Very Weak Turbulence for Certain Dispersive Equations

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The 2D cubic NLS Initial Value Problem in $\mathbb{T}^2$

We consider the defocusing initial value problem:

$$(1.1) \qquad \begin{cases} (-i\partial_t + \Delta)u = |u|^2 u \\ u(0,x) = u_0(x), \text{ where } x \in \mathbb{T}^2. \end{cases}$$

We have (see Lectures #1&#2)

Theorem (Global well-posedness for smooth data)

For any data $u_0 \in H^s(\mathbb{T}^2)$, $s \geq 1$ there exists a unique global solution $u(x,t) \in C(\mathbb{R}, H^s)$ to the Cauchy problem (1.1).

We also recall that

$$\text{Mass} = M(u) = \|u(t)\|^2 = M(0)$$

$$\text{Energy} = E(u) = \int (\frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{4}|u(x,t)|^4)\, dx = E(0).$$

Polynomial upper bounds

As mentioned in Vedran's seminar we have

Theorem (Bourgain, Zhong, Sohinger)

For the smooth global solutions of the periodic IVP (1.1) *above we have:*

$$\|u(t)\|_{\dot{H}^s} \leq C_s |t|^{s+}.$$

But are there solutions for which such a growth occur? Unfortunately so far what we can prove is much weaker and we will state the precise theorem a little later.

Can one show growth of Sobolev norms?

One should recall the following result of **Bourgain**:

Theorem

Given $m, s \gg 1$ *there exist* $\tilde{\Delta}$ *and a global solution* $u(x,t)$ *to the modified wave equation*

$$(\partial_{tt} - \tilde{\Delta})u = u^p$$

such that $\|u(t)\|_{H^s} \sim |t|^m$.

The weakness of this result is in the fact that one needs to modify the equation in order to make a solution exhibit a cascade.

More references

Recently Gerard and Grellier obtained some growth results for Sobolev norms of solutions to the periodic 1D cubic Szegö equation:

$$i\partial_t u = \Pi(|u|^2 u),$$

where $\Pi(\sum_k \hat{f}(k)e^{xk}) = \sum_{k>0} \hat{f}(k)e^{xk}$ is the Szegö projector.

- Physics: Weak turbulence theory due to Hasselmann and Zakharov.
- Numerics (d=1): Majda-McLaughlin-Tabak; Zakharov et. al.
- Probability: Benney and Newell, Benney and Saffman.

To show how far we are from actually solving the open problems proposed above I will present what is known so far for the 2D cubic defocusing NLS in $\mathbb{T}^2$.

Very weak energy transfer to high frequencies

What we can prove

Theorem (Colliander-Keel-Staffilani-Takaoka-Tao)

Let $s > 1$, $K \gg 1$ and $0 < \sigma < 1$ be given. Then there exist a global smooth solution $u(x, t)$ to the IVP (1.1) and $T > 0$ such that

$$\|u_0\|_{H^s} \le \sigma \quad \text{and} \quad \|u(T)\|^2_{\dot{H}^s} \ge K.$$

Elements of the proof of the main theorem

1. Reduction to a resonant problem *RFNLS*
2. Construction of a special finite set Λ of frequencies
3. Truncation to a resonant, finite-d Toy Model
4. "Arnold diffusion" for the Toy Model
5. Approximation result via perturbation lemma
6. A scaling argument

The Ansatz

We consider the gauge transformation

$$v(t,x) = e^{-i2Gt}u(t,x),$$

for $G \in \mathbb{R}$. If u solves *NLS* above, then v solves the equation

$$((NLS)_G) \qquad (-i\partial_t + \Delta)v = (2G + v)|v|^2.$$

We make the ansatz

$$v(t,x) = \sum_{n\in\mathbb{Z}^2} a_n(t)e^{i(\langle n,x\rangle + |n|^2 t)}.$$

Now the dynamics is all recast trough $a_n(t)$:

$$-i\partial_t a_n = 2Ga_n + \sum_{n_1 - n_2 + n_3 = n} a_{n_1}\overline{a_{n_2}}a_{n_3}e^{i\omega_4 t}$$

where $\omega_4 = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2$.

The *FNLS* system

By choosing

$$G = -\|v(t)\|_{L^2}^2 = -\sum_k |a_k(t)|^2$$

which is constant from the conservation of the mass, one can rewrite the equation above as

$$-i\partial_t a_n = -a_n|a_n|^2 + \sum_{n_1,n_2,n_3\in\Gamma(n)} a_{n_1}\overline{a_{n_2}}a_{n_3}e^{i\omega_4 t}$$

where

$$\Gamma(n) = \{n_1, n_2, n_3 \in \mathbb{Z}^2 \ / \ n_1 - n_2 + n_3 = n; n_1 \neq n; n_3 \neq n\}.$$

From now on we will be refering to this system as the *FNLS* system, with the obvious connection with the original *NLS* equation.

The *RFNLS* system

We define the set

$$\Gamma_{res}(n) = \{n_1, n_2, n_3 \in \Gamma(n) \,/\, \omega_4 = 0\},$$

where again $\omega_4 = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2$.
The geometric interpretation for this set is the following: If n_1, n_2, n_3 are in $\Gamma_{res}(n)$, then these four points represent the vertices of a rectangle in $\mathbb{Z}^2$.
We finally define the Resonant Truncation *RFNLS* to be the system

$$-i\partial_t b_n = -b_n|b_n|^2 + \sum_{n_1, n_2, n_3 \in \Gamma_{res}(n)} b_{n_1} \overline{b_{n_2}} b_{n_3}.$$

Finite dimensional resonant truncation

- A finite set $\Lambda \subset \mathbb{Z}^2$ is closed under resonant interactions if

$$n_1, n_2, n_3 \in \Gamma_{res}(n), n_1, n_2, n_3 \in \Lambda \ =: n = n_1 - n_2 + n_3 \in \Lambda.$$

- A Λ-finite dimensional resonant truncation of *RFNLS* is

$$(RFNLS_\Lambda) \qquad -i\partial_t b_n = -b_n|b_n|^2 + \sum_{(n_1,n_2,n_3)\in\Gamma_{res}(n)\cap\Lambda^3} b_{n_1}\overline{b}_{n_2}b_{n_3}.$$

- $\forall$ resonant-closed finite $\Lambda \subset \mathbb{Z}^2$, $RFNLS_\Lambda$ is an ODE.

We will construct a **special set** Λ of frequencies.

Abstract Combinatorial Resonant Set Λ

Our goal is to have a resonant-closed $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_N$, N to be fixed later, with the properties below.

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- Existence and uniqueness of spouse and children: $\forall\ 1 \leq j < M$ and $\forall\ n_1 \in \Lambda_j$ $\exists$ unique nuclear family such that $n_1, n_3 \in \Lambda_j$ are parents and $n_2, n_4 \in \Lambda_{j+1}$ are children.

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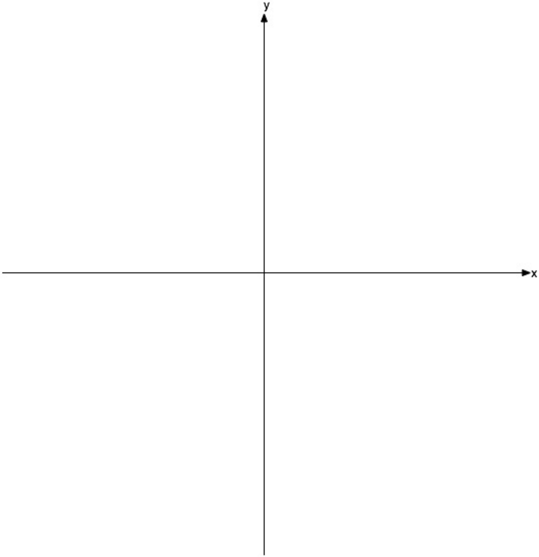
- Non degeneracy: The sibling of a frequency is never its spouse.
- Faithfulness: Besides nuclear families, Λ contains no other rectangles.
- Integenerational Equality:The function $n \longmapsto a_n(0)$ is constant on each generation Λ_j.

Cartoon Construction of Λ



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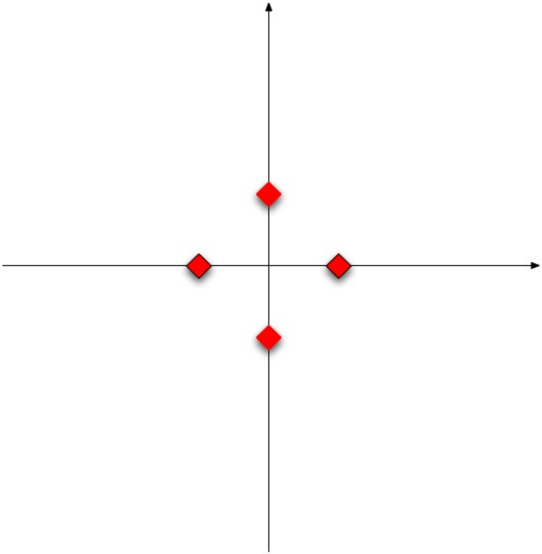

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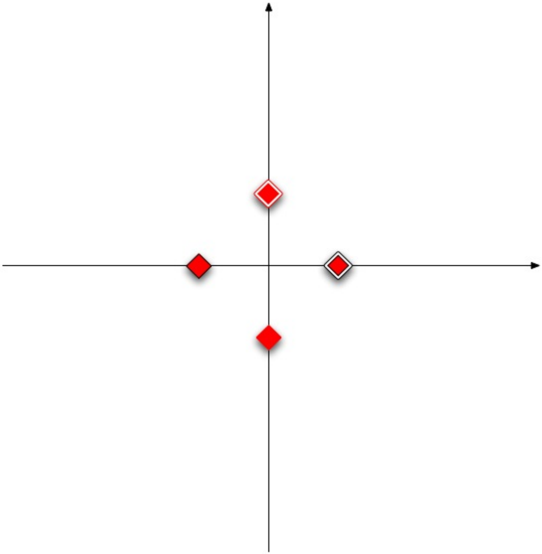

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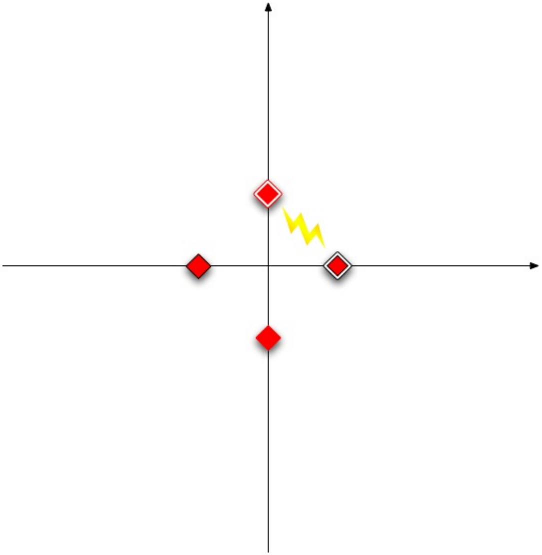

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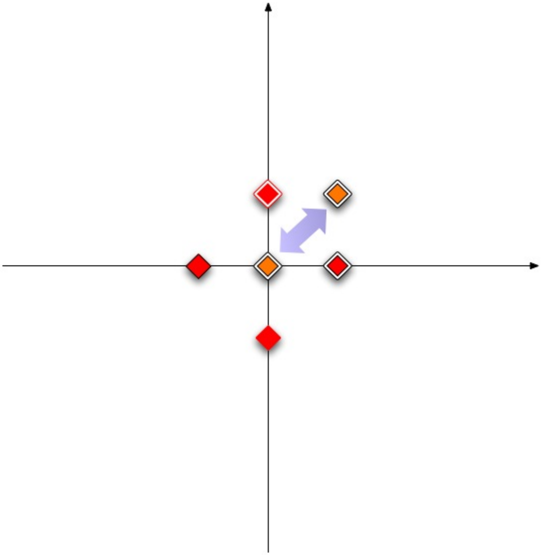

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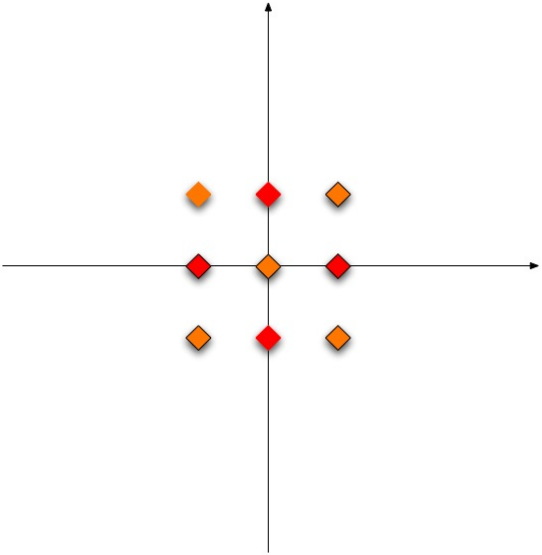

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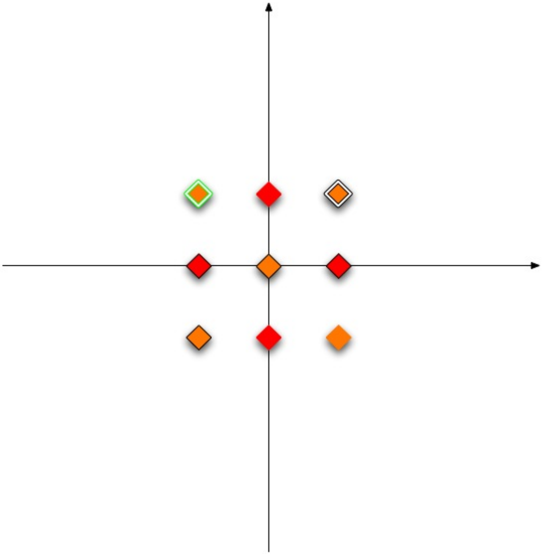

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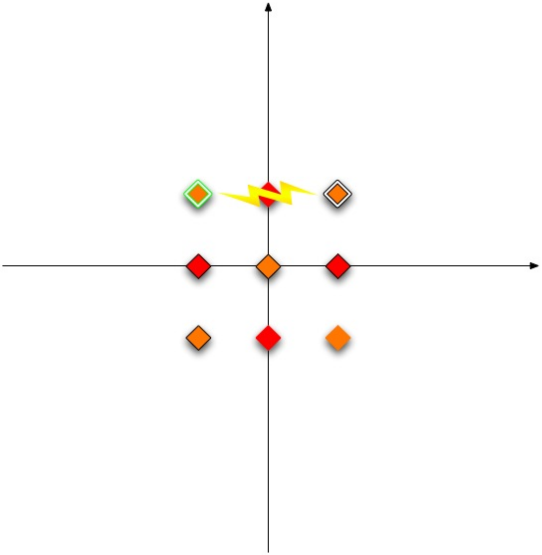

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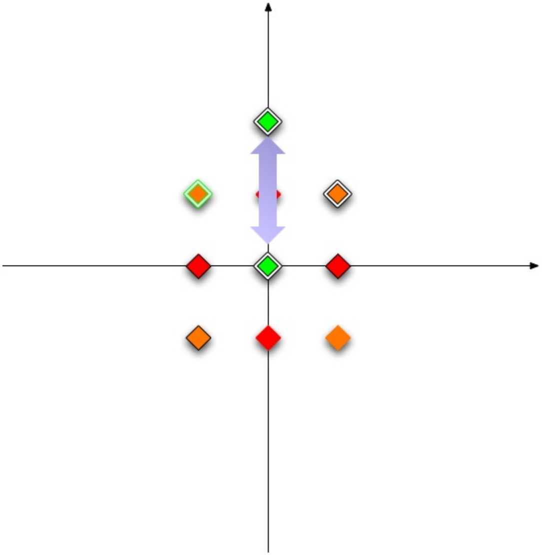

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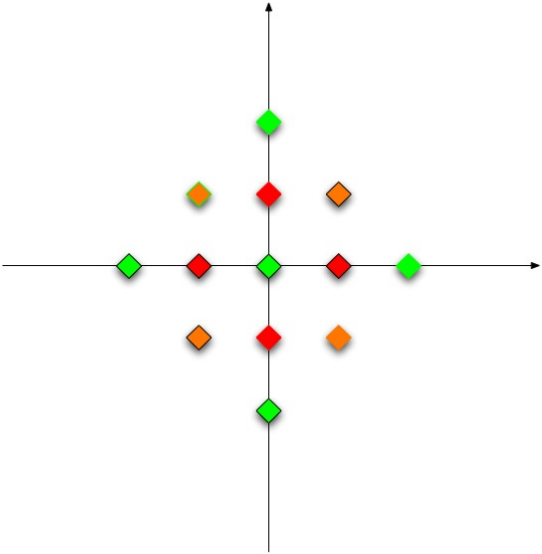

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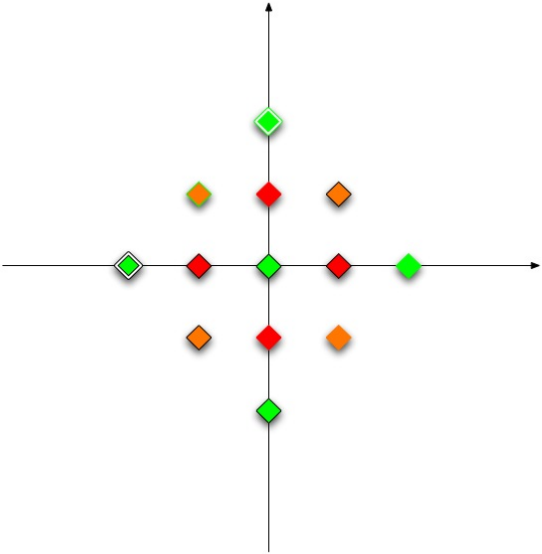

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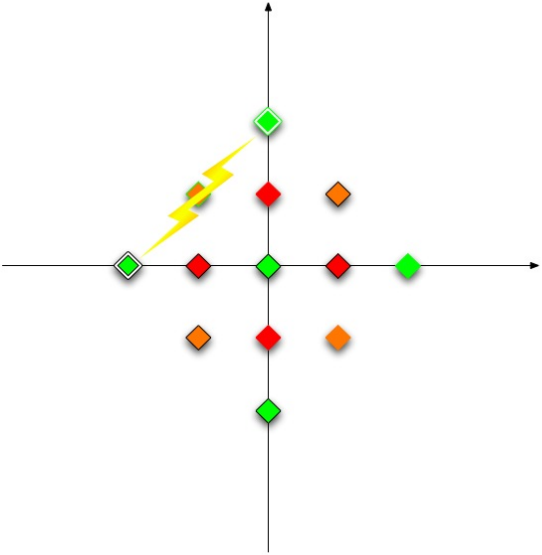

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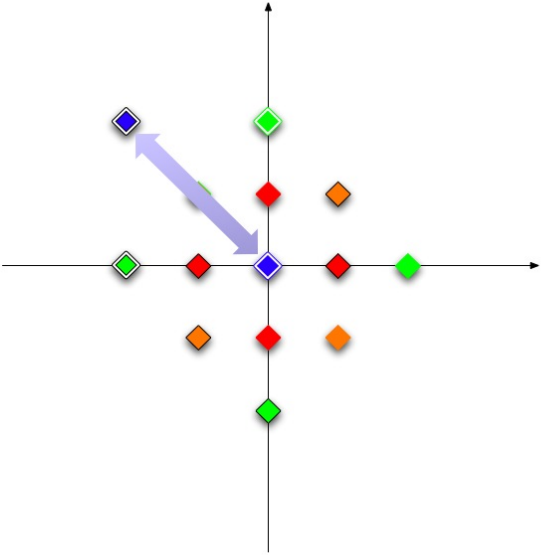

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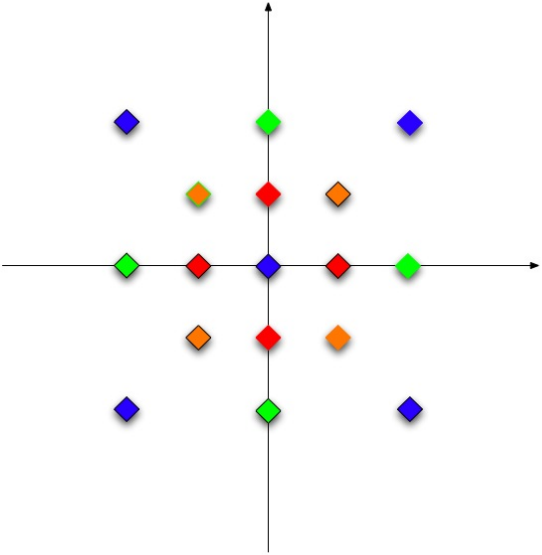

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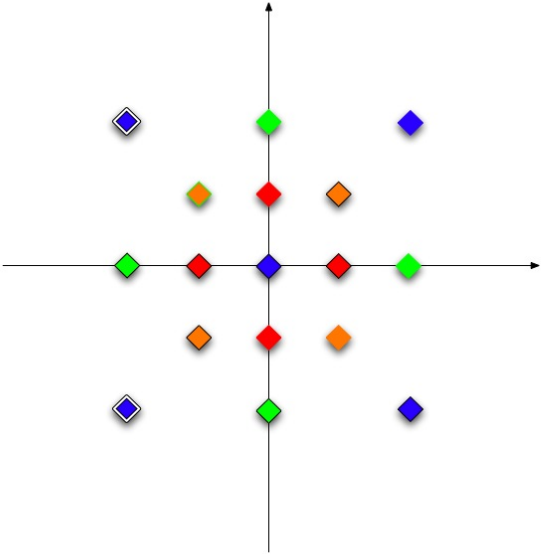

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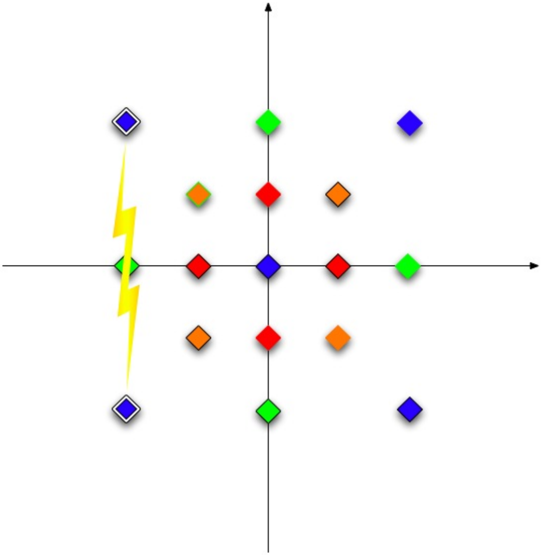

More properties for the set Λ

- **Multiplicative Structure:** If $N = N(\sigma, K)$ is large enough then Λ consists of $N \times 2^{N-1}$ disjoint frequencies n with $|n| > N = N(\sigma, K)$, the first frequency in Λ_1 is of size N and we call N the Inner Radius of Λ.
- **Wide Diaspora:** Given $\sigma \ll 1$ and $K \gg 1$, if N is large enough then $\Lambda = \Lambda_1 \cup .... \cup \Lambda_N$ as above and

$$\sum_{n \in \Lambda_N} |n|^{2s} \geq \frac{K^2}{\sigma^2} \sum_{n \in \Lambda_1} |n|^{2s}.$$

- **Approximation:** If $\text{spt}(a_n(0)) \subset \Lambda$ then *FNLS*-evolution $a_n(0) \longmapsto a_n(t)$ is nicely approximated by $RFNLS_\Lambda$-ODE $a_n(0) \longmapsto b_n(t)$.
- Given ϵ, s, K, build Λ so that $RFNLS_\Lambda$ has weak turbulence.

The Toy Model

- The truncation of *RFNLS* to the constructed set Λ is the ODE

$$(RFNLS_\Lambda) \qquad -i\partial_t b_n = -b_n|b_n|^2 + \sum_{(n_1,n_2,n_3)\in\Lambda^3\cap\Gamma_{res}(n)} b_{n_1} b_{n_2} b_{n_3}.$$

- The intergenerational equality hypothesis ($n \longmapsto b_n(0)$ is constant on each generation Λ_j.) persists under $RFNLS_\Lambda$:

$$\forall\, m, n \in \Lambda_j,\ b_n(t) = b_m(t).$$

- $RFNLS_\Lambda$ may be reindexed by generation number j.
  The recast dynamics is the Toy Model (ODE):

$$-i\partial_t b_j(t) = -b_j(t)|b_j(t)|^2 - 2b_{j-1}(t)^2\overline{b_j(t)} - 2b_{j+1}(t)^2\overline{b_j(t)},$$

  with the boundary condition

$$(BC) \qquad b_0(t) = b_{N+1}(t) = 0.$$

Conservation laws for the *ODE* system

The following are conserved quantities for (*ODE*)

$$\textit{Mass} = \sum_j |b_j(t)|^2 = C_0$$

$$\textit{Momentum} = \sum_j |b_j(t)|^2 \sum_{n\in\Lambda_j} n = C_1,$$

and if

$$\textit{Kinetic Energy} = \sum_j |b_j(t)|^2 \sum_{n\in\Lambda_j} |n|^2$$

$$\textit{Potential Energy} = \frac{1}{2}\sum_j |b_j(t)|^4 + \sum_j |b_j(t)|^2 |b_{j+1}(t)|^2,$$

then

$$\textit{Energy} = \textit{Kinetic Energy} + \textit{Potential Energy} = C_2.$$

Toy model traveling wave solution

[1] Maybe dynamical systems methods are useful here?

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Using direct calculation[1], we will prove that our Toy Model ODE evolution $b_j(0) \longmapsto b_j(t)$ is such that:

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$$(b_1(0), b_2(0), \ldots, b_N(0)) \quad \sim \quad (1, 0, \ldots, 0)$$

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&\,\cdot \\
&\,\cdot \\
&\,\cdot \\
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&\cdot \\
&\cdot \\
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Bulk of conserved mass is transferred from Λ_1 to Λ_N.

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& . & \\
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Bulk of conserved mass is transferred from Λ_1 to Λ_N. Weak turbulence lower bound follows from Wide Diaspora Property.

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Instability for the *ODE*: the set up

Global well-posedness for *ODE* is not an issue. Then we define

$$\Sigma = \{x \in \mathbb{C}^N \ / \ |x|^2 = 1\} \text{ and } W(t) : \Sigma \to \Sigma,$$

where $W(t)b(t_0) = b(t + t_0)$ for any solution $b(t)$ of *ODE*. It is easy to see that for any $b \in \Sigma$

$$\partial_t |b_j|^2 = 4\Re(i\bar{b}_j^2(b_{j-1}^2 + b_{j+1}^2)) \leq 4|b_j|^2.$$

So if

$$b_j(0) = 0 =: b_j(t) = 0, \quad \text{for all} \quad t \in [0, T].$$

If moreover we define the torus

$$\mathbb{T}_j = \{(b_1, ..., b_N) \in \Sigma \ / \ |b_j| = 1, \ b_k = 0, \ k \neq j\}$$

then

$$W(t)\mathbb{T}_j = \mathbb{T}_j \ \text{ for all } j = 1, ..., N$$

($\mathbb{T}_j$ is invariant).

Instability for the *ODE*

Theorem (Sliding Theorem)

Let $N \geq 6$. Given $\epsilon > 0$ there exist x_3 within ϵ of $\mathbb{T}_3$ and x_{N-2} within ϵ of $\mathbb{T}_{N-2}$ and a time t such that

$$W(t)x_3 = x_{N-2}.$$

Remark

$W(t)x_3$ is a solution of total mass 1 arbitrarily concentrated near mode $j = 3$ at some time t_0 and then arbitrarily concentrated near mode $j = N - 2$ at later time t.

The sliding process

To motivate the theorem let us first observe that when $N = 2$ we can easily demonstrate that there is an orbit connecting $\mathbb{T}_1$ to $\mathbb{T}_2$. Indeed in this case we have the explicit "slider" solution

$$b_1(t) := \frac{e^{-it}\omega}{\sqrt{1+e^{2\sqrt{3}t}}}; \quad b_2(t) := \frac{e^{-it}\omega^2}{\sqrt{1+e^{-2\sqrt{3}t}}} \tag{7.1}$$

where $\omega := e^{2\pi i/3}$ is a cube root of unity.

This solution approaches $\mathbb{T}_1$ exponentially fast as $t \to -\infty$, and approaches $\mathbb{T}_2$ exponentially fast as $t \to +\infty$. One can translate this solution in the j parameter, and obtain solutions that "slide" from $\mathbb{T}_j$ to $\mathbb{T}_{j+1}$. Intuitively, the proof of the Sliding Theorem for higher N should then proceed by concatenating these slider solutions......

This is a cartoon of what we have:

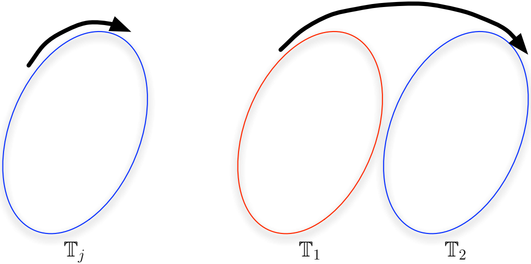

Figure: Explicit oscillator solution around $\mathbb{T}_j$ and the slider solution from $\mathbb{T}_1$ to $\mathbb{T}_2$

This though cannot work directly because each solution requires an infinite amount of time to connect one circle to the next, but it turns out that a suitably perturbed or “fuzzy” version of these slider solutions can in fact be glued together.

A Perturbation Lemma

Lemma

Let $\Lambda \subset \mathbb{Z}^2$ introduced above. Let $B \gg 1$ and $\delta > 0$ small and fixed. Let $t \in [0, T]$ and $T \sim B^2 \log B$. Suppose there exists $b(t) \in l^1(\Lambda)$ solving RFNLS$_\Lambda$ such that

$$\|b(t)\|_{l^1} \lesssim B^{-1}.$$

Then there exists a solution $a(t) \in l^1(\mathbb{Z}^2)$ of FNLS such that

$$a(0) = b(0), \quad \text{and} \quad \|a(t) - b(t)\|_{l^1(\mathbb{Z}^2)} \lesssim B^{-1-\delta},$$

for any $t \in [0, T]$.

Proof.

This is a standard perturbation lemma proved by checking that the “non resonant” part of the nonlinearity remains small enough. □

Recasting the main theorem

With all the notations and reductions introduced we can now recast the main theorem in the following way:

Theorem

For any $0 < \sigma \ll 1$ *and* $K \gg 1$ *there exists a complex sequence* (a_n) *such that*

$$\left(\sum_{n \in \mathbb{Z}^2} |a_n|^2 |n|^{2s} \right)^{1/2} \lesssim \sigma$$

and a solution $(a_n(t))$ *of* (*FNLS*) *and* $T > 0$ *such that*

$$\left(\sum_{n \in \mathbb{Z}^2} |a_n(T)|^2 |n|^{2s} \right)^{1/2} > K.$$

A Scaling Argument

In order to be able to use "instability" to move mass from lower frequencies to higher ones and start with a small data we need to introduce scaling.

Consider in $[0, \tau]$ the solution $b(t)$ of the system *RFNLS*$_\Lambda$ with initial datum b_0. Then the rescaled function

$$b^\lambda(t) = \lambda^{-1} b(\frac{t}{\lambda^2})$$

solves the same system with datum $b_0^\lambda = \lambda^{-1} b_0$.
We then first pick the complex vector $b(0)$ that was found in the "instability" theorem above. For simplicity let's assume here that $b_j(0) = 1 - \epsilon$ if $j = 3$ and $b_j(0) = \epsilon$ if $j \neq 3$ and then we fix

$$a_n(0) = \begin{cases} b_j^\lambda(0) \text{ for any } n \in \Lambda_j \\ 0 \text{ otherwise }. \end{cases}$$

Estimating the size of $(a(0))$

By definition

$$\left(\sum_{n\in\Lambda}|a_n(0)|^2|n|^{2s}\right)^{1/2}=\frac{1}{\lambda}\left(\sum_{j=1}^{M}|b_j(0)|^2\left(\sum_{n\in\Lambda_j}|n|^{2s}\right)\right)^{1/2}\sim\frac{1}{\lambda}Q_3^{1/2},$$

where the last equality follows from defining

$$\sum_{n\in\Lambda_j}|n|^{2s}=Q_j,$$

and the definition of $a_n(0)$ given above. At this point we use the proprieties of the set Λ to estimate $Q_3C(N)N^{2s}$, where N is the inner radius of Λ. We then conclude that

$$\left(\sum_{n\in\Lambda}|a_n(0)|^2|n|^{2s}\right)^{1/2}=\lambda^{-1}C(N)N^s\sim\sigma.$$

Estimating the size of $(a(T))$

By using the perturbation lemma with $B = \lambda$ and $T = \lambda^2\tau$ we have

$$\|a(T)\|_{H^s} \geq \|b^\lambda(T)\|_{H^s} - \|a(T) - b^\lambda(T)\|_{H^s} = I_1 - I_2.$$

We want $I_2 \ll 1$ and $I_1 > K$. For the first

$$I_2 \leq \|a(T) - b^\lambda(T)\|_{l^1(\mathbb{Z}^2)} \left(\sum_{n\in\Lambda} |n|^{2s}\right)^{1/2} \lesssim \lambda^{-1-\delta} \left(\sum_{n\in\Lambda} |n|^{2s}\right)^{1/2}.$$

As above

$$I_2 \lesssim \lambda^{-1-\delta} C(N) N^s$$

At this point we need to pick λ and N so that

$$\|a(0)\|_{H^s} = \lambda^{-1} C(N) N^s \sim \sigma \;\text{ and }\; I_2 \lesssim \lambda^{-1-\delta} C(N) N^s \ll 1$$

and thanks to the presence of $\delta > 0$ this can be achieved by taking λ and N large enough.

Estimating I_1

It is important here that at time zero one starts with a fixed non zero datum, namely $\|a(0)\|_{H^s} = \|b^\lambda(0)\|_{H^s} \sim \sigma > 0$. In fact we will show that

$$I_1^2 = \|b^\lambda(T)\|_{H^s}^2 \geq \frac{K^2}{\sigma^2}\|b^\lambda(0)\|_{H^s}^2 \sim K^2.$$

If we define for $T = \lambda^2 t$

$$R = \frac{\sum_{n\in\Lambda} |b_n^\lambda(\lambda^2 t)|^2 |n|^{2s}}{\sum_{n\in\Lambda} |b_n^\lambda(0)|^2 |n|^{2s}},$$

then we are reduce to showing that $R \gtrsim K^2/\sigma^2$. Now recall the notation

$$\Lambda = \Lambda_1 \cup ..... \cup \Lambda_N \quad \text{and} \quad \sum_{n\in\Lambda_j} |n|^{2s} = Q_j.$$

More on Estimating I_1

Using the fact that by the theorem on "instability" (approximately) one obtains $b_j(T) = 1 - \epsilon$ if $j = N - 2$ and $b_j(T) = \epsilon$ if $j \neq N - 2$, it follows that

$$
\begin{aligned}
R &= \frac{\sum_{i=1}^{N} \sum_{n\in\Lambda_i} |b_i^\lambda(\lambda^2 t)|^2 |n|^{2s}}{\sum_{i=1}^{N} \sum_{n\in\Lambda_i} |b_i^\lambda(0)|^2 |n|^{2s}} \\
&\geq \frac{Q_{N-2}(1-\epsilon)}{(1-\epsilon)Q_3 + \epsilon Q_1 + ... + \epsilon Q_N} \sim \frac{Q_{N-2}(1-\epsilon)}{Q_{N-2}\left[(1-\epsilon)\frac{Q_3}{Q_{N-2}} + ... + \epsilon\right]} \\
&\gtrsim \frac{(1-\epsilon)}{(1-\epsilon)\frac{Q_3}{Q_{N-2}}} = \frac{Q_{N-2}}{Q_3}
\end{aligned}
$$

and the conclusion follows from "large diaspora" of Λ_j:

$$Q_{N-2} = \sum_{n\in\Lambda_{N-2}} |n|^{2s} \gtrsim \frac{K^2}{\sigma^2} \sum_{n\in\Lambda_3} |n|^{2s} = \frac{K^2}{\sigma^2} Q_3.$$

Where does the set Λ come from?

Here we do not construct Λ, but we construct Σ, a set that has a lot of the properties of Λ. We define the *standard unit square* $S \subset \mathbb{C}$ to be the

four-element set of complex numbers

$$S = \{0, 1, 1+i, i\}.$$

We split $S = S_1 \cup S_2$, where $S_1 := \{1, i\}$ and $S_2 := \{0, 1+i\}$. The combinatorial model Σ is a subset of a large power of the set S. More precisely, for any $1 \leq j \leq N$, we define $\Sigma_j \subset \mathbb{C}^{N-1}$ to be the set of all $N-1$-tuples $(z_1, \dots, z_{N-1})$ such that $z_1, \dots, z_{j-1} \in S_2$ and $z_j, \dots, z_{N-1} \in S_1$. In other words,

$$\Sigma_j := S_2^{j-1} \times S_1^{N-j}.$$

Note that each Σ_j consists of 2^{N-1} elements, and they are all disjoint. We then set $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_N$; this set consists of $N2^{N-1}$ elements. We refer to Σ_j as the j^{th} *generation* of Σ.

For each $1 \leq j < N$, we define a *combinatorial nuclear family connecting generations* Σ_j, Σ_{j+1} to be any four-element set $F \subset \Sigma_j \cup \Sigma_{j+1}$ of the form

$$F := \{(z_1, \ldots, z_{j-1}, w, z_{j+1}, \ldots, z_N) : w \in S\}$$

where $z_1, \ldots, z_{j-1} \in S_2$ and $z_{j+1}, \ldots, z_N \in S_1$ are fixed. In other words, we have

$$F = \{F_0, F_1, F_{1+i}, F_i\} = \{(z_1, \ldots, z_{j-1})\} \times S \times \{(z_{j+1}, \ldots, z_N)\}$$

where $F_w = (z_1, \ldots, z_{j-1}, w, z_{j+1}, \ldots, z_N)$.

It is clear that

- F is a four-element set consisting of two elements F_1, F_i of Σ_j (which we call the *parents* in F) and two elements F_0, F_{1+i} of Σ_{j+1} (which we call the *children* in F).
- For each j there are 2^{N-2} combinatorial nuclear families connecting the generations Σ_j and Σ_{j+1}.

Properties of Σ

One easily verifies the following properties:

- Existence and uniqueness of spouse and children: For any $1 \leq j < N$ and any $x \in \Sigma_j$ there exists a unique combinatorial nuclear family F connecting Σ_j to Σ_{j+1} such that x is a parent of this family (i.e. $x = F_1$ or $x = F_i$). In particular each $x \in \Sigma_j$ has a unique spouse (in Σ_j) and two unique children (in Σ_{j+1}).
- Existence and uniqueness of sibling and parents: For any $1 \leq j < N$ and any $y \in \Sigma_{j+1}$ there exists a unique combinatorial nuclear family F connecting Σ_j to Σ_{j+1} such that y is a child of the family (i.e. $y = F_0$ or $y = F_{1+i}$). In particular each $y \in \Sigma_{j+1}$ has a unique sibling (in Σ_{j+1}) and two unique parents (in Σ_j).
- Nondegeneracy: The sibling of an element $x \in \Sigma_j$ is never equal to its spouse.

Example:

If $N = 7$, the point $x = (0, 1 + i, 0, i, i, 1)$ lies in the fourth generation Σ_4. Its spouse is $(0, 1 + i, 0, 1, i, 1)$ (also in Σ_4) and its two children are $(0, 1 + i, 0, 0, i, 1)$ and $(0, 1 + i, 0, 1 + i, i, 1)$ (both in Σ_5). These four points form a combinatorial nuclear family connecting the generations Σ_4 and Σ_5. The sibling of x is $(0, 1 + i, 1 + i, i, i, 1)$ (also in Σ_4, but distinct from the spouse) and its two parents are $(0, 1 + i, 1, i, i, 1)$ and $(0, 1 + i, i, i, i, 1)$ (both in Σ_3). These four points form a combinatorial nuclear family connecting the generations Σ_3 and Σ_4. Elements of Σ_1 do not have siblings or parents, and elements of Σ_7 do not have spouses or children.