ARNOLD-ZIOLKOWSKI THM FOR TONELLI HAMILTONIANS

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Hamiltonian flows restricted to Lagrangian submanifolds are very flexible:

**Prop.** Let $(M^{2n},\omega)$ be a symplectic manifold and let $\mathcal{L} \subset M$ be a closed Lagrangian submanifold. Let $(\phi^t)_{t\in\mathbb{R}}$ be any (smooth) flow on $\mathcal{L}$. Then there exists a Hamiltonian flow $(\dot{\phi}^t)_{t\in\mathbb{R}}$ on $M$ so that $\dot{\phi}^t|_{\mathcal{L}} = \phi^t$.

The classical Arnold-Ziolkowski theorem asserts that any Hamiltonian flow that preserves a Lagrangian foliation with compact leaves is in some sense linearizable (the flow restricted to every leaf is a rotation flow).

The basic example is given by $H: T^*\mathbb{R} \to \mathbb{R}$

with $\omega = d\Theta \wedge d\xi$, the Hamiltonian flow $\dot{\phi}^t(\Theta,\xi) = (\Theta + e^{\xi}d\Theta(\xi), \xi)$

The standard foliation into $\mathbb{R} = \text{conf}$ is invariant and on every leaf, the flow is conjugate to the rotation flow $\Theta \mapsto \Theta + e^\xi d\Theta(\xi)$.

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Observe that for this example, the flow is to along the leaves and tree hierarchically the same regularly as $F_A$.

In this course, we will deal with non-singular degenerate foliations. We will begin by recalling classical (smooth) Arnold—Liuville results, and then we will ask ourselves what happens for non-smooth foliations.

1. LIOUVILLE INTEGRABILITY

notation $(M,\omega)$ is a symplectic manifold that at some moment will be assumed to be $T^*M$. If $H: M \to \mathbb{R}$ is smooth, the Hamiltonian vector field $X_H$ is defined by $\omega(X_H, \cdot) = dH$ and the flow is denoted by $\text{exp}(tH)$.

Observe that $H$ is an integral: $H \circ \varphi_t = H$.

Poisson bracket : if $H, K : M \to \mathbb{R}$ are smooth

$\{H, K\}^\omega = \omega(X_H, X_K) = dH(x) \cdot X_K$

$H$ and $K$ "Poisson commute" if $\{H, K\} = 0$
We have $\{H, K\} = 0 \iff \forall t, \text{Hol}(\varphi_t^K \cdot H) = H$

$\iff \forall t, K_t \circ \varphi_t^H = K$

$\Rightarrow \varphi_t^K \circ \varphi_t^H = \varphi_t^{K \circ H}$

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Definition: a n-tuple \((\mathbf{H}_i, \mathbf{H}_j)\) of Hamiltonians is simultaneously integrable if:

- \( H_i \neq H_j \)
- \( H_i \) and \( H_j \) are degenerate almost every point, we have \( \text{rank} \{dH_i, \ldots, dH_j\} = n \).

Remarks: if is equivalent to the fact that \( RX_{H_1} + \cdots + RX_{H_n} \) is isotropic for \( u \).

Hence if \( u \) is a regular value of \((\mathbf{H}_i, \mathbf{H}_j)\), if says that \( \{\mathbf{H}_i, \mathbf{H}_j\} = 0 \)

is a deparametric submanifold and then that we have locally a deparametric foliation.

Definition: a \( C^2 \) Hamiltonian \( H \) is integrable if it is the first Hamiltonian of a simultaneously integrable system of Hamiltonians.

Definition: a level set is a connected component of \( \{H_i = c_i, \ldots, H_j = c_j\} \).

The level set \( N \) is regular if \( \{dH_i, \ldots, dH_j\} \) has rank \( n \) along \( N \).

Hence a regular level set is a \( n \)-dimensional submanifold that is Lagrangian.
II. LIOUVILLE THEOREM (Liouville, 1835)

Theorem: Assume that \((H, J, \Omega)\) is a Liouville integrable system on a symplectic manifold \((\mathcal{M}, \omega)\). Then it is regular on some compact level set \(\mathcal{N}\). Then there exists a tubular neighborhood \(\mathcal{U}\) of \(\mathcal{N}\) such that \(\mathcal{U}\) is equivalent to a Liouville's \(C^k\) free torus action of \(\mathbb{T}^n\)

\[ A: \mathbb{T}^n \times \mathcal{U} \rightarrow \mathcal{U} \]

that preserves every \(H_i\) and every \(X_i\).

and whose orbits are the regular level sets of the system.

Hence, \(\mathbb{T}^n\) is a \(n\)-torus and we can choose \(U = \mathbb{T}^n \times B^n\) with global coordinates \((\Theta_1, \ldots, \Theta_n, \varphi_1, \ldots, \varphi_n)\) such that \(H_i, X_i\) only depend on \(\Theta_1, \ldots, \Theta_n\) and hence

\[ X_i = \sum_{j=1}^{n} A_{ij}(\Theta_1, \ldots, \Theta_n) \partial_{\Theta_j} \]

Definition: A such system of coordinates is a Liouville system of coordinates if

1. every such compact level set is a Liouville level set.
2. \(A = \mathbb{T}^n\) is the Liouville torus action.
Proof If \( H = (H_1, \ldots, H_n) : U \rightarrow \mathbb{R}^n \) is a submersion on some neighborhood of \( \mathcal{N} \) and defines a trivial fibration by level sets, then has the same regularity as \( H \).

If then the vectorfields \( X_{H_1}, X_{H_m} \) generate a transitive action on the level set \( \mathcal{N} \) by

\[
\phi^t : (t, x) \in \mathbb{R}^n \rightarrow (t \cdot e^{t \cdot \xi_m} (x_m) \cdot x \in \mathcal{N}.
\]

This action has same regularity as \( H_1, H_m \) (but the transversal regularity is only regular as \( X_{H_1}, X_{H_m} \)), is a local diffeomorphism and transitive. Then \( G_x = \{ \phi_t(x) \} \) is a discrete subgroup of \( \mathbb{R}^n \) (that doesn't depend on the choice of \( x \) we took) with generators \( e_{i} (c) , e_{m} (c) \in \mathbb{R}^n \) that depend on \( c \) and the regularity of \( X_{H_1}, X_{H_m} \).

If \( e_{i} (c) = (e_{i} x(c) , e_{i} (c)) \) then

we have the group action

\[
(\theta_{x}, \phi_{n}, x) \rightarrow \phi_{n}^{H_n} \circ \phi_{m}^{H_m} (x)
\]

This group action indeed preserves \( H \) and the \( X_{H_1} \) as the fibers of the \( X_{H} \) direct
This implies that \( N \) (and hence \( N^0 \)) are tori and that we can choose coordinates \((\Theta_1, \ldots, \Theta_n, c) \in T^\ast \times B^m \rightarrow \)

\[
\left( \frac{\partial}{\partial \Theta_1}, \ldots, \frac{\partial}{\partial \Theta_n}, c \right) (x_c)
\]

in which \( H_c \), \( h_c \), etc. only depend on \( c \), because \( \mathcal{H} \left( \Phi \left( \Theta_1, \ldots, \Theta_n, c \right) \Phi^\ast \right) = c \) and

then \( X_{H_c} = \sum_{j=1}^{n} a_{y_j} \frac{\partial}{\partial \Theta_j} \).

\[ \square \]

**Remark.** Because the flow \( \Phi_{H_c} \) have the same regularity as \( \Phi_{H_0} \) and hence one regularity transversally to the leaves

\( \Phi \) along the leaves, the coordinates and the action is as regular as the \( H_c \).

We end in an essential way that the flow of \( X_{H_0} \) exists.

As \( X_{H_c} = \sum_{j=1}^{n} a_{y_j} \frac{\partial}{\partial \Theta_j} \)

we have in the \((\Theta, c)\) coordinates

\[
\Phi_{H_c} \left( \Theta, c \right) = (\Theta_1 + a_{y_1} (c) t, \ldots, \Theta_n + a_{y_n} (c) t)
\]

in another flow.
III ANGLE-ACTION COORDINATES.

We assume the same hypotheses as in Liouville's Theorem, and that the $T_k$ are $C^k$ with $k \geq 2$.

Proposition. The Liouville form action is Hamiltonian, and there exists $C^{k+1}$ angle-actions (symplectic) coordinates in which

$$\begin{align*}
\gamma^{k+1}(\theta, c) &= (\theta + \epsilon \Phi_t(c, \theta), c)
\end{align*}$$

Proof. Let obtained non-symplectic coordinates in which

$$\begin{align*}
X_{Qj} (\theta, c) &= \sum_{j=1}^{m} \alpha_{j} (c) \frac{\partial}{\partial \theta_j} \\
\alpha_{j} (c) &= \sum_{j=1}^{m} \alpha_{j} (c)
\end{align*}$$

The symplectic form is in these coordinates,

$$\begin{align*}
\omega &= \sum_{1 \leq j < k} \alpha_{j} (c) \theta_k \wedge \theta_j - \sum_{1 \leq j < k} \alpha_{j} (c) \theta_j \wedge \theta_k \\
&+ \sum_{1 \leq j < k} \gamma_{ij} (\theta, c) d\theta_j \wedge d\theta_k
\end{align*}$$

As $R_{\Phi_t(c)}^* \omega = \omega$ is a $\omega$-form, we have $\gamma_{ij} = 0$.

Hence, every $\gamma_{ij}$ is exact and their closed, where

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\[ \lambda_k \theta_k \] \[ \sum_{j \neq k} \frac{\partial \beta_{j,k}}{\partial \theta_j} (\theta_{j,k}) \text{det } (\lambda_j, \theta_{j,k}) = 0 \]

and then, as \( \text{det } (\lambda_j, \theta_{j,k}) \neq 0 \), \( \frac{\partial \beta_{j,k}}{\partial \theta_j} = 0 \)

Hence \( \beta_{j,k} \) only depends on \( j \) and
\[ w = \sum_{j \neq k} \lambda_j \beta_{j,k} (\theta_{j,k}) d \theta_{j,k} + \sum_{j \neq k} \lambda_j \beta_{j,k} (\theta_{j,k}) d \theta_{j,k} \]

Using these \( \theta_j \) is closed and looking at the coefficient of \( d\theta_{j,k} \), we obtain that
\[ \frac{\partial \beta_{j,k}}{\partial \theta_j} (\theta_{j,k}) \text{ only depends on } j \text{ and then also } \theta_{j,k} \]

the function \( \lambda_j \). Finally
\[ w = \sum_{j \neq k} \lambda_j \beta_{j,k} (\theta_{j,k}) d \theta_{j,k} + \sum_{j \neq k} \lambda_j \beta_{j,k} (\theta_{j,k}) d \theta_{j,k} \]
and the \( \lambda_j \) action is symplectic. Hence
\[ \sum_{j \neq k} \lambda_j \beta_{j,k} (\theta_{j,k}) d \theta_{j,k} \text{ is closed, and } \]

then exact on \( B^n \); \( \sum_{j \neq k} \lambda_j \beta_{j,k} (\theta_{j,k}) d \theta_{j,k} = 0 \).

We take \((z_1, \ldots, z_n)\) as coordinates:
\[ w = \sum_{j \neq k} \lambda_j \beta_{j,k} (\theta_{j,k}) d \theta_{j,k} + \sum_{j \neq k} \lambda_j \beta_{j,k} (\theta_{j,k}) d \theta_{j,k} \]

with \( \omega \) and then \( \sum_{j \neq k} \lambda_j \beta_{j,k} (\theta_{j,k}) d \theta_{j,k} \) closed and \( \omega \) exact:
\[ d (\sum_{j \neq k} \lambda_j \beta_{j,k} (\theta_{j,k}) d \theta_{j,k}) = \sum_{j \neq k} \lambda_j \beta_{j,k} (\theta_{j,k}) d \theta_{j,k} \]

that gives
\[ w = \sum_{j \neq k} \lambda_j \beta_{j,k} (\theta_{j,k}) d \theta_{j,k} + \sum_{j \neq k} \lambda_j \beta_{j,k} (\theta_{j,k}) d \theta_{j,k} \]

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If \((\Theta_i, Z_i) = (\Theta_i + A_i(z), Z_i)\), thus coordinates are symplectic and in these coordinates \(H\) only depends on \(z\).

Remark: This proposition seems to be due to Hinez (1935). The proof we presented comes from Nguyen Tien Zung 2013, a conceptual approach to the problem of angle action variables. Arxiv.

### IV. ON ENERGY LEVEL

A natural question concerns what happens when we only know that one regular energy level of a Hamiltonian \(H\) is foliated by compact Lagrangian submanifolds.

When \(M = \mathbb{R}^2\) and \(H\) is convex or when \(H\) is Tonelli and the energy level is above the so-called Mane critical level, there is a standard way to change \(H\) in such a way that:

\[
\{k = 1\} = \{ H = c_j \} \quad \text{the new Hamiltonian} \ K \text{ has one energy level in common with } H, \text{ the energy level}
\]

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that was leaked by invariant hyperplan submanifolds for \( \Phi_k \).

The energy levels \( \{ K=0 \} \) near to \( t \) are regular and the flow on \( \{ K=0 \} \) is conjugate to a reparametrization of the flow \( \Phi_k^{(1)}(K=0) \).

As two regular flows that have in common an energy level \( E \) are the same up to a time reparametrization on \( E \), we deduce

proposition if \( H \) is a Tonelli Hamiltonian that has energy level \( E \) above the critical level \( K=0 \) and is foliated by invariant compact hyperplan submanifolds, then up to a time reparametrization, the Hamiltonian flow on every such submanifold is conjugate to a rotation flow.

\[ \Delta \] This doesn't imply that this is conjugate to a rotation flow. There are examples of Shkoller (in 1960), Fayad (in 2001) - with double rotation number when this flow is weakly...