Almost sure GWP, Gibbs measures and gauge transformations

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Invariant Gibbs measures for Hamiltonian PDEs: finite dimension

Hamilton's equations of motion have the antisymmetric form

$$\text{(HE)} \qquad \dot{q}_i = \frac{\partial H(p,q)}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H(p,q)}{\partial q_i}$$

the Hamiltonian $H(p,q)$ being a first integral:

$$\frac{dH}{dt} := \sum_i \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i = \sum_i \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right) = 0$$

And by defining $y := (q_1, \ldots, q_k, p_1, \ldots, p_k)^T \in \mathbb{R}^{2k}$ $(2k = d)$ we can rewrite

$$\frac{dy}{dt} = J\nabla H(y), \qquad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

Liouville's Theorem: Let a vector field $f : \mathbb{R}^d \to \mathbb{R}^d$ be divergence free then the if the flow map Φ_t satisfies:

$$\frac{d}{dt}\Phi_t(y) \;=\; f(\Phi_t(y)),$$

then it is a volume preserving map (for all t).

In particular if f is associated to a Hamiltonian system then automatically $div\, f = 0$. Indeed

$$div\, f = \frac{\partial}{\partial q_1}\frac{\partial H}{\partial p_1} + \frac{\partial}{\partial q_2}\frac{\partial H}{\partial p_2} + \dots \frac{\partial}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial}{\partial p_1}\frac{\partial H}{\partial q_1} - \frac{\partial}{\partial p_2}\frac{\partial H}{\partial q_2} - \dots \frac{\partial}{\partial p_k}\frac{\partial H}{\partial q_k} = 0$$

by equality of mixed partial derivatives.

The Lebesgue measure on $\mathbb{R}^{2k}$ is invariant under the Hamiltonian flow (HE). Consequently from conservation of Hamiltonian H the Gibbs measures,

$$d\mu := e^{-\beta H(p,q)} \prod_{i=1}^{d} dp_i\, dq_i$$

with $\beta > 0$ are invariant under the flow of (HE); ie. for $A \subset \mathbb{R}^d$,

$$\mu(\Phi_t(A)) = \mu(A)$$

Infinite Dimension Hamiltonian PDEs

In the context of semilinear NLS $iu_t + u_{xx} \pm |u|^{p-2}u = 0$ on $\mathbb{T}$ one can think of u as the infinite dimension vector given by its Fourier coefficients:

$$\hat{u}(n) = a_n + ib_n, \quad n \in \mathbb{Z}$$

and with respect to the Hamiltonian

$$H(u) = \frac{1}{2}\int |u_x|^2\, dx \pm \frac{1}{p}\int |u|^p\, dx$$

one can think of the equation as an infinite dimension Hamiltonian system.

• Lebowitz, Rose and Speer (1988) considered the Gibbs measure *formally* given by

$$`d\mu = Z^{-1} \exp\left(-\beta H(u)\right) \prod_{x\in\mathbb{T}} du(x)'$$

and showed that μ is a well-defined probability measure on $H^s(\mathbb{T})$ for any $s < \frac{1}{2}$ but not (we will see this later) for $s = \frac{1}{2}$.

• In the focusing case the result only holds for $p \leq 6$ with the L^2-cutoff $\chi_{\|u\|_{L^2} \leq B}$ for any $B > 0$ if $p < 6$ and with small B for $p = 6$ (recall the L^2 norm is conserved for these equations.)

• Bourgain (94') proved the invariance of this measure and a.s. gwp. More precisely, in the defocusing case for example he proved:

Theorem

Consider the focusing NLS initial value problem

$$\begin{cases} (i\partial_t + \Delta)u = -|u|^4 u \\ u(0,x) = u_0(x), \ \ \textit{where } x \in \mathbb{T}. \end{cases} \tag{2.1}$$

Then the measure μ introduced above is well defined in H^s, $0 < s < 1/2$ for B small and almost surely with respect to it the problem is globally well-posed. Moreover the measure μ is invariant under the flow given by (2.1).

Two elements of the theorem above are particularly relevant: the Global Well-Posedness and the Invariance of the Measure.

Global well-posedness of dispersive equations

In the past few years two methods have been developed and applied to study the global in time existence of dispersive equations at regularities which are right below or in between those corresponding to conserved quantities:

- High-low method by J. Bourgain.
- I-method (or method of *almost conservation laws*) by J. Colliander, M. Keel, G. S., H. Takaoka and T. Tao

For many dispersive equations and systems there still remains a gap between the local in time results and those that could be globally achieved.

When these two methods fail, Bourgain's approach for periodic dispersive equations (NLS, KdV, mKdV, Zakharov system) is through

the introduction and use of the Gibbs measure derived from the PDE viewed as an infinite dimension Hamiltonian system.

- Why is this last method effective?
- There are two fundamental reasons:
  - Because failure to show global existence by Bourgain's high-low method or the I-method might come from certain 'exceptional' initial data set, and the virtue of the Gibbs measure is that it does not see that exceptional set.
  - The invariance of the Gibbs measure, just like the usual conserved quantities, can be used to control the growth in time of those solutions in its support and extend the local in time solutions to global ones almost surely.

The difficulty in this approach lies in the actual construction of the associated Gibbs measure and in showing its invariance under the flow.

This approach has recently successfully been used by:

• T. Oh (2007- PhD thesis) for the periodic KdV-type coupled systems.

• Tzevkov (2007) for subquintic radial NLW on 2d disc.

• Burq-Tzevtkov (2007-2008) for subcubic & subquartic radial NLW on 3d ball.

• T. Oh (2008-2009) Schrödinger-Benjamin-Ono, KdV on $\mathbb{T}$.

• Thomann -Tzevtkov (2010) for DNLS (only formal construction of the measure).

Gauss measure and Gibbs measures in infinite dimensions

Let's take the example in the theorem above. Note that the quantity

$$H(u) + \frac{1}{2}\int |u|^2(x)\,dx$$

is conserved, but one usually sees the Gibbs measure μ written as

$$d\mu = Z^{-1}\chi_{\|u\|_{L^2}\leq B}\exp\left(\frac{1}{6}\int |u|^6\,dx\right)\exp\left(-\frac{1}{2}\int(|u_x|^2+|u|^2)\,dx\right)\prod_{x\in\mathbb{T}} du(x)$$

where

$$d\rho = \exp\left(-\frac{1}{2}\int(|u_x|^2+|u|^2)\,dx\right)\prod_{x\in\mathbb{T}} du(x)$$

is the Gauss measure that is well understood in $H^s, s < 1/2$ and

$$\frac{d\mu}{d\rho} = \chi_{\|u\|_{L^2}\leq B}\exp\left(\frac{1}{6}\int |u|^6\,dx\right)$$

corresponding to the nonlinear term of the Hamiltonian is understood as the Radon-Nikodym derivative of μ with respect to ρ.

More about Gauss measure

Our Gauss measure ρ is defined as weak limit of the finite dimensional Gauss measures

$$d\rho_N = Z_{0,N}^{-1} \exp\Big(-\frac{1}{2} \sum_{|n|\leq N} (1+|n|^2)|\widehat{v}_n|^2 \Big) \prod_{|n|\leq N} da_n db_n \,.$$

Note that the measure ρ_N above can be regarded as the induced probability measure on $\mathbb{R}^{4N+2}$ under the map

$$\omega \longmapsto \left\{ \frac{g_n(\omega)}{\sqrt{1+|n|^2}} \right\}_{|n|\leq N} \quad \text{and} \quad \widehat{v}_n = \frac{g_n}{\sqrt{1+|n|^2}},$$

where $\{g_n(\omega)\}_{|n|\leq N}$ are independent standard complex Gaussian random variables on a probability space $(\Omega, \mathcal{F}, P)$.

In a similar manner, we can view ρ as the induced probability measure under the map

$$\omega \mapsto \left\{ \frac{g_n(\omega)}{\sqrt{1+|n|^2}} \right\}_{n\in\mathbb{Z}} .$$

What is its support?

Consider the operator $\mathcal{J}_s = (1-\Delta)^{s-1}$ then

$$\sum_n (1+|n|^2) \left|\widehat{v}_n\right|^2 = \langle v, v\rangle_{H^1} = \langle \mathcal{J}_s^{-1} v, v\rangle_{H^s} .$$

The operator $\mathcal{J}_s : H^s \to H^s$ has the set of eigenvalues $\{(1+|n|^2)^{(s-1)}\}_{n\in\mathbb{Z}}$ and the corresponding eigenvectors $\{(1+|n|^2)^{-s/2} e^{inx}\}_{n\in\mathbb{Z}}$ form an orthonormal basis of H^s.

- For ρ to be *countable additive* we need $\mathcal{J}_s$ to be of *trace class* which is true if and only if $s < \frac{1}{2}$.
  Then ρ is a countably additive measure on H^s for any $s < 1/2$ (but **not** for $s \geq 1/2$!)

Bourgain's Method

Above we stated Bourgain's theorem for the quintic focusing periodic NLS. Here we give an outline of Bourgain's idea in a general framework, and discuss how to prove almost surely GWP and the invariance of a measure from local well-posedness.
Consider a dispersive nonlinear Hamiltonian PDE with a k-linear nonlinearity possibly with derivative.

(PDE) $$\begin{cases} u_t = \mathcal{L}u + \mathcal{N}(u) \\ u|_{t=0} = u_0 \end{cases}$$

where $\mathcal{L}$ is a (spatial) differential operator like $i\partial_{xx}$, ∂_{xxx}, etc. (systems). Let $H(u)$ denote the Hamiltonian of (PDE). Then, (PDE) can also be written as

$$u_t = J\,\frac{dH}{du} \quad \text{if } u \text{ is real-valued,} \quad u_t = J\,\frac{\partial H}{\partial \overline{u}} \quad \text{if } u \text{ is complex-valued.}$$

Let μ denote a measure on the distributions on $\mathbb{T}$, whose invariance we'd like to establish. We assume that μ is a weighted Gaussian measure (formally) given by

$$\text{" } d\mu = Z^{-1} e^{-F(u)} \prod_{x \in \mathbb{T}} du(x) \text{ "}$$

where $F(u)$ is conserved[1] under the flow of (PDE) and the leading term of $F(u)$ is quadratic and nonnegative.

Now, suppose that there is a good local well-posedness theory:

There exists a Banach space $\mathcal{B}$ of distributions on $\mathbb{T}$ and a space $X_\delta \subset C([-\delta, \delta]; \mathcal{B})$ of space-time distributions in which to prove local well-posedness by a fixed point argument with a time of existence δ depending on $\|u_0\|_B$, say $\delta \sim \|u_0\|_B^{-\alpha}$ for some $\alpha > 0$.

[1] $F(u)$ could be the Hamiltonian, but not necessarily!

In addition, suppose that the Dirichlet projections P_N – the projection onto the spatial frequencies $\leq N$ – act boundedly on these spaces, uniformly in N.

Consider the finite dimensional approximation to (PDE)

(FDA)
$$\begin{cases} u_t^N = \mathcal{L}u^N + P_N(\mathcal{N}(u^N)) \\ u^N|_{t=0} = u_0^N := P_N u_0(x) = \sum_{|n|\leq N} \widehat{u}_0(n)e^{inx}. \end{cases}$$

Then, for $\|u_0\|_B \leq K$ one can see (FDA) is LWP on $[-\delta, \delta]$ with $\delta \sim K^{-\alpha}$, independent of N.

Two more important assumptions on (FDA):

(1) (FDA) is Hamiltonian with $H(u^N)$ i.e.

$$u_t^N = J \frac{dH(u^N)}{d\overline{u}^N}$$

(2) $F(u^N)$ is still conserved under the flow of (FDA)

Note: (1) holds for example when the symplectic form J commutes with the projection P_N. (e.g. $J = i$ or ∂_x.).

In general however (1) and (2) are **not** guaranteed and may not necessarily hold! (more later).

From this point on the argument goes through the following steps:

- By Liouville's theorem and (1) above the Lebesgue measure

$$\prod_{|n|\leq N} da_n db_n,$$

where $\widehat{u^N}(n) = a_n + ib_n$, is invariant under the flow of (FDA).

- Then, using (2) - the conservation of $F(u^N)$- we have that the finite dimensional version μ_N of μ:

$$d\mu_N = Z_N^{-1} e^{-F(u^N)} \prod_{|n|\leq N} da_n db_n$$

is also invariant under the flow of (FDA)!

- Next ingredient we need is:

Lemma [Fernique-type tail estimate]

For K suff. large, we have

$$\mu_N(\{\|u_0^N\|_{\mathcal{B}} > K\}) < Ce^{-CK^2}, \text{indep of } N.$$

Proof.

Here I will show the main ingrediets in the case of the quintic NLS equation introduced above:

- **Step 1:** If B is small enough then

$$\exp \left\| \sum_{|n| \leq N} \frac{g_n(\omega)}{(1+n^2)^{1/2}} e^{i2\pi xn} \right\|_{L^6}^6 \chi_{\{\sum_{|n| \leq N} \frac{|g_n(\omega)|^2}{1+n^2} < B\}} \in L^1(d\omega)$$

  and the bound is uniform in N.

□

Now define

$$\Omega_{N,K} := \left\{ (a_n) \ / \ \left\| \sum_{|n|\leq N} a_n e^{inx} \right\|_{H^s} > K \quad \text{and } L^2 \text{ restriction)} \right\}$$

then

- **Step 2:**

$$\begin{aligned} \mu_n(\Omega_{N,K}) &= \int_{\Omega_{NK}} \exp \| \sum_{|n|\leq N} a_n e^{i2\pi xn} \|_{L^6}^6 \, d\rho_N \\ &\leq C\mathbb{P}_{\rho_N}[\{\omega \ / \ \| \sum_{|n|\leq N} a_n e^{inx} \|_{H^s} > K\}]^{1/2} \leq Ce^{-CK^2} \end{aligned}$$

  since ρ_N is a Gaussian measure and ρ too since we assume $s < 1/2$.

- The lemma we just presented + invariance of μ_N imply the following estimate controlling the growth of solution u^N to (FDA).

Main Proposition: Bourgain '94

Given $T < \infty, \varepsilon > 0$, there exists $\Omega_N \subset \mathcal{B}$ s.t.

- $\mu_N(\Omega_N^c) < \varepsilon$
- for $u_0^N \in \Omega_N$, (FDA) is well-posed on $[-T, T]$ with the growth estimate:

$$\|u^N(t)\|_{\mathcal{B}} \lesssim \left(\log \frac{T}{\varepsilon}\right)^{\frac{1}{2}}, \text{ for } |t| \leq T.$$

Proof.

Let $\Phi_N(t)$ = flow map of (FDA), and define

$$\Omega_N = \cap_{j=-[T/\delta]}^{[T/\delta]} \Phi_N(j\delta)(\{\|u_0^N\|_{\mathcal{B}} \leq K\}).$$

- By invariance of μ_N,

$$\mu(\Omega_N^c) = \sum_{j=-[T/\delta]}^{[T/\delta]} \mu_N \Phi_N(j\delta)(\{\|u_0^N\|_{\mathcal{B}} > K\}) = 2[T/\delta]\mu_N(\{\|u_0^N\|_{\mathcal{B}} > K\})$$

  This implies $\mu(\Omega_N^c) \lesssim \frac{T}{\delta}\mu_N(\{\|u_0^N\|_{\mathcal{B}} > K\}) \sim TK^\theta e^{-cK^2}$, and by choosing $K \sim \left(\log \frac{T}{\varepsilon}\right)^{\frac{1}{2}}$, we have $\mu(\Omega_N^c) < \varepsilon$.

- By its construction, $\|u^N(j\delta)\|_{\mathcal{B}} \leq K$ for $j = 0, \cdots, \pm[T/\delta]$ and by local theory,

$$\|u^N(t)\|_{\mathcal{B}} \leq 2K \sim \left(\log \frac{T}{\varepsilon}\right)^{\frac{1}{2}} \text{ for } |t| \leq T.$$

□

- One then needs to prove that μ_N converges weakly to μ.

This is standard and one can go back to the work of **Zhidkov** for example.

One defines

$$E_N = [e^{2\pi inx} \;/\; |n| \leq N],$$

and then shows that if $U \subset H^s$, $s < 1/2$ is open the two limits below are defined:

$$\begin{aligned} \rho(U) &:= \lim_{N\to\infty} \rho_N(U \cap E_N) \\ \mu(U) &:= \lim_{N\to\infty} \mu_N(U \cap E_N). \end{aligned}$$

Going back to PDE

Essentially as a corollary of the Main Proposition one can then prove:

Corollary

(a) *Given $\varepsilon > 0$, there exists $\Omega_\varepsilon \subset \mathcal{B}$ with $\mu(\Omega_\varepsilon^c) < \varepsilon$ such that for $u_0 \in \Omega_\varepsilon$, (PDE) is globally well-posed with the growth estimate:*

$$\|u(t)\|_{\mathcal{B}} \lesssim \left(\log \frac{1+|t|}{\varepsilon}\right)^{\frac{1}{2}}, \text{ for all } t \in \mathbb{R}.$$

(b) *The uniform convergence lemma: $\|u - u^N\|_{C([-T,T];\mathcal{B}')} \to 0$ as $N \to \infty$ uniformly for $u_0 \in \Omega_\varepsilon$, where $\mathcal{B}' \supset \mathcal{B}$.*

Note (a) implies that (PDE) is a.s. GWP, since $\widetilde{\Omega} := \bigcup_{\varepsilon>0} \Omega_\varepsilon$ has probability 1.

One can prove (a) and (b) by estimating the difference $u - u^N$ using the LWP theory + an Approximation Lemma and applying the Main Proposition above to u^N.

Finally, putting all the ingredients together, we obtain the invariance of μ.

Derivative NLS Equation

Now we would like to introduce another infinite dimensional system:

(DNLS) $$\begin{cases} u_t - i\,u_{xx} = \lambda(|u|^2 u)_x \\ u\big|_{t=0} = u_0 \end{cases}$$

where either $(x,t) \in \mathbb{R} \times (-T,T)$ or $(x,t) \in \mathbb{T} \times (-T,T)$ and λ is real.

- We take $\lambda = 1$ for convenience and note DNLS is a Hamiltonian PDE with conservation of *mass* and *'energy'*. In fact, it is completely integrable.

The first three conserved quantities of time are:

• Mass: $m(u) = \frac{1}{2\pi} \int_{\mathbb{T}} |u(x,t)|^2\, dx$

• 'Energy': $E(u) = \int_{\mathbb{T}} |u_x|^2\, dx + \frac{3}{2}\mathrm{Im} \int_{\mathbb{T}} u^2 \overline{u u_x}\, dx + \frac{1}{2} \int_{\mathbb{T}} |u|^6\, dx$

• Hamiltonian: $H(u) = \mathrm{Im} \int_{\mathbb{T}} u \bar{u}_x\, dx + \frac{1}{2} \int_{\mathbb{T}} |u|^4\, dx$ (at $\dot{H}^{\frac{1}{2}}$ level).

We would like now to explore the possibility of extending Bourgain's approach for the 1D periodic DNLS.

- **Goal 1**: Construct an associated invariant weighted Wiener measure and establish GWP for data living in its support. In particular almost surely for data living in a Fourier-Lebesgue space (defined later) scaling like $H^{\frac{1}{2}-\epsilon}(\mathbb{T})$, for small $\epsilon > 0$. Joint with:

  Andrea Nahmod (UMass Amherst)

  Tadahiro Oh (Princeton U)

  Luc Rey Bellet (UMass Amherst).

- **Goal 2**: Show that the ungauged invariant Wiener measure associated to the periodic derivative NLS obtained above is absolutely continuous with respect to the weighted Wiener measure constructed by Thomann and Tzvetkov. We prove a general result on absolute continuity of Gaussian measures under certain gauge transformations. Joint with:

  Andrea Nahmod (UMass Amherst)

  Luc Rey Bellet (UMass Amherst)

  Scott Sheffield (MIT)

More about the DNLS

- The equation is scale invariant for data in L^2: if $u(x,t)$ is a solution then $u_a(x,t) = a^{\alpha} u(ax, a^2 t)$ is also a solution iff $\alpha = \frac{1}{2}$. Thus *a priori* one expects some form of existence and uniqueness for data in $H^{\sigma}, \sigma \geq 0$.
- Many results are known for the Cauchy problem with smooth data, including data in H^1 (Tsutsumi-Fukada 80's; N.Hayashi, N. Hayashi- T. Ozawa and T. Ozawa 90's)

In looking for solutions to (DNLS) we face a derivative loss arising from the nonlinear term and hence for low regularity data the key is to somehow make up for this loss.

The non-periodic case ($x \in \mathbb{R}$)

- Takaoka (1999) proved sharp local well-posedness (LWP) in $H^{\frac{1}{2}}(\mathbb{R})$ via a gauge transformation (Hayashi and Ozawa) + sharp multilinear estimates for the gauged equivalent equation in the Fourier restriction norm spaces $X^{s,b}$.
- Colliander, Keel, S., Takaoka and Tao (2001-2002) established global well-posedness (GWP) in $H^{\sigma}(\mathbb{R})$, $\sigma > \frac{1}{2}$ of small L^2 norm using the so-called I-Method on the gauge equivalent equation. (Small in L^2 means $\lesssim \sqrt{\frac{2\pi}{\lambda}}$: 'energy' to be positive via Gagliardo-Nirenberg inequality.).
- Miao, Wu and Xu recently extended GWP to $H^{\sigma}(\mathbb{R})$, $\sigma \geq \frac{1}{2}$.
- The Cauchy initial value problem is ill-posed for data in $H^{\sigma}(\mathbb{R})$ and $\sigma < 1/2$; i.e. data map fails to be C^3 or uniformly C^0. (Takaoka, Biagioni-Linares)

Periodic (DNLS)

- S. Herr (2006) showed that the Cauchy problem associated to periodic DNLS is locally well-posed for initial data $u(0) \in H^\sigma(\mathbb{T})$, if $\sigma \geq \frac{1}{2}$.

  Proof based on an adaptation of the gauge transformation above to the periodic setting + sharp multilinear estimates for the gauged equivalent equation in periodic Fourier restriction norm spaces $X^{s,b}$.
  By use of conservation laws, the problem is also shown to be globally well-posed for $\sigma \geq 1$ and data which is small in L^2-as in [CKSTT].

- Y. Y. Su Win (2009- PhD thesis) applied the I-Method to prove GWP in $H^\sigma(\mathbb{T})$ for $\sigma > 1/2$.
- Also in the periodic case the problem is believed to be ill-posed in $H^\sigma(\mathbb{T})$ for $\sigma < 1/2$ in the sense that fixed point theorem cannot be used with Sobolev spaces.

Periodic Gauged Derivative NLS Equation

- Why do we need to gauge? Because the nonlinearity:

$$(|u|^2 u)_x = u^2\,\overline{u}_x + 2\,|u|^2\,u_x \quad \text{hard to control.}$$

Periodic Gauge Transformation (S. Herr, 2006): For $f \in L^2(\mathbb{T})$

$$G(f)(x) := \exp(-iJ(f))\,f(x)$$

where

$$J(f)(x) := \frac{1}{2\pi}\int_0^{2\pi}\int_\theta^x \left(|f(y)|^2 - \frac{1}{2\pi}\|f\|^2_{L^2(\mathbb{T})} \right) dy\,d\theta$$

is the **unique** 2π**-periodic mean zero primitive** of the map

$$x \longrightarrow |f(x)|^2 - \frac{1}{2\pi}\|f\|^2_{L^2(\mathbb{T})}.$$

Then, for $u \in C([-T,T];L^2(\mathbb{T}))$ the adapted periodic gauge is defined as

$$\mathcal{G}(u)(t,x) := G(u(t))(x - 2\,t\,m(u))$$

We have that

$$\mathcal{G} : C([-T,T];H^\sigma(\mathbb{T})) \to C([-T,T];H^\sigma(\mathbb{T}))$$

is a homeomorphism for any $\sigma \geq 0$. Moreover,

$\mathcal{G}$ is locally bi-Lipschitz on subsets of functions in $C([-T,T]; H^{\sigma}(\mathbb{T}))$ with prescribed L^2-norm. The same is true if we replace $H^{\sigma}(\mathbb{T})$ by $\mathcal{F}L^{s,r}$, the Fourier-Lebesgue spaces (later).

• Local well-posedness for (GDNLS) in H^{σ} **implies** local existence and uniqueness for (DNLS) in H^{σ}; but don't necessarily have all the auxiliary estimates coming from the LWP result on (GDNLS).

If u is a solution to (DNLS) and $v := \mathcal{G}(u)$ we have that v solves:

$$\text{(GDNLS)} \qquad v_t - i v_{xx} = -v^2 \overline{v}_x + \frac{i}{2}|v|^4 v - i\psi(v)v - i m(v)|v|^2 v$$

with initial data $v(0) = \mathcal{G}(u(0))$ and where

$$m(u) = m(v) := \frac{1}{2\pi}\int_{\mathbb{T}} |v|^2(x,t)dx = \frac{1}{2\pi}\int_{\mathbb{T}} |v(x,0)|^2(x)dx$$

$$\psi(v)(t) := -\frac{1}{\pi}\int_{\mathbb{T}} \mathrm{Im}(v\overline{v}_x)\, dx + \frac{1}{4\pi}\int_{\mathbb{T}} |v|^4 dx - m(v)^2$$

Note both $m(v)$ and $\psi(v)(t)$ are real.

What's the energy for GDNLS?

For v the solution to the periodic (GDNLS) define

$$\mathcal{E}(v) := \int_{\mathbb{T}} |v_x|^2\,dx - \frac{1}{2}\mathrm{Im}\int_{\mathbb{T}} v^2\overline{v\,v_x}\,dx + \frac{1}{4\pi}\left(\int_{\mathbb{T}} |v(t)|^2\,dx\right)\left(\int_{\mathbb{T}} |v(t)|^4\,dx\right).$$

$$\mathcal{H}(v) := \mathrm{Im}\int_{\mathbb{T}} v\overline{v}_x - \frac{1}{2}\int_{\mathbb{T}} |v|^4\,dx + 2\pi m(v)^2$$

$$\tilde{\mathcal{E}}(v) := \mathcal{E}(v) + 2m(v)\mathcal{H}(v) - 2\pi\, m(v)^3$$

We prove:

$$\frac{d\tilde{\mathcal{E}}(v)}{dt} = 0.$$

In fact one can show that $E(u) = \tilde{\mathcal{E}}(v)$.

We refer to $\tilde{\mathcal{E}}(v)$ from now on as the *energy* of (GDNLS).

- A. Grünrock and S. Herr (2008) showed that the Cauchy problem associated to DNLS is locally well-posed for initial data $u_0 \in \mathcal{F}L^{s,r}(\mathbb{T})$ and $2 \leq r < 4$, $s \geq 1/2$.

$$\|u_0\|_{\mathcal{F}L^{s,r}(\mathbb{T})} := \| \langle n \rangle^s \hat{u}_0 \|_{\ell^r_n}(\mathbb{Z}) \qquad r \geq 2$$

These spaces scale like the Sobolev $H^\sigma(\mathbb{T})$ ones where $\boxed{\sigma = s + 1/r - 1/2}$.

For example for $s = 2/3-$ and $r = 3$ $\quad \sigma < 1/2$.

Proof based on Herr's adapted periodic gauge transformation and new **sharp** multilinear estimates for the gauged equivalent equation in an appropriate variant of Fourier restriction norm spaces $X^{s,b}_{r,q}$ introduced by Grünrock-Herr.

$$\|u\|_{X^{s,b}_{r,q}} := \|\langle n\rangle^s \langle \tau - n^2\rangle^b \widehat{u}(n,\tau)\|_{\ell^r_n L^q_\tau}$$

where first take the L^q_τ norm and then the ℓ^r_n one.

For $\delta > 0$ fixed, the restriction space $X^{s,b}_{r,q}(\delta)$ is defined as usual

$$\|v\|_{X^{s,b}_{r,q}(\delta)} := \inf\{\|u\|_{X^{s,b}_{r,q}} \ : \ u \in X^{s,b}_{r,q} \ \text{and} \ v = u\big|_{[-\delta,\delta]}\}.$$

For $q = 2$ we simply write $X^{s,b}_{r,2} = X^{s,b}_{r}$. Note $X^{s,b}_{2,2} = X^{s,b}$.

Later we will also use the space

$$Z^s_r(\delta) := X^{s,\frac{1}{2}}_{r,2}(\delta) \cap X^{s,0}_{r,1}(\delta).$$

In particular,

$$Z^s_r(\delta) \subset C([-\delta,\delta], \mathcal{F}L^{s,r}).$$

A.S. Global well-posedness for DNLS

Our Goal 1:

- Establish the a.s GWP for the periodic DNLS in a Fourier Lebesgue space $\mathcal{F}L^{s,r}$ scaling below $H^{1/2}(\mathbb{T})$. and the invariance of the associate Gibbs measure μ.

- Invariance μ: if $\Phi(t)$ is the flow map associated to the nonlinear equation; then for reasonable F

$$\int F(\Phi(t)(\phi))\,\mu(d\phi) = \int F(\phi)\mu(d\phi)$$

Method:

Construct μ so that LWP of periodic DNLS in some space B containing supp(μ) holds. Then show a.s. GWP as well as the invariance of μ via Bourgain's argument (and Zhidkov's) (for the Gibbs meas of NLS, KdV, mKdV, Bourgain '94) + some new ingredients !.

Finite dimensional approximation of (GDNLS)

Recall

(GDNLS) $$v_t = iv_{xx} - v^2\bar{v}_x + \frac{i}{2}|v|^4 v - i\psi(v)v - im(v)|v|^2 v$$

where

$$\psi(v) = -\frac{1}{\pi}\int \mathrm{Im}\, v\bar{v}_x + \frac{1}{4\pi}\int |v|^4\, dx - m(v)^2$$

and $m(v) = \frac{1}{2\pi}\int |v|^2\, dx$.

Our finite dimensional approximation is (FGDNLS):

$$v_t^N = iv_{xx}^N - P_N((v^N)^2\overline{v_x^N}) + \frac{i}{2}P_N(|v^N|^4 v^N) - i\psi(v^N)v^N - im(v^N)P_N(|v^N|^2 v^N)$$

with initial data $v_0^N = P_N\, v_0$.

- Note $m(v^N)(t) := \frac{1}{2\pi}\int_{\mathbb{T}} |v^N(x,t)|^2 dx$ is also conserved under the flow of (FGDNLS).

Lemma [Local well-posedness]

Let $2 < r < 4$ and $s \geq \frac{1}{2}$. **Then** for every

$$v_0^N \in B_R := \{v_0^N \in \mathcal{F}L^{s,r}(\mathbb{T}) / \|v_0^N\|_{\mathcal{F}L^{s,r}(\mathbb{T})} < R\}$$

and $\delta \lesssim R^{-\gamma}$, for some $\gamma > 0$, there exists a unique solution

$$v^N \in Z_r^s(\delta) \subset C([-\delta, \delta]; \mathcal{F}L^{s,r}(\mathbb{T}))$$

of (FGDNLS) with initial data v_0^N . Moreover the map

$$\left(B_R, \|\cdot\|_{\mathcal{F}L^{s,r}(\mathbb{T})}\right) \longrightarrow C([-\delta, \delta]; \mathcal{F}L^{s,r}(\mathbb{T})) : \quad v_0^N \to v^N$$

is real analytic.

- The proof essentially follows from Grünrock-Herr's LWP estimates.
  - P_N acts on a multilinear nonlinearity and it is a bounded operator on $L^p(\mathbb{T})$ commuting with D^s.

Lemma [Approximation lemma]

Let $v_0 \in \mathcal{F}L^{s,r}(\mathbb{T})$, $s \geq \frac{1}{2}$, $r \in (2,4)$ be such that $\|v_0\|_{\mathcal{F}L^{s,r}(\mathbb{T})} < A$, for some $A > 0$, and let N be a large integer. Assume the solution v^N of (FGDNLS) with initial data $v_0^N(x) = P_N v_0$ satisfies the bound

$$\|v^N(t)\|_{\mathcal{F}L^{s,r}(\mathbb{T})} \leq A, \text{ for all } t \in [-T, T],$$

for some given $T > 0$. **Then** the IVP **(GDNLS)** with initial data v_0 is well-posed on $[-T, T]$ and there exists $C_0, C_1 > 0$, such that its solution $v(t)$ satisfies the following estimate:

$$\|v(t) - v^N(t)\|_{\mathcal{F}L^{s_1,r}(\mathbb{T})} \lesssim \exp[C_0(1+A)^{C_1} T] N^{s_1 - s},$$

for all $t \in [-T, T], 0 < s_1 < s$.

Construction of the Weighted Wiener Measure

We need to construct probability spaces on which we establish well-posedness.

To construct these measures we will make use of the conserved quantity: $\tilde{\mathcal{E}}(v)$ as well as the L^2-norm.

To construct the measures on infinite dimensional spaces we consider conserved quantities of the form $\exp(-\frac{\beta}{2}\tilde{\mathcal{E}}(v))$. But can't construct a finite measure *directly*! using this quantity since:

(a) the nonlinear part of $\tilde{\mathcal{E}}(v)$ is not bounded below

(b) the linear part is only non-negative but not positive definite.

To resolve this issue we proceed as follows.

As we learned above we use the conservation of L^2-norm and consider instead the quantity

$$\chi_{\{\|v\|_{L^2}\leq B\}} e^{-\frac{\beta}{2}\mathcal{N}(v)} e^{-\frac{\beta}{2}\int(|v|^2+|v_x|^2)dx}$$

where $\mathcal{N}(v)$ is the nonlinear part of the energy $\tilde{\mathcal{E}}(v)$, i.e.

$$\begin{aligned} \mathcal{N}(v) &= -\frac{1}{2}\mathrm{Im}\int_{\mathbb{T}} v^2\overline{v}\overline{v}_x\,dx - \frac{1}{4\pi}\left(\int_{\mathbb{T}}|v|^2\,dx\right)\left(\int_{\mathbb{T}}|v|^4\,dx\right) + \\ &+ \frac{1}{\pi}\left(\int_{\mathbb{T}}|v|^2\,dx\right)\left(\mathrm{Im}\int_{\mathbb{T}} v\overline{v}_x\,dx\right) + \frac{1}{4\pi^2}\left(\int_{\mathbb{T}}|v|^2\,dx\right)^3. \end{aligned}$$

and B is a (suitably small) constant.

Then we would like to construct the measure (with $v(x) = u(x) + iw(x)$)

$$\text{“ } d\mu_\beta = Z^{-1}\chi_{\{\|v\|_{L^2}\leq B\}} e^{-\frac{\beta}{2}\mathcal{N}(v)} e^{-\frac{\beta}{2}\int(|v|^2+|v_x|^2)dx} \prod_{x\in\mathbb{T}} du(x)dw(x) \text{ ”}$$

This is a purely formal, although suggestive, expression since it is impossible to define the Lebesgue measure on an infinite-dimensional space as countably additive measure. Moreover as it will turn out that $\int|u_x|^2 = \infty$, μ almost surely.

We learned that one uses instead a Gaussian measure as reference measure and the weighted measure μ is constructed in two steps:

- First one constructs a Gaussian measure ρ as the limit of the finite-dimensional measures on $\mathbb{R}^{4N+2}$ given by

$$d\rho_N = Z_{0,N}^{-1} \exp\Big(-\frac{\beta}{2} \sum_{|n|\leq N} (1+|n|^2)|\widehat{v}_n|^2 \Big) \prod_{|n|\leq N} da_n db_n$$

where $\widehat{v}_n = a_n + ib_n$.

The construction of such Gaussian measures on Hilbert spaces is a classical subject. But we need to realize this measure as a measure supported on a suitable Banach space, the Fourier-Lebesgue space $\mathcal{F}L^{s,r}(\mathbb{T})$ in view of the local well-posedness result by Grünrock-Herr . Since $\mathcal{F}L^{s,r}$ is not a Hilbert space, we need to construct ρ as a measure supported on a Banach space. This needs some extra work but it is possible by relying on L. Gross 65' and H. Kuo 75' theory of abstract Wiener spaces, (from here the name of Weighted Wiener Measures).

- In particular, we prove that for $2 \le r < \infty$ and $(s-1)r < -1$:

(1) $(i, H^1, \mathcal{F}L^{s,r})$ is an abstract Wiener space. (i=inclusion map)

(2) The Wiener measure ρ can be realized as a countably additive measure supported on $\mathcal{F}L^{s,r}$ and

(3) Have an exponential tail estimate : there exists $c > 0$ (with $c = c(s,r)$) such that

$$\rho(\|v\|_{\mathcal{F}L^{s,r}} > K) \le e^{-cK^2}.$$

Note: For (r, s) as above $\underbrace{s + \frac{1}{r} - \frac{1}{2}}_{=:\sigma} < \frac{1}{2}$ (recall $\mathcal{F}L^{s,r}$ scales as H^σ)

Here we assume again that v is of the form

$$v(x) = \sum_n \frac{g_n(\omega)}{\sqrt{1+|n|^2}} e^{inx},$$

where $\{g_n(\omega)\}$ are independent standard complex Gaussian random variables as above

- Once this measure ρ has been constructed, with a nontrivial amount of work, and using also some estimates in Thomann and Tzvetkov one then obtains the weighted Wiener measure μ. More precisely

$$R(v) := \chi_{\{\|v\|_{L^2} \leq B\}} e^{-\frac{1}{2}\mathcal{N}(v)}\,, \qquad R_N(v) \;:=\; R(v^N)$$

where $\mathcal{N}(v)$ is the nonlinear part of the energy $\tilde{\mathcal{E}}$. Here $v^N = P_N(v)$ for some generic function v in our F-L spaces.
We obtain

$$d\mu = Z^{-1} R(v) d\rho\,,$$

for sufficiently small B, as is the weak limit of the finite dimensional weighted Wiener measures μ_N on $\mathbb{R}^{4N+2}$ given by

$$\begin{aligned} d\mu_N &= Z_N^{-1} R_N(v) d\rho_N \\ &= \hat{Z}_N^{-1} \chi_{\{\|\widehat{v}^N\|_{L^2} \leq B\}} e^{-\frac{1}{2}(\tilde{\mathcal{E}}(\hat{v}^N) + \|\hat{v}^N\|_{L^2})} \prod_{|n| \leq N} da_n db_n \end{aligned}$$

for suitable normalizations Z_N, $\hat{Z}_N$. More precisely we have:

Lemma [Convergence]

$R_N(v)$ converges in measure to $R(v)$.

Moreover we have

Proposition [Existence of weighted Wiener measure]

(a) For sufficiently small $B > 0$, we have $R(v) \in L^2(d\rho)$. In particular, the weighted Wiener measure μ is a probability measure, *absolutely continuous* with respect to the Wiener measure ρ.

(b) We have the following tail estimate. Let $2 \leq r < \infty$ and $(s-1)r < -1$; then there exists a constant c such that

$$\mu(\|v\|_{\mathcal{F}L^{s,r}} > K) \leq e^{-cK^2}$$

for sufficiently large $K > 0$.

(c) The finite dim. weighted Wiener measure μ_N converges weakly to μ.

Example of an estimate

Recall that

$$R_N(v) := \chi_{\{\|v^N\|_{L^2} \le B\}} e^{-\frac{1}{2}\mathcal{N}(v^N)},$$

and

$$\mathcal{N}(v) = -\frac{1}{2}\mathrm{Im}\int_{\mathbb{T}} v^2 \overline{v}\overline{v}_x\, dx - \frac{1}{4\pi}\left(\int_{\mathbb{T}} |v|^2\, dx\right)\left(\int_{\mathbb{T}} |v|^4\, dx\right) + $$
$$+ \frac{1}{\pi}\left(\int_{\mathbb{T}} |v|^2\, dx\right)\left(\mathrm{Im}\int_{\mathbb{T}} v\overline{v}_x\, dx\right) + \frac{1}{4\pi^2}\left(\int_{\mathbb{T}} |v|^2\, dx\right)^3.$$

Here we concentrate on the term $X_N(v) := \int_{\mathbb{T}} v^N \overline{v}_x^N$. We have the following

Lemma

For any $N \le M$ and $\varepsilon > 0$ we have

$$\|X_M(v) - X_N(v)\|_{L^4} \lesssim \frac{1}{N^{\frac{1}{2}}}.$$

Proof of the lemma

We start by recalling that $v^N(\omega, x) := \sum_{|n|\leq N} \frac{g_n(\omega)}{\langle n\rangle} e^{inx}$. Then by Plancherel

$$X_N(v) = -i \sum_{|n|\leq N} n \frac{|g_n(\omega)|^2}{\langle n\rangle^2} \quad \text{and} \quad X_M(v) - X_N(v) = -i \sum_{N\leq|n|\leq M} n \frac{|g_n(\omega)|^2}{\langle n\rangle^2},$$

and

$$|X_M(v) - X_N(v)|^2 = \sum_{N\leq|n_1|,|n_2|\leq M} n_1 n_2 \frac{|g_{n_1}(\omega)|^2 |g_{n_2}(\omega)|^2}{\langle n_1\rangle^2 \langle n_2\rangle^2} =: Y^1_{N,M} + Y^2_{N,M} + Y^3_{N,M},$$

$$\begin{aligned}
Y^1_{N,M} &:= \sum_{N\leq|n_2|,|n_1|\leq M} n_1 n_2 \frac{(|g_{n_1}(\omega)|^2 - 1)(|g_{n_2}(\omega)|^2 - 1)}{\langle n_1\rangle^2 \langle n_2\rangle^2} \\
Y^2_{N,M} &:= \sum_{N\leq|n_2|,|n_1|\leq M} n_1 n_2 \frac{(|g_{n_1}(\omega)|^2 - 1) + (|g_{n_2}(\omega)|^2 - 1)}{\langle n_1\rangle^2 \langle n_2\rangle^2} \\
Y^3_{N,M} &:= \sum_{N\leq|n_2|,|n_1|\leq M} \frac{n_1 n_2}{\langle n_1\rangle^2 \langle n_2\rangle^2}.
\end{aligned}$$

By symmetry

$$Y^3_{N,M} = \sum_{N\leq|n_2|,|n_1|\leq M} \frac{n_1 n_2}{\langle n_1\rangle^2\langle n_2\rangle^2} = 0,$$

hence

$$\|X_M(v) - X_N(v)\|^4_{L^4} \lesssim \|Y^1_{N,M}\|^2_{L^2} + \|Y^2_{N,M}\|^2_{L^2}.$$

We now proceed as in **Thomann** and **Tzvetkov**: denote by

$$G_n(\omega) := |g_n(\omega)|^2 - 1$$

and note that by the definition of $g_n(\omega)$

$$\mathbb{E}[G_n(\omega)G_m(\omega)] = 0 \quad \text{for } n \neq m.$$

Since

$$|Y^1_{N,M}|^2 = \sum_{N\leq|n_1|,|n_2|,|n_3|,|n_4|,\leq M} n_1 n_2 n_3 n_4 \frac{G_{n_1} G_{n_2} G_{n_3} G_{n_4}}{\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2\langle n_4\rangle^2},$$

when we compute $\mathbb{E}[|Y^1_{N,M}|^2]$ the only contributions come from ($n_1 = n_3$ and $n_2 = n_4$), ($n_1 = n_2$ and $n_3 = n_4$) or ($n_2 = n_3$ and $n_1 = n_4$).

Hence by symmetry we have:

$$\|Y^1_{N,M}\|^2_{L^2} = E[|Y^1_{N,M}|^2] \leq C \sum_{N\leq |n_1|,|n_2|\leq M} \frac{n_1^2 n_2^2}{\langle n_1\rangle^4 \langle n_2\rangle^4} \lesssim \frac{1}{N^2}.$$

On the other hand, since

$$|Y^2_{N,M}|^2 = \sum_{N\leq |n_1|,|n_2|,|n_3|,|n_4|,\leq M} n_1 n_2 n_3 n_4 \frac{(G_{n_1}+G_{n_2})(G_{n_3}+G_{n_4})}{\langle n_1\rangle^2 \langle n_2\rangle^2 \langle n_3\rangle^2 \langle n_4\rangle^2},$$

by symmetry it is enough to consider a single term of the form

$$\sum_{N\leq |n_1|,|n_2|,|n_3|,|n_4|,\leq M} n_1 n_2 n_3 n_4 \frac{G_{n_j} G_{n_k}}{\langle n_1\rangle^2 \langle n_2\rangle^2 \langle n_3\rangle^2 \langle n_4\rangle^2},$$

with $1 \leq j \neq k \leq 4$, which we set without any loss of generality to be $j = 1,\ k = 3$. We then have

$$\|Y^2_{N,M}\|^2_{L^2} = E[|Y^2_{N,M}|^2] \leq C \sum_{N\leq |n_1|,|n_2|,|n_4|\leq M} \frac{n_1^2 n_2 n_4}{\langle n_1\rangle^4 \langle n_2\rangle^2 \langle n_4\rangle^2} = 0.$$

Analysis of the (FGDNLS): necessary estimates

The key step now is to prove the analogue of Bourgain's Main Proposition above controlling the growth of solutions v^N to (FGDNLS).

Obstacles we have to face:

- The symplectic form associated to the periodic gauged derivative nonlinear Schrödinger equation GDNLS does not commute with Fourier modes truncation and so the truncated finite-dimensional systems are not necessarily Hamiltonian. This entails two problems:

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  - (1) A mild one: need to show the invariance of Lebesgue measure associated to (FGDNLS) ('Liouville's theorem') by hand directly .

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  - (1) A mild one: need to show the invariance of Lebesgue measure associated to (FGDNLS) ('Liouville's theorem') by hand directly .
  - (2) A more serious one and at the heart of this work. The energy $\tilde{\mathcal{E}}(v^N)$ is **no** longer conserved. In other words, the finite dimensional weighted Wiener measure μ_N is **not** invariant any longer.

Almost conserved energy

- Zhidkov faced a similar problem but unlike Zhidkovs work on KdV we do not have a priori knowledge of global well posedness.

- We show however that it is *almost* invariant in the sense that we can control the growth in time of $\tilde{\mathcal{E}}(v^N)(t)$.

  - This idea is reminiscent of the I-method. **However**:

    In the I-method one needs to estimate the variation of the energy of *solutions to the infinite dimensional equation* at time t smoothly projected onto frequencies of size up to N.

    Here one needs to control the variation of the energy $\tilde{\mathcal{E}}$ of the solution v^N to the *finite dimensional approximation equation*.

More precisely we have the following estimate controlling the growth of $\tilde{\mathcal{E}}(v^N)(t)$

Theorem [Energy Growth Estimate]

Let $v^N(t)$ be a solution to (FGDNLS) in $[-\delta, \delta]$, and let $K > 0$ be such that $\|v^N\|_{X_3^{\frac{2}{3}-,\frac{1}{2}}(\delta)} \leq K$. Then there exists $\beta > 0$ such that

$$|\mathcal{E}(v^N(\delta)) - \mathcal{E}(v^N(0))| = \left| \int_0^\delta \frac{d}{dt}\mathcal{E}(v^N)(t)dt \right| \lesssim C(\delta)N^{-\beta} \max(K^6, K^8).$$

Remark This estimate may still hold for a different choice of $X_r^{s,\frac{1}{2}}(\delta)$ norm, with $s \geq \frac{1}{2},\ 2 < r < 4$ so that the local well-posedness holds.

On the other hand the pair (s, r) should also be such that $(s-1) \cdot r < -1$ since this regularity is low enough to contain the support of the Wiener measure.

Our choice of $s = \frac{2}{3}-$ and $r = 3$ allows us to prove the energy growth estimate while satisfying *both* the conditions for local well-posedness *and* the support of the measure.

On the energy estimate

We start by writing

$$\frac{d\tilde{\mathcal{E}}}{dt}(v^N) = -2\text{Im}\int v^N \overline{v^N v_x^N} P_N^\perp((v^N)^2\overline{v_x^N}) + \text{Re}\int v^N\overline{v^N v_x^N} P_N^\perp(|v^N|^4 v^N)$$
$$- 2m(v^N)\text{Re}\int v^N\overline{v^N v_x^N} P_N^\perp(|v^N|^2 v^N)$$
$$+ 2m(v^N)\text{Re}\int v^N\overline{v^N}^2 P_N^\perp((v^N)^2\overline{v_x^N})$$
$$+ m(v^N)\text{Im}\int v^N\overline{v^N}^2 P_N^\perp(|v^N|^4 v^N)$$
$$- 2m(v^N)^2\text{Im}\int v^N\overline{v^N}^2 P_N^\perp(|v^N|^2 v^N) + \ldots\ldots,$$

The first term is the worst term since it has two derivatives. Also it looks like the unfavorable structure of the nonlinearity $(v^N)^2\overline{v_x^N})$ is back!

The dangerous term

Let's now concentrate on the first term coming from the expression above. It is essentially:

$$I_1 = \int_0^{\delta} \int_{\mathbb{T}} v^N \overline{v^N v_x^N} P_N^{\perp}((v^N)^2 \overline{v_x^N})\, dx\, dt.$$

We start by discussing how to absorb the rough time cut-off. Assume ϕ is any function in $X_3^{\frac{2}{3}-,\frac{1}{2}}$ such that

$$\phi|_{[-\delta,\delta]} = v^N;$$

then we write

$$I_1 = \int_{\mathbb{T}\times\mathbb{R}} \chi_{[0,\delta]}(t)\ P_N^{\perp}((v^N)^2 \partial_x \overline{v^N})\ v^N \overline{v^N v_x^N} dxdt$$

$$= \int_{\mathbb{T}\times\mathbb{R}} P_N^{\perp}((\chi_{[0,\delta]}\phi)^2\, \chi_{[0,\delta]}\overline{\phi_x})\ \chi_{[0,\delta]}\phi\, \chi_{[0,\delta]}\overline{\phi}\, \chi_{[0,\delta]}\overline{\phi_x} dxdt.$$

By denoting

$$w := \chi_{[0,\delta]}\phi, \qquad \text{hence} \qquad w = P_N(w),$$

we will in fact show that

$$|I_1| = \left| \int_{\mathbb{T}\times\mathbb{R}} P_N^{\perp}((w)^2 \partial_x \overline{w})\, w\overline{w w_x} dxdt \right|$$

$$\leq C(\delta) N^{-\beta} \|w_1\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}} \|w_2\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}} \|w_3\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}$$

$$\times \|w_4\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}} \|w_5\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}} \|w_6\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}$$

where $w_1 = w_2 = w_4 = w$ and $\overline{w_3} = \overline{w_5} = \overline{w_6} = \overline{w}$.

To go back to v^N then one uses the fact that for $b < b_1 < 1/2$, there exists $C'(\delta) > 0$ such that

$$\|w\|_{X_3^{\frac{2}{3}-,b}} \leq C'(\delta)\, \|\phi\|_{X_3^{\frac{2}{3}-,b_1}} \leq C'(\delta)\, \|v^N\|_{X_3^{\frac{2}{3}-,\frac{1}{2}}(\delta)}$$

where w, ϕ and v^N are as above.

Ingredients for the proof

We now list some of the ingredients for the proof of the estimate:

- A trilinear refinement of Bourgain's $L^6(\mathbb{T})$ Strichartz estimate:

  Let $u, v, w \in X^{\epsilon,\frac{1}{2}-}$ for some $\epsilon > 0$. Then

  $$\|uv\overline{w}\|_{L^2_{xt}} \lesssim \|u\|_{X^{\epsilon,\frac{1}{2}-}} \|v\|_{X^{\epsilon,\frac{1}{2}-}} \|w\|_{X^{0,\frac{1}{2}-}}$$

- Certain arithmetic identities that relate frequencies to the distance to the parabola

  $$P = \{(n,\tau) \ : \ \tau = n^2\}$$

  where the solution of the linear problem lives.

  These estimates are important since one would like to trade derivatives, e.i. powers of frequencies like $|n|^\alpha$, with powers of $|\tau - n^2|$.

Ingredient for the proof

We notice that we can write

$$I_1 = \int_{\mathbb{T}\times\mathbb{R}} P_N^{\perp}(w^2\partial_x\overline{w})\, w\,\overline{ww_x}dxdt$$

$$= \int \sum_{|n|>N} \left(\int_{\tau=\tau_1+\tau_2-\tau_3} \sum_{n=n_1+n_2-n_3} \widehat{w}(n_1,\tau_1)\widehat{w}(n_2,\tau_2)(-in_3)\overline{\widehat{w}}(n_3,\tau_3)d\tau_1 d\tau_2 \right)$$

$$\times \left(\int_{-\tau=\tau_4-\tau_5-\tau_6} \sum_{-n=n_4-n_5-n_6} \widehat{w}(n_4,\tau_4)\overline{\widehat{w}}(n_5,\tau_5)(-in_6)\overline{\widehat{w}}(n_6,\tau_6)d\tau_4 d\tau_5 \right) d\tau$$

and from here one has

$$\tau - n^2 - (\tau_1 - n_1^2) - (\tau_2 - n_2^2) - (\tau_3 + n_3^2) = -2(n-n_1)(n-n_2),$$
$$\tau - n^2 + (\tau_4 - n_4^2) + (\tau_5 + n_5^2) + (\tau_6 + n_6^2) = -2(n+n_5)(n+n_6).$$

This is the kind of relationships that we want to exploit!

Ingredient for the proof

More precisely, if we let $\tilde{\sigma}_j := \tau_j \pm n_j^2$ we have

$$\sum_{j=1}^{6} \tilde{\sigma}_j = -2\,(\,n\,(n_1 + n_2 + n_5 + n_6) - n_1 n_2 + n_5 n_6\,)$$

This in turn can also be rewritten using $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = 0$ or $n = n_1 + n_2 + n_3$ and $-n = n_4 + n_5 + n_6$ as:

$$\sum_{j=1}^{6} \tilde{\sigma}_j = 2(\,n\,(n_3 + n_4) + n_1 n_2 - n_5 n_6\,).$$

In addition, since $\tau_1 + \tau_2 + \tau_3 + \tau_4 + \tau_5 + \tau_6 = 0$, adding and subtracting $n_j^2,\ j = 1, \ldots, 6$ in the appropriate fashion, we obtain:

$$\sum_{j=1}^{6} \tilde{\sigma}_j = (n_3^2 + n_5^2 + n_6^2) - (n_1^2 + n_2^2 + n_4^2)$$

We now write

$$|I_1| = \Big| \sum_{N_i \le N;\, i=1,\dots 6} \int_{\mathbb{R}} \int_{\mathbb{T}} P_N^{\perp} \left(w_{N_1}\, w_{N_2}\, \partial_x \overline{w_{N_3}} \right) w_{N_4}\, \overline{w_{N_5}}\, \partial_x \overline{w_{N_6}}\, dxdt \Big|$$

$$= \Big| \sum_{N_i \le N;\, i=1,\dots 6} \sum_{|n| \ge N} \int_\tau \Big(\int_{\tau = \sum_{i=1}^3 \tau_i} \sum_{n=\sum_{i=1}^3 n_i} \widehat{w_{N_1}}\, \widehat{w_{N_2}}\, (in_3)\, \widehat{\overline{w_{N_3}}}\; d\tau_1 d\tau_2 \Big) \quad \times$$

$$\Big(\int_{-\tau = \sum_{j=4}^6 \tau_j} \sum_{-n=\sum_{j=4}^6 n_j} \widehat{w_{N_4}} \widehat{\overline{w_{N_5}}}\, (in_6)\, \widehat{\overline{w_{N_6}}}\; d\tau_4 d\tau_5 \Big)\, d\tau \Big|$$

$$\le \sum_{N \le |n| \le 3N} \sum_{N_i \le N;\, i=1,\dots 6} \int_\tau \Big(\int_{\tau = \sum_{i=1}^3 \tau_i} \sum_{n=\sum_{i=1}^3 n_i} |\widehat{w_{N_1}}|\, |\widehat{w_{N_2}}|\, |n_3|\, |\widehat{\overline{w_{N_3}}}|\; d\tau_1 d\tau_2 \Big)$$

$$\Big(\int_{-\tau = \sum_{j=4}^6 \tau_j} \sum_{-n=\sum_{j=4}^6 n_j} |\widehat{w_{N_4}}|\, |\widehat{\overline{w_{N_5}}}|\, |n_6|\, |\widehat{\overline{w_{N_6}}}|\; d\tau_4 d\tau_5 \Big)\, d\tau.$$

Above we always think of N_j, N as dyadic; more precisely $N_j := 2^{K_j}$, $N := 2^K$ where $K_j < K$. Moreover we denote by w_{N_j} the function such that $\widehat{w_{N_j}}(n_j) = \chi_{\{|n_j|\sim N_j\}}\widehat{w}_j(n_j)$.

From the expression above we then have,

$$|n_j| \leq N, \qquad N \leq |n| \leq 3N, \qquad n = n_1 + n_2 + n_3, \quad \text{and} \quad -n = n_4 + n_5 + n_6,$$

$$N \sim \max(N_1, N_2, N_3) \sim \max(N_4, N_5, N_6),$$

We start by laying out all possible cases and organizing them according to the sizes of the two derivative terms.

Types:

I. $N_3 \sim N$, $N_6 \sim N$

II. $N_3 \sim N$ and $N_6 \ll N$

III. $N_6 \sim N$ and $N_3 \ll N$

IV. $N_3 \ll N$; $N_6 \ll N$

Now we subdivide into all subcases in each situation and group them according to how many low frequencies (ie. $N_j \ll N$) we have overall.

All Cases for each type:

IA. $N_3 \sim N,\ N_6 \sim N$ and 4 lows: $N_1, N_2, N_4, N_5 \ll N$

IB. $N_3 \sim N,\ N_6 \sim N$ and 3 lows

(i) $N_1, N_2, N_4 \ll N$ and $N_5 \sim N$

(ii) $N_1, N_2, N_5 \ll N$ and $N_4 \sim N$

(iii) $N_1, N_4, N_5 \ll N$ and $N_2 \sim N$

(iv) $N_2, N_4, N_5 \ll N$ and $N_1 \sim N$

IC. $N_3 \sim N,\ N_6 \sim N$ and 2 lows

(i) $N_1, N_2 \ll N$ and $N_4, N_5 \sim N$

(ii) $N_1, N_4 \ll N$ and $N_2, N_5 \sim N$

(iii) $N_1, N_5 \ll N$ and $N_2, N_4 \sim N$

(iv) $N_2, N_4 \ll N$ and $N_1, N_5 \sim N$

(v) $N_2, N_5 \ll N$ and $N_1, N_4 \sim N$

(vi) $N_4, N_5 \ll N$ and $N_1, N_2 \sim N$

ID. $N_3 \sim N,\ N_6 \sim N$ and 1 low

(i) $N_1 \ll N$ and $N_2, N_4, N_5 \sim N$

(ii) $N_2 \ll N$ and $N_1, N_4, N_5 \sim N$

(iii) $N_4 \ll N$ and $N_1, N_2, N_5 \sim N$

(iv) $N_5 \ll N$ and $N_1, N_2, N_4 \sim N$

IE. $N_3 \sim N,\ N_6 \sim N$ and $N_1, N_2, N_4, N_5 \sim N$

IIA. $N_3 \sim N$ and $N_6 \ll N$ and 3 lows

(i) $N_1, N_2, N_4 \ll N$ and $N_5 \sim N$

(ii) $N_1, N_2, N_5 \ll N$ and $N_4 \sim N$

IIB. $N_3 \sim N$ and $N_6 \ll N$ and 2 lows

(i) $N_1, N_2 \ll N$ and $N_4, N_5 \sim N$

(ii) $N_1, N_4 \ll N$ and $N_2, N_5 \sim N$

(iii) $N_1, N_5 \ll N$ and $N_2, N_4 \sim N$

(iv) $N_2, N_4 \ll N$ and $N_1, N_5 \sim N$

(v) $N_2, N_5 \ll N$ and $N_1, N_4 \sim N$

IIC. $N_3 \sim N$ and $N_6 \ll N$ and 1 low

(i) $N_1 \ll N$ and $N_2, N_4, N_5 \sim N$

(ii) $N_2 \ll N$ and $N_1, N_4, N_5 \sim N$

(iii) $N_4 \ll N$ and $N_1, N_2, N_5 \sim N$

(iv) $N_5 \ll N$ and $N_1, N_2, N_4 \sim N$

IID. $N_3 \sim N$ and $N_6 \ll N$ and $N_1, N_2, N_4, N_5 \sim N$

IIIA. $N_6 \sim N$ and $N_3 \ll N$ and 3 lows

(i) $N_2, N_4, N_5 \ll N$ and $N_1 \sim N$

(ii) $N_1, N_4, N_5 \ll N$ and $N_2 \sim N$

IIIB. $N_6 \sim N$ and $N_3 \ll N$ and 2 lows

(i) $N_4, N_5 \ll N$ and $N_1, N_2 \sim N$

(ii) $N_1, N_4 \ll N$ and $N_2, N_5 \sim N$

(iii) $N_1, N_5 \ll N$ and $N_2, N_4 \sim N$

(iv) $N_2, N_4 \ll N$ and $N_1, N_5 \sim N$

(v) $N_2, N_5 \ll N$ and $N_1, N_4 \sim N$

IIIC. $N_6 \sim N$ and $N_3 \ll N$ and 1 low

(i) $N_1 \ll N$ and $N_2, N_4, N_5 \sim N$

(ii) $N_2 \ll N$ and $N_1, N_4, N_5 \sim N$

(iii) $N_4 \ll N$ and $N_1, N_2, N_5 \sim N$

(iv) $N_5 \ll N$ and $N_1, N_2, N_4 \sim N$

IIID. $N_6 \sim N$ and $N_3 \ll N$ and $N_1, N_2, N_4, N_5 \sim N$

IVA. $N_3 \ll N, N_6 \ll N$ and 2 lows

(i) $N_1, N_4 \ll N$ and $N_2, N_5 \sim N$

(ii) $N_1, N_5 \ll N$ and $N_2, N_4 \sim N$

(iii) $N_2, N_4 \ll N$ and $N_1, N_5 \sim N$

(iv) $N_2, N_5 \ll N$ and $N_1, N_4 \sim N$

IVB. $N_3 \ll N, N_6 \ll N$ and 1 low

(i) $N_1 \ll N$ and $N_2, N_4, N_5 \sim N$

(ii) $N_2 \ll N$ and $N_1, N_4, N_5 \sim N$

(iii) $N_4 \ll N$ and $N_1, N_4, N_5 \sim N$

(iv) $N_5 \ll N$ and $N_1, N_2, N_4 \sim N$

IVC. $N_3 \ll N, N_6 \ll N$ and $N_1, N_2, N_4, N_5 \sim N$

Ingredient of the proof

- It is fundamental the following lemma

Lemma

If $0 < \beta < 2$*, then*

$$\|J_\beta w_M\|_{X^{0,\rho}} \lesssim C_T\, A(\beta, M)^{\frac{1}{6}} M^{\rho\beta+} \|w_M\|_{X_3^{0,\frac{1}{6}}},$$

where

$$\begin{aligned} &(i) \quad \operatorname{supp} w_M(\cdot, x) \subset [-T, T] \qquad (x \in \mathbb{T}). \\ &(ii) \quad \widehat{J_\beta w_M}(\tau, n) = \chi_{\{|n|\sim M\}} \chi_{\{|\tau - n^2| \le M^\beta\}} |\widehat{w_M}(\tau, n)|. \end{aligned}$$

Here, if

$$S(\tau, M, \beta) := \{n \in \mathbb{Z} \, : \, |n| \sim M \text{ and } |\tau - n^2| \le M^\beta\}$$

and $|S|$ represents the counting measure of the set S, then one can show that

$$A(M, \beta) := \sup_\tau |S(\tau, M, \beta)| \le 1 + M^{\beta - 1}.$$

The estimate of $A(M,\beta)$

If $S := S(\tau, M, \beta) \neq \emptyset$, then there exists $n_0 \in S$ and hence

$$
\begin{aligned}
|S| &\leq 1 + |\{l \in \mathbb{Z} / |n_0 + l| \sim M, \; |\tau - (n_0 + l)^2| \leq M^\beta\}| \\
&\leq 1 + |\{l \in \mathbb{Z} \, / \, |l| \leq M, \; |2n_0 l + l^2| \lesssim M^\beta\}|.
\end{aligned}
$$

Now we note that $|2n_0 l + l^2| = |(l + n_0)^2 - n_0^2| \lesssim M^\beta$ if and only if

$$-CM^\beta + n_0^2 \leq (l + n_0)^2 \leq n_0^2 + CM^\beta.$$

Hence we need $|\, l \,| \leq M$ to satisfy

$$
\begin{gathered}
-\sqrt{n_0^2 + CM^\beta} \leq (l + n_0) \leq \sqrt{n_0^2 + CM^\beta}, \\
(l + n_0) \geq \sqrt{n_0^2 - CM^\beta} \quad \text{and} \quad (l + n_0) \leq -\sqrt{n_0^2 - CM^\beta}.
\end{gathered}
$$

In other words we need to know the size of

$$[-\sqrt{n_0^2 + CM^\beta}, -\sqrt{n_0^2 - CM^\beta}] \cup [\sqrt{n_0^2 - CM^\beta}, \sqrt{n_0^2 + CM^\beta}]$$

which is of the order of $\frac{M^\beta}{|n_0|}$. Hence since $|n_0| \sim M$, we have that $|S| \leq 1 + M^{\beta - 1}$ as claimed.

Growth of solutions to (FGDNLS)

Armed with the Energy Growth Estimate we count on the almost invariance of the finite-dimensional measure μ_N under the flow of (FGDNLS) to control the growth of its solutions (our analogue of Bourgain's Main Proposition)

Proposition [Growth of solutions to FGDNLS]

For any given $T > 0$ and $\varepsilon > 0$ there exists an integer $N_0 = N_0(T, \varepsilon)$ and sets $\widetilde{\Omega}_N = \widetilde{\Omega}_N(\varepsilon, T) \subset \mathbb{R}^{2N+2}$ such that for $N > N_0$

(a) $\mu_N\left(\widetilde{\Omega}_N\right) \geq 1 - \varepsilon$.

(b) For any initial condition $v_0^N \in \widetilde{\Omega}_N$, (FGDNLS) is well-posed on $[-T, T]$ and its solution $v^N(t)$ satisfies the bound

$$\sup_{|t| \leq T} \|v^N(t)\|_{\mathcal{F}L^{\frac{2}{3}-,3}} \lesssim \left(\log \frac{T}{\varepsilon}\right)^{\frac{1}{2}} .$$

A.S GWP of solution to (GDNLS)

Combining the Approximation Lemma of v by v^N with the previous Proposition on the growth of solutions to (FGDNLS) we can prove a similar result for solutions v to (GDNLS):

Proposition ['Almost almost ' sure GWP for (GDNLS)]

For any given $T > 0$ and $\varepsilon > 0$ there exists a set $\Omega(\varepsilon, T)$ such that

(a) $\mu\left(\Omega(\varepsilon, T)\right) \geq 1 - \varepsilon$.

(b) For any initial condition $v_0 \in \Omega(\varepsilon, T)$ the IVP (GDNLS) is well-posed on $[-T, T]$ with the bound

$$\sup_{|t|\leq T} \|v(t)\|_{\mathcal{F}L^{\frac{2}{3}-,3}} \lesssim \left(\log \frac{T}{\varepsilon}\right)^{\frac{1}{2}} .$$

All in all we now have:

Theorem 1 [Almost sure global well-posedness of (GDNLS)]

There exists a set Ω, $\mu(\Omega^c) = 0$ such that for every $v_0 \in \Omega$ the IVP (GDNLS) with initial data v_0 is globally well-posed.

Theorem 2 [Invariance of μ]

The measure μ is invariant under the flow $\Phi(t)$ of (GDNLS)

Finally: The last step is going back to the ungauged (DNLS) equation. By pulling back the gauge, it follows easily from Theorems 1 and 2 that we have:

The ungauged DNLS equation

Theorem 3 [Almost sure global well-posedness of (DNLS)]

There exists a subset Σ of the space $\mathcal{F}L^{\frac{2}{3}-,3}$ with $\nu(\Sigma^c) = 0$ such that for every $u_0 \in \Sigma$ the IVP (DNLS) with initial data u_0 is globally well-posed.

Recall that for μ is a measure on Ω and $\mathcal{G}^{-1} : \Omega \to \Omega$ measurable, the measure $\nu = \mu \circ \mathcal{G}$ is defined

$$\nu(A) \,:=\, \mu(\mathcal{G}(A)) \,=\, \mu(\{v \,:\, \mathcal{G}^{-1}(v) \in A\})\,.$$

for all measurable sets A or equivalently - for integrable F- by

$$\int F d\nu = \int F \circ \varphi \, d\mu$$

Finally we show that the measure ν is invariant under the flow map of DNLS.

Theorem 4 [Invariance of measure under (DNLS) flow]

The measure $\nu = \mu \circ \mathcal{G}$ is invariant under the (DNLS) flow.

Our Goal 2:

What is $\nu = \mu \circ \mathcal{G}$ really? Is this absolutely continuous with respect to the measure that can be naturally constructed for DNLS by using its energy E,

$$\begin{aligned} E(u) &= \int_{\mathbb{T}} |u_x|^2\, dx + \frac{3}{2}\mathrm{Im} \int_{\mathbb{T}} u^2 \overline{u u_x}\, dx + \frac{1}{2} \int_{\mathbb{T}} |u|^6\, dx \\ &=: \int_{\mathbb{T}} |u_x|^2\, dx + \mathcal{K}(u) \end{aligned}$$

as done by Thomann-Tzevtkov?

We know ν is invariant and that the ungauged (DNLS) equation is GWP a.s with respect to ν. Treating the weight is easy. The problem is ungauging the Gaussian measure ρ.

Question: What is $\tilde{\rho} := \rho \circ \mathcal{G}$? Is (its restriction to a sufficiently small ball in L^2) absolutely continuous with respect to ρ? If so, what is its Radon-Nikodym derivative?

We would like to compute $\tilde{\rho}$ explicitly.

The ungauged measure: absolute continuity

In order to finish this step one should stop thinking about the solution v as a infinite dimension vector of Fourier modes and start thinking instead about v as a (periodic) complex Brownian path in $\mathbb{T}$ (Brownian bridge) solving a certain stochastic process.

We recall that to ungauge we need to define

$$\mathcal{G}^{-1}(v)(x) := \exp(iJ(v))\, v(x)$$

where

$$J(v)(x) := \frac{1}{2\pi}\int_0^{2\pi}\int_\theta^x |v(y)|^2 - \frac{1}{2\pi}\|v\|^2_{L^2(\mathbb{T})}\, dy\, d\theta$$

It will be important later that $J(v)(x) = J(|v|)(x)$. Then, if v satisfies

$$dv(x) = \underbrace{dB(x)}_{\text{Brownian motion}} + \underbrace{b(x)dx}_{\text{drift terms}}$$

by Ito's calculus and since $\exp(iJ(v))$ is differentiable we have:

$$d\mathcal{G}^{-1}v(x) = \exp(iJ(v))\, dv + iv\exp(iJ(v))\left(|v(x)|^2 - \frac{1}{2\pi}\|v\|^2_{L^2}\right)dx + \dots$$

What one may think it saves the day...

Substituting above one has

$$d\mathcal{G}^{-1}v(x) = \exp(iJ(v))\,[dB(x) + a(v,x,\omega))\,dx] + \dots$$

where

$$a(v,x,\omega) = iv\ \left(|v(x)|^2 - \frac{1}{2\pi}\|v\|_{L^2}^2\right).$$

What could help?

- The fact that $\exp(iJ(v))$ is a unitary operator
- The fact that one can prove Novikov's condition:

$$E\left[\exp\left(\frac{1}{2}\int a^2(v,x,\omega)dx\right)\right] < \infty.$$

In fact this last condition looks exactly like what we need for the following theorem:

"Theorem" [Girsanov]

If we change the drift coefficient of a given Ito process in an appropriate way, then the law of the process will not change dramatically. In fact the new process law will be absolutely continuous with respect to the law of the original process and we can compute explicitly the Radon-Nikodym derivative.

Why Girsanov's theorem doesn't save the day

If one reads the theorem carefully one realizes that an important condition is that $a(v, x, \omega)$ is *non anticipative*; in the sense that it only depends on the BM v up to "time" x and not further. This unfortunately is not true in our case! The new drift term $a(v, x, \omega)$ involves the L^2 norm of $v(x)$ (periodic case!) and hence it is *anticipative*. A different strategy is needed ...

Conformal invariance of complex BM comes to the rescue!

We use the well known fact that if $W(t) = W_1(t) + iW_2(t)$ is a complex Brownian motion, and if ϕ is an analytic function then $Z = \phi(W)$ is, after a suitable time change, again a complex Brownian motion.

(In what follows one should think of $Z(t)$ to play the role of our complex BM $v(x)$)

For $Z(t) = \exp(W(s))$ the time change is given by

$$t = t(s) = \int_0^s |e^{W(r)}|^2 dr, \qquad \frac{dt}{ds} = |e^{W(s)}|^2,$$

equivalently

$$s(t) = \int_0^t \frac{dr}{|Z(r)|^2}, \qquad \frac{ds}{dt} = \frac{1}{|Z(t)|^2}.$$

We are interested in $Z(t)$ for the interval $0 \leq t \leq 1$ and thus we introduce the stopping time

$$\mathcal{S} = \inf \left\{ s\,;\, \int_0^s |e^{W(r)}|^2 dr = 1 \right\}$$

and remark the important fact that the stopping time $\mathcal{S}$ depends only on the real part $W_1(s)$ of $W(s)$ (or equivalently only $|Z|$). If we write $Z(t)$ in polar coordinate $Z(t) = |Z(t)|e^{i\Theta(t)}$ we have

$$W(s) = W_1(s) + iW_2(s) = \log |Z(t(s)| + i\Theta(t(s))$$

and W_1 and W_2 are real independent Brownian motions.

If we define

$$\tilde{W}(s) := W_1(s) + i\left[W_2(s) + \int_0^{t(s)} h(|Z|)(r)dr\right]$$
$$= W_1(s) + i\left[W_2(s) + \int_0^{t(s)} h(e^{W_1})(r)dr,\right]$$

(in our case, essentially $h(|Z|)(\cdot) = |Z(\cdot)|^2 - \|Z\|_{L^2}^2)$

we then have

$$e^{\tilde{W}(s)} = \tilde{Z}(t(s)) = \mathcal{G}^{-1}(Z)(t(s)).$$

- In terms of W, the gauge transformation is now easy to understand: it gives a complex process such that:
  - The real part is left unchanged.
  - The imaginary part is translated by the function $J(Z)(t(s))$ which depends only on the real part (ie. on $|Z|$, which has been fixed) and in that sense is deterministic.
  - It is now possible to use Cameron-Martin-Girsanov's theorem only for the law of the imaginary part and conclude the proof.

Conclusion

Then if η denotes the probability distribution of W and $\tilde{\eta}$ the distribution of $\tilde{W}$ we have the absolute continuity of $\tilde{\eta}$ and η whence the absolute continuity between $\tilde{\rho}$ and ρ follows with the *same Radon-Nikodym derivative* (re-expressed back in terms of t).

All in all then we prove that our ungauged measure ν is in fact essentially (up to normalizing constants) of the form

$$d\nu(u) = \chi_{\|u\|_{L^2} \leq B} e^{-\mathcal{K}(u)} d\rho,$$

the weighted Wiener measure associated to DNLS (constructed by Thomann-Tzvetkov). In particular we prove its invariance.

- The above needs to be done carefully for complex Brownian bridges (periodic BM) by conditioning properly.

Conclusion

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- The above needs to be done carefully for complex Brownian bridges (periodic BM) by conditioning properly.
  - $W(s)$ is a BM conditioned to end up at the same place when the total variation time $t = t(s)$ reaches 2π. The time when this occurs is our $\mathcal{S}$.
  - Conditioned on $\mathrm{Re}W$ we have that $\mathrm{Im}W$ is just a regular real-valued BM conditioned to end at the same place (up to multiple of 2π) where it started at time $\mathcal{S}$
  - Conditioned on $\mathrm{Re}W$ and the total winding (multiple of 2π above) $\mathrm{Im}W$ is regular real-valued BM bridge on $[0, \mathcal{S}]$.