ARNOLD-LIOUVILLE THM FOR TONELLI HAMILTONIANS

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Hamiltonian flows restricted to Lagrangian submanifolds are very flexible:

$\boxed{\text{PROP}}$ Let (M^{2n}, ω) be a symplectic manifold and let $\mathcal{L} \subset M$ be a closed Lagrangian submanifold. Let $(f_t)_{t\in\mathbb{R}}$ be any (smooth) flow on $\mathcal{L}$. Then there exists a Hamiltonian flow $(\varphi_t^H)_{t\in\mathbb{R}}$ on M so that $\varphi_t^H|_{\mathcal{L}} = f_t$.

The classical Arnol'd-Liouville theorem asserts that any Hamiltonian flow that preserves a Lagrangian foliation with compact leaves is in some sense linearizable (the flow restricted to every leaf is a rotation flow).

<u>The basic example</u> is given by $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$, $(\theta, r) \mapsto h(r)$

with $\omega = d\theta \wedge dr$, the Hamiltonian flow is:

$$\varphi_t^H(\theta, r) = (\theta + t\nabla h(r), r)$$

The standard foliation into $\{r = cste\}$ is invariant and on every leaf, the flow is conjugate to the rotation flow $\theta \mapsto \theta + t\nabla h(r_0)$

Observe that for this example, the flow is C^∞ along the leaves and has transversally the same regularity as ∇h.

In this course, we will deal with non-singular Lagrangian foliations. We will begin by recalling classical (smooth) Arnold-Liouville results ; and then we will ask ourselves what happens for non-smooth foliations.

(I) LIOUVILLE INTEGRABILITY.

<u>notations</u> (M,ω) is a symplectic manifold that at some moment will be assumed to be $\mathbb{T}^n \times \mathbb{R}^n$. If $H : M \to \mathbb{R}$ is C^2, the Hamiltonian vectorfield X_H is defined by $\omega(X_H, \cdot) = dH$ and the flow is denoted by (φ_t^H).

Observe that H is an integral: $H \circ \varphi_t^H = H$.

<u>Poisson bracket</u>: if $H, K : M \to \mathbb{R}$ are C^2,

$$\{H,K\} = \omega(X_H, X_K) = dH \cdot X_K.$$

H and K "Poisson commute" if $\{H,K\} = 0$.

We have $\{H,K\} = 0 \iff \forall t,\ H \circ \varphi_t^K = H$

$$\iff \forall t,\ K \circ \varphi_t^H = K$$

$$\Rightarrow \varphi_t^K \circ \varphi_s^H = \varphi_s^H \circ \varphi_t^K,\ \forall t,s$$

Definition a n-uple $(H_1, \dots, H_n)$ of Hamiltonians is Liouville integrable if

i) $\forall i, j \ \{H_i, H_j\} = 0$,

ii) at Lebesgue almost every point, we have $\operatorname{rank}\{dH_1, \dots, dH_n\} = n$.

Remarks i) is equivalent to the fact that $\mathbb{R}X_{H_1} + \dots + \mathbb{R}X_{H_n}$ is involutive for ω.

Hence if c is a regular value of $(H_1, \dots, H_n)$, it says that $\{(H_1, \dots, H_n) = c\}$ is a Lagrangian submanifold and then that we have locally a Lagrangian foliation.

Definition a C^2 Hamiltonian H_1 is integrable if it is the first Hamiltonian of a Liouville integrable system of Hamiltonians.

Definition: a level set is a connected component of $\{H_1 = c_1, \dots, H_n = c_n\}$.

. the level set N is regular if $(dH_1, \dots, dH_n)$ has rank n along N.

Hence a regular level set is a n-dimensional submanifold that is Lagrangian.

II LIOUVILLE THEOREM (Liouville ~ 1855)

<u>Theorem</u> assume that $(H_1, \dots, H_n)$ is a Liouville integrable system on a symplectic manifold (M, ω). that is regular on some compact level set N. Then there exists a tubular neighborhood $\mathcal{U}$ of N and unique up to automorphisms C^1 free torus action of $\mathbb{T}^n$

$$A : \mathbb{T}^n \times \mathcal{U} \longrightarrow \mathcal{U}$$

that preserves every H_i and every X_{H_i} and whose orbits are the regular level sets of the system.

Hence N is a n-torus and we can choose $\mathcal{U} \simeq \mathbb{T}^n \times B^n$ with global coordinates $(\theta_1, \dots, \theta_n, c_1, \dots, c_n)$ such that $H_1, \dots, H_n$ only depend on $c_1, \dots, c_n$ and

$$X_{H_i} = \sum_{j=1}^{n} a_{ij}(c_1, \dots, c_n) \frac{\partial}{\partial \theta_j}$$

<u>Definition</u> 1) such a system of coordinates is a Liouville system of coordinates

2) every such regular compact level set is a Liouville torus.

3) A is the Liouville torus action

<u>Proof</u> 1/ $\mathcal{H} = (H_1, \dots, H_n) : \mathcal{U} \longrightarrow \mathbb{R}^n$ is a submersion on some tubular neighborhood of N and defines a trivial fibration by level sets, that has the same regularity as $\mathcal{H}$.

2/ Then the vectorfields $X_{H_1}, \dots, X_{H_n}$ generate a transitive action on the level sets $N_c \ni x_c$ by

$$\phi_c : (t_1, \dots, t_n) \in \mathbb{R}^n \longmapsto \varphi_{t_1}^{H_1} \circ \dots \circ \varphi_{t_n}^{H_n}(x_c) \in N_c.$$

This action has same regularity as $H_1, \dots, H_n$ (but the transversal regularity is only as regular as $X_{H_1}, \dots, X_{H_n}$), is a local diffeomorphism and transitive. Then $G_c = \phi_c^{-1}(\{x_c\})$ is a discrete subgroup of $\mathbb{R}^n$ (that doesn't depend on the choice of x_c we did) with generators $e_1(c), \dots, e_n(c) \in \mathbb{R}^n$ that depend on c with the regularity of $X_{H_1}, \dots, X_{H_n}$. If $e_i(c) = (e_{i,1}(c), \dots, e_{i,n}(c))$, then we have a group action

$$(\theta_1, \dots, \theta_n, x) \xmapsto{A} \varphi^{H_1}_{\sum_{j=1}^n \theta_j e_{j,1}(\mathcal{H}(x))} \circ \dots \circ \varphi^{H_n}_{\sum_{j=1}^n \theta_j e_{j,n}(\mathcal{H}(x))}(x)$$

[observe that it is defined on $\mathbb{T}^n \times \mathcal{U}$ because $e_i(\mathcal{H}(x)) \in G_{\mathcal{H}(x)}$.]

This group action indeed preserves $\mathcal{H}$ and the X_{H_i} as the flows of the X_{H_i} did.

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This implies that N (and the N_c) are tori and that we can choose coordinates $(\theta_1, \dots, \theta_n, c) \in \mathbb{T}^n \times B^N \xrightarrow{\Phi}$

$$\varphi^{H_1}_{\sum_{j=1}^n \theta_j e_{j1}(c)} \circ \dots \circ \varphi^{H_n}_{\sum_{j=1}^n \theta_j e_{jn}(c)} (x_c)$$

in which $H_1, \dots, H_n$ only depend on c because $\mathcal{H}(\Phi(\theta_1, \dots, \theta_n, c)) = c$ and then $X_{H_i} = \sum_{j=1}^n a_{i,j}(c) \cdot \frac{\partial}{\partial \theta_j}$. ▨

<u>Remark</u> • Because the flows $(\varphi_t^{H_i})$ have the same regularity as ∇H_i, we loose one regularity transversally to the leaves

• Along the leaves, the coordinates and the action is as regular as the H_i's.

• We end in an essential way that the flow of X_{H_i} exists, ie that H_i is at least $C^{1,1}$.

• As $X_{H_i} = \sum_{j=1}^n a_{i,j}(c) \frac{\partial}{\partial \theta_j}$ we have in the (θ, c) coordinates:

" $\varphi_t^{H_i}(\theta, c)$ " $= (\theta_1 + a_{i,1}(c)t, \dots, \theta_n + a_{i,n}(c)t)$

is a rotation flow

III ANGLE - ACTION COORDINATES.

We assume the same hypotheses as in Liouville theorem, and that the H_i are C^k with $k \geq 2$.

<u>Proposition</u>. The Liouville torus action is Hamiltonian and there exists C^{k-1} angle-actions (symplectic) coordinates in which

$$\varphi_t^{H_i}(\theta, c) = (\theta + t \nabla H_i(c), c)$$

<u>Proof</u> we obtained non symplectic coordinates in which we have

$$X_{H_i}(\theta, c) = \sum_{j=1}^{n} a_{ij}(c) \frac{\partial}{\partial \theta_j}$$

The symplectic form is in these coordinates

$$\omega = \sum_{i<j} \alpha_{ij}(\theta, c)\, dc_i \wedge c_j + \sum_{i,j} \beta_{ij}(\theta, c)\, d\theta_i \wedge d$$

$$+ \sum_{i<j} \gamma_{i,j}(\theta, c)\, d\theta_i \wedge d\theta_j$$

As $\mathbb{R}X_{H_1} + \dots + \mathbb{R}X_{H_n}$ is Lagrangian, we have $\gamma_{ij} = 0$

Moreover, every $i_{X_{H_i}} \omega$ is exact and then closed, where

$i_{X_i}\omega = \sum_{j,\ell} \beta_{j,\ell}(\theta,c)\, a_{ij}(c)\, dc_\ell$ is closed, hence

$\forall i, \ell, k$ $$\sum_j \frac{\partial \beta_{j,\ell}}{\partial \theta_k}(\theta,c)\, a_{ij}(c) = 0$$

and then, as $\det\left(a_{ij}(c)\right)_{i,j} \neq 0$: $\frac{\partial \beta_{j,\ell}}{\partial \theta_k} = 0$

Hence $\beta_{j,\ell}$ only depends on c and

$$\omega = \sum_{i<j} \alpha_{ij}(\theta,c)\, dc_i \wedge dc_j + \sum_{i,j} \beta_{ij}(c)\, d\theta_i \wedge dc_j$$

Using that ω is closed and looking at the coefficient of $d\theta_\ell \wedge dz_i \wedge dz_j$, we obtain that $\frac{\partial \alpha_{ij}}{\partial \theta_\ell}(\theta,c)$ only depends on c and then also the function α_{ij}. Finally

$$\omega = \sum_{i<j} \alpha_{ij}(c)\, dc_i \wedge dc_j + \sum_{i,j} \beta_{ij}(c)\, d\theta_i \wedge dc_j$$

and the $\mathbb{A}$ action is symplectic. Hence $i_{\frac{\partial}{\partial \theta_i}}\omega = \sum_j \beta_{ij}(c)\, dc_j$ is closed, and then exact on B^n : $i_{\frac{\partial}{\partial \theta_i}}\omega = d z_i$.

We take $(z_1, \dots, z_n)$ as coordinates :

$$\omega = \sum_{i<j} \delta_{ij}(z)\, dz_i \wedge dz_j + \sum_i d\theta_i \wedge dz_i$$

with ω and then $\sum_{i<j} \delta_{ij}(z)\, dz_i \wedge dz_j$ closed and then exact : $d\left(\sum a_i(z)\, dz_i\right) = \sum_{i<j} \delta_{ij}\, dz_i \wedge dz_j$

that gives $$\omega = \sum_i d\left(\theta_i + a_i(z)\right) \wedge dz_i$$

If $(\Theta_i, z_i) = (\theta_i + a_i(z), z_i)$, this coordinates are symplectic and in these coordinates, τ_i only depends on z ▨

<u>Remark</u> this proposition seems to be due to Mineur (~1935). The proof we presented comes from Nguyen Tien Zung 2017, "a conceptual approach to the problem of angle action variables" ArXiv

IV ON ENERGY LEVEL

A natural question concerns what happens when we only know that one regular energy level of a Hamiltonian H is C^2-foliated by compact Lagrangian submanifolds.

When $M = \mathbb{R}^{2n}$ and H is convex, or when H is Tonelli and the energy level is above the so-called Mane critical level, there is a standard way to change H in such a way that:

$\rightarrow$ $\{K=1\} = \{H=c\}$: the new Hamiltonian K has one energy level in common with H, the energy level

that was foliated by invariant Lagrangian submanifolds for (φ_t^H).

$\rightarrow$ the energy levels $\{K=c\}$ with c close to 1 are regular and the flow on $\{K=c\}$ is conjugated to a reparametrization of the flow $(\varphi_t^K{}_{|\{K=1\}})$.

As two regular flows that have in common an energy level $\mathcal{E}$ are the same up to a time reparametrization on $\mathcal{E}$, we deduce

<u>Proposition</u> If H is a Tonelli Hamiltonian that has an energy level $\mathcal{E}$ above the critical level that is foliated by invariant compact Lagrangian submanifolds, then <u>up to a time reparametrization</u> the Hamiltonian flow on every such submanifold is conjugated to a rotation flow.

⚠ This doesn't imply that this is conjugate to a rotation flow. There are examples of Shklover (~1960), Fayad (~2001)... with Liouville rotation number where this flow is weakly