{J}(t)$ given by $(n - 1) \times (n - 1)$ matrix solutions of the Jacobi equation

$$\mathcal{J}''(t) + K(t)\mathcal{J}(t) = 0$$

where the derivatives are covariant derivatives of (M, g) along $\gamma_\theta(t)$, and $K(t)$ is the matrix of sectional curvatures

$$K_{ij}(t) = g(R(\gamma_\theta'(t), e_i(t))\gamma_\theta'(t), e_j(t)),$$

$R(X, Y)Z$ the curvature tensor of (M, g).

When $\mathcal{J}(t)$ is invertible, the matrix $\mathcal{J}'(t)\mathcal{J}^{-1}(t)$ gives a solution of the matrix Riccati equation

$$U'(t) + U^2(t) + K(t) = 0.$$

Now, if $W(t)$ is a Lagrangian invariant subbundle, then $L(t) = W(t) \cap N_{\varphi_t(\theta)}$ is a subbundle defined along the orbit $\varphi_t(\theta)$ and there exist $n - 1$ linearly independent Jacobi fields J_i along the geodesic γ_θ such that $L(t)$ is generated by $(J_i(t), J_i'(t))$, $i = 1, ..., n-1$, for every $t \in \mathbb{R}$. Such Jacobi fields are perpendicular to $\gamma_\theta'(t)$ for every t, so there exists a matrix solution of the Jacobi equation associated to this basis and hence, whenever this solution is invertible it defines a solution of the Riccati equation $U(t)$ associated to a Lagrangian bundle $W(t)$ At these points, the subspace $L(t)$ is the graph of $U(t)$ viewed as a linear operator $U(t) : \mathcal{H}_{\varphi_t(\theta)} \longrightarrow \mathcal{V}_{\varphi_t(\theta)}$. The graph representation of $W(t)$ fails whenever $W(t)$ intersects $V_{\varphi_t(\theta)}$ nontrivially. Equivalently, when there exists a nontrivial linear combination $J(t)$ of the $J_i(t)$'s in γ_θ such that $J(t_0) = 0$ for some t_0. The Riccati equation is a well known tool to study conjugate points along geodesics.

Definition 1.1. Let γ_θ be a geodesic of (M, g). Two points $\gamma_\theta(t)$, $\gamma_\theta(s)$, $t < s$ are conjugate along γ_θ if there exists a nontrivial Jacobi field $J(r)$ of γ which vanishes at $r = t$ and $r = s$. The geodesic $\gamma_\theta : (a, b) \longrightarrow M$ is said to have no conjugate points if every Jacobi field of γ_θ has at most one zero in (a, b). Equivalently, $\gamma_\theta : (a, b) \longrightarrow M$ has no conjugate points if for each $\eta = \phi_t(\theta)$, where $t \in (a, b)$, the intersection of $D_\eta\phi_r \mid_{V_\eta}$ with the vertical bundle $V_{\phi_t(\theta)}$ occurs only at $r = 0$.

The next two results are considered classical in the theory of geodesics without conjugate points. The first one is a straightforward consequence of the work of Green ([25] pages 1, 2).

Theorem 1.2. *Let (M, g) be a complete C^∞ Riemannian manifold . A geodesic $\gamma_\theta : (a, b) \longrightarrow M$ has no conjugate points if and only if there exists a matrix solution of the Jacobi equation $J(t)$ along γ that is invertible for every $t \in (a, b)$. Equivalently, $\gamma_\theta : (a, b) \longrightarrow M$ has no conjugate points if there exists a solution of the Riccati equation $U(t)$ along γ that is defined for every $t \in (a, b)$.*

The second result is due to Eberlein ([22], Lemma 2.8) for Riemannian metrics (see Contreras-Iturriaga [20] for a Hamiltonian version).

Theorem 1.3. *Let (M, g) be a compact, C^∞ Riemannian manifold, Let $k_0 > 0$ be a constant such that all sectional curvatures of (M, g) are bounded below by $-k_0$. Let $\gamma : I \longrightarrow M$ be a unit speed geodesic where $I = (a, b)$ is an open interval. Then, any matrix solution $U(t)$ of the Riccati equation $U'(t) + U^2(t) + K(t) = 0$ that is defined for every $t \in I$ satisfies*

$$-k_0 coth(k_0(b - t)) \le g(U(t)X, X) \le k_0 coth(k_0(t - a)).$$

In particular,

(1) *For any $\epsilon > 0$ that is smaller than the length of $I = (a, b)$, there exists $C(\epsilon, k_0) > 0$ such that*

$$\mid g(U(t)X, X) \mid \le C(\epsilon, k_0)$$

for every $t > a + \epsilon$.

(2) *if $U(t)$ is defined for every $t \in \mathbb{R}$ we have that*

$$sup_{t \in \mathbb{R}} \mid g(U(t)X, X) \mid \le k_0.$$

1.3. **Global minimizers and conjugate points.** We recall the definition of minimizing Lagrangian submanifold.

The action of the Lagrangian L in an absolutely continuous curve $c : I \to M$ is $A_L(c) = \int_I L(c(t), c'(t))dt$. By Tonelli's Theorem, local minimizers of the action are the solutions of the Euler-Lagrange equation of L, in particular when $L(p, v) = \frac{1}{2}g_p(v, v)$ such local minimizers are the geodesics of (M, g).

In order to define the notion of minimizing submanifold, let us consider the abelian covering M_a of a manifold M that is the quotient of the universal covering by the subgroup of $\pi_1(M)$ generated by commutators $xyx^{-1}y^{-1}$. The abelian covering projection $\pi_a : M_a \longrightarrow M$ is a local homeomorphism and the first integer homology group has a natural representation in the set of automorphisms of M_a. Any Lagrangian $L : TM \longrightarrow \mathbb{R}$ lifts to a Lagrangian $L_a : TM_a \longrightarrow \mathbb{R}$ in a natural way. In particular, if (M, g) is a Riemannian manifold, the metric g lifts to a metric g_a in M_a that is locally isometric to g. As in the case of the universal covering, the first homology group acts a discrete subgroup of isometries of (M_a, g_a). Of course, if the fundamental group is abelian (as in the torus case), M_a coincides with $\tilde{M}$.

We say that an absolutely continuous curve $\gamma : \mathbb{R} \longrightarrow M$ is a **global minimizer** if for every $a < b \in \mathbb{R}$, and any lift $\tilde{\gamma}$ of γ in M_a, given an absolutely continuous curve $\delta : [a, b] \longrightarrow M_a$ with $\delta(a) = \tilde{\gamma}(a)$, $\delta(b) = \tilde{\gamma}(b)$, we have that the action $A_{L_a}(\delta) = \int_a^b L_a(\delta(t), \delta'(t))dt$ is at least the action in $\tilde{\gamma}$: $A_{L_a}(\delta) \ge \int_a^b L_a(\tilde{\gamma}(t), \tilde{\gamma}'(t))dt$.

Definition 1.4. An invariant submanifold W is called minimizing if any lift to the abelian cover of the canonical projection of every trajectory in W is a global minimizer.

Following a notation given by Morse in [34], we say that an absolutely continuous curve $\gamma : \mathbb{R} \longrightarrow M$ is a **type A minimizer** if for every $a < b \in \mathbb{R}$, and any lift $\tilde{\gamma}$ of γ in $\tilde{M}$, given an absolutely continuous curve $\delta : [a, b] \longrightarrow \tilde{M}$ with $\delta(a) = \tilde{\gamma}(a)$, $\delta(b) = \tilde{\gamma}(b)$, we have that the action $A_{\tilde{L}}(\delta) = \int_a^b \tilde{L}(\delta(t), \delta'(t))dt$ is at least the action in $\tilde{\gamma}$: $A_{\tilde{L}}(\delta) \ge \int_a^b \tilde{L}(\tilde{\gamma}(t), \tilde{\gamma}'(t))dt$. For Riemannian Lagrangians , type A minimizers of the Lagrangian action are globally minimizing geodesics in the universal covering $(\tilde{M}, \tilde{g})$. It is easy to show that global minimizers are type A minimizers, the converse is not true in general. This implies that global minimizers have no conjugate points.

In particular, every trajectory in W projects onto a geodesic that has no conjugate points.

2. RECURRENCE AND REGULARITY OF THE CANONICAL PROJECTION RESTRICTED TO A LAGRANGIAN MINIMIZING TORUS

The purpose of this section is to show that nontrivial dynamics in an invariant Lagrangian torus combined with minimizing properties imply regularity of the canonical projection. We start by recalling what happens at recurrent points of tori which are homologous to the zero section. Based on the results in this case we shall look at the limit set of Lagrangian minimizing tori and extend such results by different methods.

Recall that a point $\theta \in T_1M$ is forward (resp. backward) recurrent for the geodesic flow if given any open neighborhood B of θ there exists $t_B > 1$ (resp. $t_B < -1$) such that $\phi_{t_B}(\theta) \in B$. A point θ is recurrent if it is positively and negatively recurrent. We shall analyze the problem in each of the assumptions of the main Theorem.

2.1. **The Maslov cycle.** One of the classical approaches to study the points of a Lagrangian invariant submanifold where the canonical projection is not regular is the theory of the Maslov cycle.

Recall that $\Lambda(n) = \frac{U(n)}{O(n)}$, the set of all Lagrangian subspaces in a symplectic space is called the Lagrangian Grassmann manifold.

For a symplectic manifold (N, ω) we can construct the Lagrangian Grassmann bundle $\Pi : \Lambda(N) \to N$, the bundle over N whose fibers consist of all the Lagrangian subspaces. That is, for each point $x \in N$ we consider all Lagrangian subspaces of T_xN. Let $\Sigma \subset N$ be a $2n - 1$ connected submanifold. Let $\Lambda(\Sigma)$ be the induced bundle $\pi^{-1}(\Sigma)$. For a fixed section $\mathbb{E}$, the Maslov cycle is the subbundle of $\Lambda(\Sigma)$ whose fiber over a point x consists of all Lagrangian subspaces which intersects the fiber E_θ non trivially.

In our case, the symplectic form on TM induced by the Legendre transform defined the Riemannian metric. Let us consider the Lagrangian Grassmannn $\Lambda(TM)$ and the subbundle $\Lambda(T_1M)$ over the unit tangent bundle.

Definition 2.1. The Maslov cycle Λ_V of T_1M is defined by
$$\Lambda_V = \bigcup_{k \in [1, n-1]} \Lambda_k(T_1M)$$
where $\Lambda_k(T_1M)$ is the bundle of Lagrangian subspaces of N whose intersection with the vertical subspace has dimension k.

The Maslov cycle Λ_V is a subset of the Grassmannian $\mathcal{G}$ of Lagrangian subspaces of the fiber bundle N. Each of the sets $\Lambda_k(TM)$ is a smooth submanifold of codimension $\frac{k(k+1)}{2}$, so Λ is a stratified manifold.

There is a natural lift of the geodesic flow ϕ_t to the Grassmannian $\mathcal{G}$. Namely, if $(\theta, E) \in \mathcal{G}$, where $\theta \in T_1M$, $E \subset N_\theta$ is Lagrangian, then $\phi_t^G(\theta, E) = (\phi_t(\theta), D_\theta\phi_t(E))$ defines a flow in $\mathcal{G}$. This flow commutes with the projection $P : \mathcal{G} \longrightarrow T_1M$ given by $P(\theta, E) = \theta$: $P \circ \phi_t^G = \phi_t \circ P$. A C^k invariant Lagrangian submanifold $\Sigma \subset T_1M$ lifts to a C^{k-1}, ϕ_t^G-invariant submanifold $\bar\Sigma$ of the same dimension. The next result pointed out by Polterovich [37] is an important application of the work of Viterbo [39]:

Theorem 2.2. *Let W be a C^2 Lagrangian torus invariant by the geodesic flow of (T^n, g). If W is homologous to the zero section then recurrent orbits of the lifted flow ϕ_t^G restricted to the lift $\bar{W}$ of W in the Grassmannian do not meet the Maslov cycle.*

A brief sketch of the proof of Theorem 2.2 is the following: Viterbo shows that the Maslov cycle of a Lagrangian invariant torus homologous to the zero section is trivial. From this fact it is possible to deduce that the tangent space along compact, "almost closed" pieces of recurrent orbits of a C^2 Lagrangian invariant torus homologous to the zero section cannot cross the vertical subbundle. Therefore, such orbits are regular for the canonical projection.

2.2. **Lagrangian, invariant, minimizing tori.** Now, let us consider a C^2 Lagrangian minimizing torus. Viterbo's result might not apply in this case since we know nothing about the triviality of the Maslov cycle of the torus. Instead, we have the following Arnold's generalization to Lagrangian invariant subspaces of the Sturm-Liouville theorem for second order, ordinary differential equations, that describes the intersections of a Lagrangian invariant subspace with the Maslov cycle. We state the result as quoted in [18].

Theorem 2.3. *Let $H = e$ be a regular energy level of a Tonelli Hamiltonian $H : T^*M \longrightarrow \mathbb{R}$ defined in a compact, n-dimensional manifold M, and let λ_θ and λ'_θ be two Lagrangian subspaces. Let ϕ_t^G be the lift of the geodesic flow in the Grassmannian. If $t \mapsto d\phi_t^G(\lambda_\theta)$ has $n + 1$ points of intersection with the Maslov cycle (counted with multiplicity) in an interval $[t_1, t_2]$ then $t \mapsto d\phi_t^G(\lambda'_\theta)$ has at least one point of intersection with the Maslov cycle in the same interval.*

Corollary 2.4. *Let (M, g) be a compact Riemannian manifold. Let $l > 0$ be such that every geodesic $\gamma : (a, b) \longrightarrow M$ has no conjugate points whenever $0 < b - a \leq l$. Then, for every $\theta \in T_1 M$, and any Lagrangian subspace $L_\theta \subset T_\theta T_1 M$, the number of times t_i in (a, b) where $d_\theta \phi_{t_i}(L_\theta) \cap V_{\phi_{t_i}(\theta)}$ is not zero is bounded above by n. In particular, if W is a C^1 minimizing, Lagrangian torus invariant by the geodesic flow of (T^n, g), we have that for every $\theta \in W$, the number of times where $T_{\phi_t(\theta)} W$ meets the vertical subspace is at most n.*

Proof. Recall briefly that the existence of $l > 0$ is a consequence of Rauch comparison Theorem.

According to Theorem 2.3, if $T_{\phi_t(\theta)} W = D_\theta \phi_t T_\theta W$ meets non trivially the vertical bundle more than $n + 1$ times t_i in an interval (a, b), then the subspaces $D_\theta \phi_t(V_\theta)$ will meet the vertical bundle at least twice in the same interval (a, b). This contradicts the absence of conjugate points in the geodesic $\gamma_\theta(t)$. $\square$

Corollary 2.5. *Let W be a C^1 minimizing, Lagrangian torus invariant by the geodesic flow of (T^n, g). There exists $\alpha > 0$ with the following property: given $\theta \in W$ there exists $b_\theta \in \mathbb{R}$ such that the angle subtended by the vertical subspace $V_{\phi_t(\theta)}$ and $T_{\phi_t(\theta)} W$ is at least α for every $t > b_\theta + 1$.*

Proof. Let $\theta \in W$. By Corollary 2.4 there exists a compact interval $[a_\theta, b_\theta]$ which contains all the intersections of $T_{\phi_t(\theta)} W$ with the vertical bundle. Let us suppose that b_θ is the maximum of the values of t where $T_{\phi_t(\theta)} W$ meets the vertical bundle. Let $U(t)$ be the solution of the Riccati equation associated to W defined along

$\phi_t(\theta)$. By Eberlein's Theorem 1.3, if $-k_0 < 0$ is a lower bound for the sectional curvatures of (T^n, g), we have that $\tilde{U}(t) = U(b_\theta + t)$ satisfies

$$| \, g(\tilde{U}(t)X, X) \, | \le k_0 coth(k_0 t)$$

for every $t > 0$, and every unit vector $X \in T_{\gamma(b_\theta+t)}M$ perpendicular to $\gamma'(t)$. Since the subspace $T_{\phi_t(\theta)}W \cap N_{\gamma(t)}$ is the graph of $U(t)$, which is a bounded operator for $t > 1$, elementary linear algebra implies that there exists $\alpha > 0$ such that for every $t > 1$ we have that the angle between the vertical bundle and $T_{\phi_t(\theta)}W$ is at least α, for every $t > b_\theta + 1$. $\qquad\square$

Lemma 2.6. *Let W be a C^1 minimizing, Lagrangian torus invariant by the geodesic flow of (T^n, g). Then every recurrent orbit is regular for the canonical projection. Moreover, if $\alpha > 0$ is the number defined in Corollary 2.5, the angle from $T_{\phi_t(\theta)}W$ and $V_{\phi_t(\theta)}$ is at least α for every $t \in \mathbb{R}$.*

Proof. Let $\theta \in W$ be a forward recurrent point. Since every point of its orbit is recurrent, then every point is accumulated by positive iterates of $\phi_t(\theta)$ for $t > 0$ arbitrarily large. According to Corollary 2.5, for $t > b_\theta + 1$ the angle subtended by $T_{\phi_t(\theta)}W$ and $V_{\phi_t(\theta)}$ is at least $\alpha > 0$. Therefore, there is no t_0 such that $T_{\phi_{t_0}(\theta)}W \cap V_{\phi_{t_0}(\theta)} \ne \{0\}$, for otherwise, there would be a sequence $t_n \to \infty$, with $\phi_{t_n}(\theta) \to \phi_{t_0}(\theta)$, where $T_{\phi_{t_n}(\theta)}W$ tends to a Lagrangian subspace that is transversal to $V_{\phi_{t_0}(\theta)}$. Since TW is a continuous bundle, the subspaces $T_{\phi_{t_n}(\theta)}W$ tend to $T_{\phi_{t_0}(\theta)}W$ and therefore, the latter subspace must be transversal to the vertical bundle. The argument extends easily to backward recurrent points. $\qquad\square$

3. Graph property of Lagrangian minimizing tori in a neighborhood of the limit set

Combining the results of the previous section we get,

Proposition 3.1. *Let (T^n, g) be a C^∞ Riemannian metric in the torus and let W be a C^1 Lagrangian invariant torus. Suppose that W is either homologous to the zero section or minimizing. Then*

(1) *The Riccati solution $U_\theta(t)$ associated to a recurrent orbit $\phi_t(\theta)$ is well defined for every $t \in \mathbb{R}$.*
(2) *There exists $r > 0$ such that the r-tubular neighborhood $B_r \subset W$ of the closure of the set of recurrent orbits of W is regular.*

Proof. Theorems 1.2, 1.3 together with Theorem 2.2 and Lemma 2.6 imply that recurrent points of W are regular and the underlying geodesics have no conjugate points. Moreover, the tangent space of W never meets the vertical bundle along recurrent orbits and there is a positive lower bound α for the angle subtended by TW and the vertical bundle along recurrent orbits. Since W is C^1 and compact, this implies that there exists an open neighborhood of radius $r > 0$ of the closure of the set of recurrent points of W where TW is transversal to the vertical fibration. Therefore, the canonical projection restricted to the r-tubular neighborhood B_r of recurrent points is regular. $\qquad\square$

Next, let us extend the above result to the limit set of a Lagrangian minimizing torus.

Recall that the limit set $L(W)$ is the union of the forward or ω-limit set and the backward or α-limit set of W by the action of the geodesic flow. The ω-limit

set of W, $\omega(W)$, is the union of the ω-limit sets of points, namely, given $\theta \in W$, the ω-limit set of θ, $\omega(\theta)$, is the set of points η such that there exists a sequence of numbers $t_n \to +\infty$ with $\eta = \lim_{n \to +\infty} \phi_{t_n}(\theta)$. The α-limit set $\alpha(W)$ is defined analogously replacing $t_n \to +\infty$ by $t_n \to -\infty$.

Lemma 3.2. *Let (T^n, g) be a C^∞ Riemannian metric in the torus and let W be a C^1 Lagrangian invariant minimizing torus. Then*

(1) *The Riccati solution $U_\theta(t)$ associated to an orbit $\phi_t(\theta)$ in the limit set of W is well defined for every $t \in \mathbb{R}$.*

(2) *There exists $r > 0$ such that the r-tubular neighborhood $B_r \subset W$ of the closure of the limit set $L(W)$ of W is regular.*

Proof. So let $\theta \in L(W) \cap S(W)$, and let $\eta \in W$ such that the orbit of η accumulates θ, namely, there is a sequence $t_n \to \infty$ such that $\lim_{n \to \infty} \phi_{t_n}(\eta) = \theta$.

By Corollary 2.4 the subspaces $D_\eta \phi_t(V_\eta)$ just meet the vertical bundle at a finite number of t_k, $0 \leq k \leq n$. So there exists $t_P > 0$ such that

$$D_\eta \phi_t(V_\eta) \cap V_{\phi_t(\eta)} = \{0\}$$

for every $t \geq t_P$. Since $t_n \to \infty$ there exists $t_{n_0} > t_P$ such that $t_n - t_{n_0} > 1$ for every $n > n_0$. Therefore, by Theorem 1.3 there exists $C(1, k_0) > 0$ such that the norm of the Riccati operator $U(\phi_{t_{n_0}+t}(\eta))$ representing $T_{\phi_{t_{n_0}}(\eta)} W$ is bounded above by $C(1, k_0)$ for every $t > t_{n_0}$. Hence, by the the continuity of the tangent bundle of W we conclude that the same happens with the Riccati operator of W at θ. Since every point of the orbit of θ is in the limit set, the same happens at every point of the orbit. So the solution of the Riccati equation $U_\theta(t)$ is well defined for every $t \in \mathbb{R}$ and hence (by Corollary 2.5) the tangent space of W at θ forms an angle greater than α with the vertical bundle, where $\alpha > 0$ is given in Corollary 2.5. By the compactness and the smoothness of W, the transversality of the tangent space of W with respect to the vertical fibres extends to the closure of $L(W)$. Since the closure of the limit set is compact, there exists $r > 0$ such that in an open neighborhood of radius $r > 0$ of the closure every point is regular for the canonical projection. $\qquad \square$

3.1. **On the geometry of Lagrangian minimizing tori at the limit set.** In this subsection we look at the following problem: in what extent the geometry of a Lagrangian invariant torus as an isometrically embedded submanifold of $T_1 T^n$ depends on the geometry of its canonical projection? The question is natural and motivated by the fact that the Riccati operator associated to the Lagrangian torus is related to sectional curvatures of (T^n, g) at places where the torus is a graph. Let us explain in detail.

Lemma 3.3. *Let (M, g) be a compact C^∞ Riemannian manifold of dimension n and let $k_0 > 0$ be such that $-k_0$ is a lower bound for the sectional curvatures of (M, g). Then, given $C > 0$ there exists a constant $K_C = K_C(k_0) > 0$ such that every C^3 Lagrangian minimizing submanifold $W \subset T_1 M$ whose C^3 norm is bounded above by C has the following property:*

Suppose that $\theta \in W$ is a point in the limit set. Then the supremum of the absolute values of sectional curvatures of W at θ with respect to the Sasaki metric is bounded by K_C.

Proof. By assumption, there exists an open neighborhood $U \subset W$ of θ where the canonical projection is a diffeomorphism. So we have a local solution of the Hamilton-Jacobi equation $H(q, d_q f) = 1$ where $d_q f = \alpha_q$ is the canonical projection of the canonical one form of the geodesic flow restricted to W. The canonical projection $\pi(U)$ of U is an open subset of M where the form α is exact, its Reeb flow is a unit flow by geodesics, the projections of the orbits of U. Let X be the unit vector field in $\pi(U)$ tangent to the Reeb flow, then the levels $f = c$ are co-dimension one C^3 submanifolds that are perpendicular to X and invariant by the Reeb flow in the following sense: if $\psi_t = \pi(\phi_t)$ is the Reeb flow (the projection of the restriction of ϕ_t to U) then for $p \in \pi(U)$ we have $\psi_t(f^{-1}(p)) = f^{-1}(\psi_t(p))$ as long as $\psi_t(p)$ belongs to $\pi(U)$.

Claim: The matrix of the second fundamental form of a level $f^{-1}(c)$ at a point $x = \pi(x, v)$, $(x, v) \in U$, is the matrix solution of the Riccati equation associated to the geodesic $\gamma_{(x,v)}$.

This is perhaps well known by specialists, but we recall briefly the proof of this fact. The unit vector field X is the unit normal field of $f^{-1}(c)$, and the geometry of the family of levels is quite similar to the geometry of horospheres in manifolds without conjugate points (where the role of the function f is played by a Busemann function). Pesin in [36], Theorem 6.1, shows that the matrix solution $U(t)$ of the Riccati equation of a geodesic $\gamma_{(p,v)}(t)$ in the universal covering is the matrix of the normal curvatures of the horosphere of $\gamma_{(p,v)}(t)$, namely, the matrix of the second fundamental form of the immersion $f^{-1}(c)$ where f a Busemann function. The same proof extends without changes to our case since the picture of the levels $f^{-1}(c)$ and the Reeb flow is totally analogous.

The sets $\pi^{-1}(f^{-1}(c))$ define a C^2 foliation of $U \subset W$ by codimension one submanifolds. Since the canonical projection is a Riemannian submersion from $T_1 M$ endowed with the Sasaki metric to (M, g), we can apply the curvature formulae of Riemannian submersions (see Kowalsky [28]) to estimate the sectional curvatures of $\pi^{-1}(f^{-1}(c))$ at θ with respect to the Sasaki metric. These sectional curvatures depend on the sectional curvatures of $f^{-1}(c)$ and its first derivatives. By the bounds on the Riccati operator given in Theorem 1.3, the fact that $f^{-1}(c)$ is C^3 and the compactness of $\pi(L(W))$ we conclude the statement. $\square$

Corollary 3.4. *Let (T^n, g) be a compact C^∞ Riemannian manifold and let $K_C = K_C(k_0, C) > 0$ be given in Lemma 3.3. Then given a C^3 Lagrangian minimizing torus, the number $r > 0$ given in Lemma 3.2 depends only on K_C.*

3.2. **Proof of Theorem A.** We now conclude the proof of Theorem A:

Proposition 3.1, Corollary 3.4 and the assumption in Theorem A yield that W is regular for the canonical projection. Hence, it is a covering map. The following well known result due to Arnold [26] implies that W is a Lagrangian graph.

Theorem 3.5. *Let $S \subset T_1 T^n$ be a closed C^2 Lagrangian submanifold such that the restriction of the canonical projection to S is a covering map. Then the restriction of the canonical projection to S is a diffeomorphism.*

4. On the structure of the singular set

Now let us proceed to prove Theorem C. As we mentioned before, we already know that the singular set of the canonical projection restricted to a Lagrangian invariant submanifold of $T_1 M$ can be represented by the intersections of the orbits of the lifted geodesic flow in the Grassmannian of Lagrangian subspaces with the Maslov cycle. The Maslov cycle is not a manifold in general as in the case of surfaces, so it is not clear that the Maslov cycle provides a cross section of the lifted geodesic flow.

We shall suppose through the section that W is a C^1 Lagrangian invariant submanifold for the geodesic flow of (M, g) where M is a compact C^∞ manifold with dimension n.

A **topological cross section** $C \subset W$ for the geodesic flow is a C^0, $n-1$-dimensional submanifold of W with the following properties:

(1) Given $\theta \in C$ there exists an open neighborhood $B(\theta) \subset W$ and numbers $\delta > 0$, $\epsilon > 0$ such that $B(\theta) \subset \phi_{[-\delta, \delta]} C$ or the orbit of every $\eta \in B(\theta)$ intersects C for some time t with $| t | < \delta$.
(2) $\phi_{[-\epsilon, \epsilon]}(\theta) \cap C = \{\theta\}$

We say that a subset $S \subset N$ of a smooth manifold N is a C^0 k-dimensional submanifold if S is a Hausdorff topological space endowed with the relative topology and if there exists an atlas of S whose local parametrizations are homeomorphisms defined in open subsets of $\mathbb{R}^k$.

Let us denote by **singular set** $S(W)$ the set of points of W where the canonical projection is not regular. The first simple remark about the singular set is

Lemma 4.1. *The singular set $S(W)$ is compact.*

Proof. The Gauss map $\theta \in W \mapsto T_\theta W$ is continuous, therefore $S(W)$, the pre-image of the Maslov cycle, is closed.

$\square$

Theorem 1.3 gives us an estimate for the geometry of solutions of the Riccati equation associated to invariant Lagrangian subspaces that meet the vertical subbundle. It tells us that the geometry of certain reference functions (namely, $f_a(t) = \sqrt{k_0} coth(k_0(a - t)))$ "tames" the behavior of the solution of the Riccati equation in neighborhoods of singularities.

Lemma 4.2. *Let (M, g) be a C^∞ compact Riemannian manifold. Let $\theta \in T_1 M$, let $L_\theta \in H_\theta \oplus V_\theta$ be Lagrangian subspace and let $U(t)$ be the Riccati operator associated to $d\phi_t(L_\theta)$ (wherever it is defined). Suppose that for two numbers $b < c$ we have consecutives intersections of $d\phi_t(L_\theta)$ with the vertical bundle, namely,*

$$d\phi_b(L_\theta) \cap V_{\phi_b(\theta)} \neq \{0\}, \ d\phi_c(L_\theta) \cap V_{\phi_c(\theta)} \neq \{0\},$$

and $d\phi_b(L_\theta) \cap V_{\phi_b(\theta)} = \{0\}$ for $t \in (b, c)$. Then given $0 < \varepsilon < \frac{c-b}{8}$ there exists $C(b, c, \varepsilon) > 0$ such that

$$\| U(t) \| \leq C(b, c, \varepsilon)$$

for every $t \in [b + \varepsilon, c - \varepsilon]$.

Proof. The proof is straigthforward from Theorem 1.3 but is important for the sequel. Let

$$C_{b,c,\varepsilon}^+ = max_{t \in (b+\varepsilon, c-\varepsilon)} \sqrt{k_0} coth(k_0(t-b))$$

and

$$C_{b,c,\varepsilon}^- = min_{t \in (b+\varepsilon, c-\varepsilon)} \sqrt{k_0} coth(k_0(t-c)).$$

Since the function $f(t) + \sqrt{k_0} coth(k_0(t-b))$ is strictly decreasing for every $t \neq b$, it is positive if $t > b$ and negative if $t < b$, then we have

$$C_{b,c,\varepsilon}^+ = \sqrt{k_0} coth(k_0(\epsilon))$$

$$C_{b,c,\varepsilon}^- = \sqrt{k_0} coth(k_0(-\epsilon))$$

for every $\epsilon < \frac{b-c}{8}$. Since the function $f(t)$ is odd we get $C(b,c,\varepsilon) = \sqrt{k_0} coth(k_0(\epsilon))$. The dependence on b, c is implicit because ϵ depends on b, c.

$\square$

Lemma 4.3. *Let (M, g) be a compact Riemannian manifold and let $W \subset T_1 M$ be a compact C^1 Lagrangian invariant submanifold. Given $\theta \in S(W)$ there exist $\epsilon > 0$, a local cross section Σ_θ for the geodesic flow containing $\phi_{-\epsilon}(\theta)$, an open neighborhood $B(\theta) \subset T_1 M$ of θ homeomorphic to $\Sigma_\theta \times (-\epsilon, 2\epsilon)$, and a continuous function $t^1 : \Sigma_\theta \longrightarrow \mathbb{R}$ such that*

1. *$t^1(\theta) = \epsilon$,*
2. *$\phi_{t^1(\eta)}(\eta) \in S(W)$,*
3. *If $\phi_t(\eta) \in S(W)$ for some $\eta \in \Sigma_\theta$ and $t \in (-\epsilon, 2\epsilon)$ then $t \geq t^1(\eta)$.*
4. *The orbit of σ intersects $S(W)$ if and only if there exist $\theta(\sigma) \in S(W)$, $r_0 \in \mathbb{R}$, such that $\phi_{r_0}(\sigma) \in B(\theta(\sigma))$.*

Proof. Let us start the proof by the following remark:

Claim 1: Given $\theta \in S(W)$ there exists an open neighborhood $B_0(\theta)$ of θ in W such that the orbit of every point η in the neighborhood hits $S(W)$ at some time $t(\eta)$.

Suppose by contradiction that such neighborhood does not exist. This would imply the existence of a sequence of points θ_n converging to θ, and the existence of $\tau > 0$ such that $\phi_t(\theta_n)$ does not intersect $S(W)$ for every $t \in (-\tau, \tau)$. By Lemma 1.3 the Riccati operator $U_{\theta_n}(t)$ associated to the tangent space of W along the orbit $\phi_t(\theta_n)$ satisfies

$$\| U_{\theta_n}(t) \| \leq C(-\tau, \tau, \frac{\tau}{8})$$

for every $t \in (-\frac{7\tau}{8}, \frac{7\tau}{8})$. Since the Riccati operators associated to W depend continuously on $\eta \in W$ we would get that

$$\| U_\theta(t) \| \leq C(-\tau, \tau, \frac{\tau}{8})$$

which yields that the tangent space of W at θ is "far" from the vertical subbundle, contradicting the assumption that $\theta \in S(W)$.

Once $B_0(\theta)$ exists, by Lemma 2.4 there exists $\epsilon > 0$ such that the point θ is the unique point of intersection of $\phi_{(-3\epsilon, 2\epsilon)}(\theta)$ with $S(W)$.

Let $\Sigma \subset B_0(\theta)$ be a cross section for the geodesic flow containing θ and define $\Sigma_1 = \phi_{-\epsilon}(\Sigma)$, and $B_1(\theta)$ by

$$B_1(\theta) = \cup_{t \in (-\epsilon, 2\epsilon)} \phi_t(\Sigma_1).$$

Let $t^1 : \Sigma_1 \longrightarrow \mathbb{R}$ be given by the first positive time $t^1(\eta) \in (-\epsilon, 2\epsilon)$ where $\phi_{t^1(\eta)}(\eta) \in S(W)$.

By Claim 1 and the choice of Σ_1 we know that for every $\eta \in \Sigma_1$ there exists some $t(\eta)$ close to ϵ such that $\phi_{t_\eta}(\eta) \in S(W)$. Since for each η the intersections of its orbit with $S(W)$ are discrete (Lemma 2.4) there would be a first positive time of intersection of the orbit of $\eta \in \Sigma_1$ with $S(W)$. So t_1 is well defined in Σ_1.

Claim 2: The function $t^1 : \Sigma_1 \longrightarrow \mathbb{R}$ is continuous at $\phi_{-\epsilon}(\theta)$.

Otherwise, let $\eta_n \in \Sigma_1$ be a sequence converging to $\phi_{-\epsilon}(\theta)$ such that $-\epsilon < t^1(\eta_n) \leq \epsilon - a$ for some $a > 0$ suitably small. Since $S(W)$ is closed we have that $\phi_{t^1(\eta_{n_k})}(\eta_{n_k})$ converges to some $\phi_{\bar{t}}(\theta) \in S(W)$ where $-2\epsilon \leq \bar{t} \leq -a$. But the only point of $\phi_t(\theta)$ in $S(W)$ in the interval $t \in (-3\epsilon, 2\epsilon)$ is $t = 0$. This yields that $\bar{t} \leq -3\epsilon$ and hence $t^1(\eta_{n_k})$ does not belong to the interval $(-\epsilon, \epsilon)$ for some n_k large, which contradicts the fact that $t^1(\eta)$ is well defined in Σ_1.

Claim 3: There exists an open subset $\Sigma_\theta \subset \Sigma_1$ where the function t^1 is continuous.

By Claim 2 there exists an open subset $\Sigma_\theta \subset \Sigma_1$ such that $| t^1(\eta) | < \frac{\epsilon}{2}$ and any other point $\phi_t(\eta) \in S(W)$ with $-2\epsilon < t < t^1(\eta)$ satisfies $t < -\frac{3}{2}\epsilon$. Then, the same argument of the proof of Claim 2 applies to show the continuity of t^1 in an open neighborhood of η.

So the open neighborhood of the statement is $B(\theta) = \cup_{t \in (-\epsilon, 2\epsilon)} \phi_t(\Sigma_\theta)$ and items (1), (2), (3) follow from the Claims. Item (4) follows from the existence of the local cross section Σ_θ. Indeed, an orbit that intersects $B(\theta)$ must intersect the cross section Σ_θ and by items (1), (2), (3) it must intersect the singular set $S(W)$. $\square$

Corollary 4.4. *Under the assumption of Lemma 4.3 the function $\psi_\theta : \Sigma_\theta \longrightarrow S(W)$ given by $\psi_\theta(\eta) = \phi_{t^1(\eta)}(\eta)$ is a homeomorphism onto its image $S_\theta \subset S(W)$. Moreover, in local coordinates $\Sigma_\theta \times (-\epsilon, \epsilon)$ the set S_θ is the graph of ψ_θ and is a cross section for the geodesic flow.*

Proof. It is clear that ψ_θ is a continuous bijection due to the continuity of the function t^1 (Lemma 4.3) and the injectivity of the geodesic flow. The image S_θ of ψ_θ is the graph of t^1, namely

$$S_\theta = \{(\eta, t^1(\eta)), \ \eta \ \in \ \Sigma_\theta\}$$

in the local coordinates $\Sigma_\theta \times (-\epsilon, \epsilon)$. This yields that S_θ is a C^0 submanifold of codimension 1 in W. And since the intersections of each orbit of W with the singular set are isolated (Corollary 2.4) the set S_θ is a local cross section by the definition. $\square$

Corollary 4.4 gives a local description of part of the singular set $S(W)$ for every C^1 Lagrangian, invariant torus. Around every point $\theta \in S(W)$ there is an open neighborhood $B(\theta)$ of W such that $B(\theta) \cap S(W)$ containing a $n - 1$ continuous submanifold that is a cross section for the geodesic flow. The structure of the singular set might be more complicated, the following result (that won't be really

needed for the proof of the main theorem) gives a more precise description of the local profile of $S(W)$.

Lemma 4.5. *Let (M, g) be a compact n-dimensional Riemannian manifold and let $W \subset T_1 M$ be a C^1 Lagrangian invariant submanifold. Let $l > 0$ be given in Corollary 2.4 and suppose that the number $\epsilon = \epsilon(\theta)$ defined in Lemma 4.3 satisfies $5\epsilon \leq l$ for every $\theta \in S(W)$. Then there exists at most n cross sections S_i in $B(\theta)$ such that*

$$S(W) \cap B(\theta) = \cup_i S_i.$$

Moreover, there exist functions $t_i : A_i \subset \Sigma_\theta \longrightarrow \mathbb{R}$ where A_i is an open subset (in the relative topology) such that for every $\eta \in A_i$, $\phi_{t_i(\eta)}(\eta)$ is the i-th intersection of the piece orbit of η contained in $B(\theta)$ with $S(W)$ (following the orientation of the orbit).

Proof. We obtain the sections Σ_i recursively, taking $\Sigma_1 = \Sigma_\theta$. If the piece of orbit in $B(\theta)$ of every point in Σ_θ just meets $S(W)$ at Σ_θ then the lemma is proved, $i = 1$. Otherwise, there is a point $\eta \in \Sigma_\theta$ such that $\phi_{(-2\epsilon, 2\epsilon)}(\eta)$ meets $S(W) \cap B(\theta)$ at least twice. Let $A_2 \subset \Sigma_\theta$ be this set of points and define

$$t_2 : A_2 \longrightarrow \mathbb{R}$$

$$t_2(\eta) = \inf\{t > t_1(\eta) \ s.t. \ \phi_t(\eta) \in S(W)\}.$$

The same ideas in the proof of Lemma 4.3 lead to show that t_2 is a continuous function and that A_2 is an open subset (in the relative topology) of Σ_θ. The function t_2 can be extended to the boundary of A_2 in Σ_θ if and only if there exists a sequence η_n converging to some $\alpha \in \Sigma_\theta$ such that

$$\inf\{t > t_1(\eta_n) \ s.t. \ \phi_t(\eta_n) \in S(W)\} = 0.$$

The point α could be regarded as a sort of ramification point of the singular set $S(W)$. Notice that A_2 might not be connected, and by the definition of cross section the set

$$\Sigma_2 = \cup_{\eta \in A_2}\{\phi_{t_2(\eta)}(\eta)\}$$

is again a cross section. We can go on by taking $A_3 \subset A_2$ such that $\phi_{(-2\epsilon, 2\epsilon)}(\eta)$ meets $S(W) \cap B(\theta)$ at least three times. By the assumption on the bound of ϵ depending on l, there are at most n steps in this process according to Corollary 2.4. $\square$

5. Nonwandering points and the proof of Theorem B

The purpose of the section is to show Theorem B. A point $\theta \in T_1 M$ is nonwandering for the geodesic flow if given an open neighborhood N of θ there exists $t_N \in \mathbb{R}$ with $\mid t_N \mid \geq 1$ such that $\phi_{t_N}(N) \cap N$ is nonempty. Every recurrent point is of course nonwandering but the converse is not true in general.

The main idea is to apply Theorem C to look at the singular set $S(W)$ as a sort of weak or generalized cross section.

Suppose that $S(W)$ is nonempty, let $B(\theta)$ for $\theta \in S(W)$ be the set given by Lemma 4.3 and let $S_\theta \subset B(\theta)$ be the singular cross section given in Corollary 4.4. By the compactness of $S(W)$, there exists a finite covering by open sets of the form $B(\theta_i)$ homeomorphic to $\Sigma_{\theta_i} \times (-\epsilon_i, \epsilon_i)$. Let $\bar{\epsilon}$ be the supremum of the $\epsilon_i's$.

Lemma 5.1. *Let W be a C^1 Lagrangian minimizing torus invariant by the geodesic flow of (T^n, g). If the singular set of the canonical projection $S(W)$ is not empty, then there exists a point $\theta \in S(W)$ and an open neighborhood $\nu(\theta)$ such that every orbit of a point in $\nu(\theta)$ does not meet $S(W)$ set for every $t > \bar\epsilon$.*

Proof. Let us argue by contradiction. Suppose that such a point $\theta \in S(W)$ does not exist. By continuity of the geodesic flow the set of points $\theta \in B(\theta_{i_1})$ whose orbits $\phi_t(\theta)$ intersect any $B(\theta_{i_2})$ for $t \geq \bar\epsilon$ is an open subset $R_{i_1, i_2} \subset B(\theta_{i_1})$. By the choice of $\bar\epsilon$ we have that i_1 is different from i_2, and by the contradiction assumption we have that the union

$$\Gamma_1 = \cup_{i,j} R_{i,j}$$

is open and dense in $B(\theta_i)$ for every i and represents the set of points in the union of the balls $B(\theta_i)$ that meet the singular set at least twice. Let $P_{i,j} : R_{i,j} \longrightarrow \Sigma_{\theta_j}$ be the Poincaré map giving the first intersection of an orbit of a point in the domain $R_{i,j}$ of $P_{i,j}$ with the cross section Σ_{θ_j}. The union of the images of the maps $P_{i,j}$ is

$$\mathfrak{R}_1 = \cup_{i,j} P_{i,j}(R_{i,j})$$

and by assumption, this set is also open in the union of the sections Σ_{θ_i}'s and therefore their orbits intersect the union of the balls $B(\theta_i)$'s in an open set. Now, by the contradiction assumption, the set

$$\Gamma_1 \cap \mathfrak{R}_1 = \cup_{i,j}(P_{i,j}(R_{i,j}) \cap \Gamma_1) = \cup_{i,j} \tilde{R}_{i,j}$$

is open and dense in $\mathfrak{R}_1$ and therefore, the set

$$\Gamma_2 = \cup P_{i,j}^{-1}(\tilde{R}_{i,j})$$

is open and dense in the union of the balls $B(\theta_i)$ and represents the set of points that meet the singular set at least 3 times.

We can continue by induction defining the open sets

$$R_{i_1, i_2, i_3} = P_{i_1, i_2}(R_{i_1, i_2}) \cap B(\theta_{i_3})$$

$$\mathfrak{R}_2 = \cup_{i,j,k} P_{j,k}(R_{i,j,k})$$

$$\Gamma_1 \cap \mathfrak{R}_2 = \cup_{i,j,k}(P_{i,j}(R_{i,j,k}) \cap \Gamma_1) = \cup_{i,j,k} \tilde{R}_{i,j,k}$$

that is an open set by the contradiction assumption,

$$\Gamma_3 = \cup_{i,j,k} P_{i,j}^{-1}(\tilde{R}_{i,j,k})$$

that is open and dense in the union of the balls $B(\theta_i)$ and consists of the points whose orbits meet at least 4 times the singular set, and so on.

By this procedure we get that given any $m \in \mathbb{N}$ the set of orbits of W that meet the singular set at least m times is open and dense in the union of the balls $B(\theta_i)$. This clearly contradicts the fact that the torus W is minimizing since we know that this number must be less than n. $\qquad\square$

Proof of Theorem B

Suppose that the singular set $S(W)$ is nonempty. By Lemma 5.1 there exists an open neighborhood $\nu(\theta)$ of a point $\theta \in S(W)$ such that every orbit in the neighborhood stays for $t > \bar{\epsilon}$ in a regular component. Combining this fact with Lemma 4.3 we get

Claim: There exists an open neighborhood of θ of wandering points.

Indeed, Lemma 4.3 gives us an open neighborhood $B(\theta)$ of θ such that every orbit meeting $B(\theta)$ has to meet the singular set. Since the positive orbits of points in $B(\theta) \cap \nu(\theta)$ do not meet the singular set again, none of these orbits meet $B(\theta) \cap \nu(\theta)$ for $t > 0$ thus proving that the points of $B(\theta) \cap \nu(\theta)$ are wandering points.

To finish the proof of Theorem B, the assumption and the Claim imply that $S(W)$ is empty and therefore by Theorem 3.5 the torus W is a graph.

References

[1] Arnaud, M. C.: On a Theorem Due to Birkhoff. GAFA, Vol. 20, n. 6 (2010) 1307-1316.

[2] Arnaud, M.C.: When are invariant sumanifolds of symplectic dynamics Lagrangian?, Discrete and Continuos Dynamical Systems-A (2014), 34, 5, 1811-1827.

[3] Arnold, V. I.: Mathematical methods of classical mechanics. Springer-Verlag 1978, Berlin, Heidelberg, New York.

[4] Amorim,L.,Oh,Y-G,Dos Santos,J.O; Exact Lagrangian submanifolds, Lagrangian Spectral Invariants and Aubry-Mather Theory, https://arxiv.org/abs/1603.06966v4

[5] Bangert, V.: Mather sets for twist maps and geodesic on tori, In U. Kirchgraber, H. O. Walther (eds.) Dynamics Reported. Vol. 1, B. G. Teubner, J. Wiley, 1988, 1-56.

[6] Bangert, V.: Minimal geodesics. Erg. Th. Dyn. Sys. (1989) 10, 263-286.

[7] Bernard,P., dos Santos, J.O.: A geometric definition of the Mañé-Mather set an a theorem of Marie-Claude Arnaud.Math Proc. Cambridge Philos.Soc. 152,(2012) n.1,167-178.

[8] Bialy, M.: Aubry-Mather sets and Birkhoff's theorem for geodesic flows on the two dimensional torus. Communications in Math. Physics, 126 (1989) 13-24.

[9] Bialy, M., Polterovich, L.: Geodesic flows on the two dimensional torus and phase transitions "commensurability-noncommensurability". Funk. Anal. Appl. 20 (1986) 223-226.

[10] Bialy, M., Polterovich, L.: Lagrangian singularities of invariant tori of Hamiltonian systems with two degrees of freedom. Invent. math. 97 (1989) 291-303.

[11] Bialy, M., Polterovich, L.: Hamiltonian systems, Lagrangian tori and Birkhoff's theorem. Math. Ann. 292 (1992) 619-627.

[12] Bialy, M., Polterovich, L.: Invariant tori and symplectic topology. Amer. Math. Soc. Transl. (2) vol. 171 (1996) 23-33.

[13] Candel, A., Conlon, L.: Foliations I. Graduate Studies in Mathematics vol. 23. Amer. Math. Soc. Providence, Rhode Island, 2000.

[14] Carneiro, M. J., Ruggiero, R.: On Variational and topological properties of C^1 invariant Lagrangian tori. Ergodic Theory and Dynamical systems, vol. 24 (2004) 1-27.

[15] Carneiro, M. J., Ruggiero, R.: On Birkhoff Theorems for Lagrangian invariant tori with closed orbits. Manuscripta Mathematica, vol. 119 (2006) 411-432.

[16] Conley, C., Easton, R.: Isolated invariant sets and isolating blocks. Trans. Amer. Math. Soc. 158 (1971), 35-61.

[17] Contreras, G., Delgado, J., Iturriaga, R.: Lagrangian flows: the dynamics of globally minimizing orbits II. Bol. Soc. Bras. Mat. 28 (1997) n. 2, 155-196.

[18] Contreras, G., Gambaudo, J.M., Iturriaga, R., Paternain, G.: The asymptotic Maslov index and its applications. Ergodic Theory Dynam. Systems 23 (2003), no. 5, 14151443.

[19] Contreras, G., Iturriaga, R., Paternain, G., Paternain, M.: Lagrangian graphs, minimizing measures, and Mañé's critical values. Geom. Funct. Anal. 8 (1998) n. 5, 788-809.

[20] Contreras, G., Iturriaga, R.: Convex Hamiltonians without conjugate points. Ergod. Th. Dyn. Sys. (1999), 19, 901-952.

[21] Contreras, G., Iturriaga, R.: Global minimizers of autonomous Lagrangians. 22 Colóquio Brasileiro de Matemática, IMPA, Rio de Janeiro (1999).

[22] Eberlein, P.: When is a geodesic flow of Anosov type? I, J. Differential Geometry, 8 (1973), 437–463.

[23] Fathi, A.: Weak KAM Theorem in Lagrangian Dynamics. Lyon (notes) 2000.

[24] Green, L. W.: Surfaces without conjugate points, Transactions of the American Mathematical Society **76** (1954), 529–546.

[25] Green, L. W.: A theorem of E. Hopf. Michigan Math. J. Vol. 5 (1) (1958) 31-34.

[26] Herman, M.: Inégalités a priori pour des tores lagrangiens invariants par des difféomorphismes symplectiques. Publications Mathématiques de l'Institut des Hautes Études Scientifiques (1989), Volume 70, Issue 1, pp 47-101.

[27] Hopf, E.: Closed surfaces without conjugate points, Proceedings of the National Academy of Sciences **34** (1948), 47–51.

[28] O. Kowalski: Curvature of the Induced Riemannian Metric on the Tangent Bundle of a Riemannian Manifold, J. Reine Angew. Math. 250 (1971), 124-129.

[29] McDuff, D., Salamon, D.: Introduction to Symplectic Topology. Claredon Press, Oxford (1995).

[30] Mañé, R.: On the minimizing measures of Lagrangian dynamical systems. Nonlinearity 5 (1992) n. 3, 623-638.

[31] Mañé, R.: Global variational methods in conservative dynamics. IMPA, Rio de Janeiro (1993).

[32] Mather, J.: Action minimizing measures for positive definite Lagrangian systems. Math. Z. 207 (1991) n. 2, 169-207.

[33] Morse, J. M.: A fundamental class of geodesics on any closed surface of genus greater than one. Trans. Amer. Math. Soc. 26, n. 1 (1924) 25-60.

[34] Morse, J. M.: Calculus of variations in the large. AMS Colloquium Publications XVIII, Providence RI.

[35] Paternain, G.: Geodesic flows. Progress in Mathematics vol. 180, Birkhauser, Boston, Mass. 1999.

[36] Pesin, J.: Geodesic flows on closed Riemmanian manifolds without focal points. Math. USSR Izvestija, Vol. 11 (1977) n. 6, 1195-1228.

[37] Polterovich, L.: The second Birkhoff theorem for optical Hamiltonian systems. Proceedings of the AMS 113 (1991) n. 2, 513-516.

[38] Ruggiero, R.: Manifolds admitting continuous foliations by geodesics. Geometriae Dedicata, 78 (1999) 161-170.

[39] Viterbo, C.: A new obstruction to embedding Lagrangian tori. Invent. Math. 100 (1990) n. 2, 301-320.

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