Asymptotic Behavior of Massless Dirac Waves in Schwarzschild Geometry

Chunjing XIE

University of Michigan

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Schwarzschild Geometry and Regge-Wheeler Coordinates

The vacuum Einstein equations $R_{ij} = 0$ describe the evolution of spacetime with no sources, where R_{ij} is the Ricci tensor of the Lorentzian metric g_{ij}. The Schwarzschild solution, a static and spherically symmetric black hole, can be described as

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

where the coordinates (t, r, θ, φ) are in the range

$$-\infty < t < \infty,\ 2M < r < \infty,\ 0 < \theta < \pi,\ 0 < \varphi < 2\pi.$$

The Regge-Wheeler coordinate is defined as

$$x = r + 2M \ln(r - 2M).$$

The range of x is the whole real line.

Massless Dirac Equations in the Schwarzschild Geometry

The massless Dirac equations in the Schwarzschild geometry are

$$i\frac{\partial}{\partial t}\Psi = H\Psi,$$

where

$$H = \begin{pmatrix} -\mathcal{E} & 0 & 0 & 0 \\ 0 & \mathcal{E} & 0 & 0 \\ 0 & 0 & \mathcal{E} & 0 \\ 0 & 0 & 0 & -\mathcal{E} \end{pmatrix} + \begin{pmatrix} 0 & -\mathcal{M}_+ & 0 & 0 \\ -\mathcal{M}_- & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{M}_+ \\ 0 & 0 & \mathcal{M}_- & 0 \end{pmatrix},$$

and

$$\mathcal{E} = i\frac{\partial}{\partial x}, \quad \mathcal{M}_\pm = \frac{\sqrt{\Delta}}{r^2}\left(i\frac{\partial}{\partial\theta} + i\frac{\cot\theta}{2} \pm \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right), \quad \Delta = r^2 - 2Mr.$$

Separation of Variables

Let

$$\Psi_1 = R_-(t,x)Y_-(\theta,\varphi),\ \Psi_2 = R_+(t,x)Y_+(\theta,\varphi),$$
$$\Psi_3 = R_+(t,x)Y_-(\theta,\varphi),\ \Psi_4 = R_-(t,x)Y_+(\theta,\varphi).$$

Then $Y_\pm(\theta,\varphi)$ satisfy

$$\left(\frac{\partial}{\partial\theta} + \frac{\cot\theta}{2} \mp i\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right) Y_\pm = \mp\lambda Y_\mp, \qquad (1)$$

and thus Y_+ and Y_- are spin weighted spherical harmonics $_{-1/2}Y_{jm}$ and $_{-1/2}Y_{jm}$, respectively, where $j = \frac{1}{2}, \frac{3}{2}, \cdots$. The eigenvalues λ in (1) are of the form

$$\lambda = j + \frac{1}{2} = 1, 2, 3, \cdots$$

Decay Rate

The radial functions R_- and R_+ solve the following first order equations depending on λ

$$(\frac{\partial}{\partial t}+\frac{\partial}{\partial x})R_{-,\lambda}=\lambda\frac{\sqrt{\Delta}}{r^2}R_{+,\lambda},\ (\frac{\partial}{\partial t}-\frac{\partial}{\partial x})R_{+,\lambda}=-\lambda\frac{\sqrt{\Delta}}{r^2}R_{-,\lambda}. \quad (2)$$

Theorem As $t\to\infty$, any solution $R_{\pm,\lambda}$ of the Cauchy problem for (2) with initial date

$$R_{-,\lambda}(0,x)=g_{-,\lambda}(x),\quad R_{+,\lambda}(0,x)=g_{+,\lambda}(x) \quad (3)$$

satisfies

$$\begin{aligned}&\|\langle\cdot\rangle^{-3\lambda-1}R_{\pm,\lambda}(t,\cdot)\|_{L^\infty(\mathbb{R})}\\ \leq & Ct^{-2\lambda}\left\|\langle\cdot\rangle^{3\lambda+1}\sum_{i=0}^{1}\left(\left|\frac{d^i g_{+,\lambda}}{dx^i}\right|+\left|\frac{d^i g_{-,\lambda}}{dx^i}\right|\right)(\cdot)\right\|_{L^1(\mathbb{R})},\end{aligned}$$

where we always denote $\langle x\rangle=(1+|x|^2)^{1/2}$.

Known Results on Linear Waves in Black Geometry

- Decay of massive Dirac particles in Kerr-Newman geometry (Finster, Kamran, Smoller, Yau)
- Decay rate of scalar waves in Schwarzschild geometry (Dafermos & Rodnianski, Tataru, Donninger, Schlag, & Soffer, Luk, $\cdots$)
- Decay of scalar waves in Kerr geometry (Finster et al, Andersson & Blue, Dafermos & Rodnianski, Tataru, $\cdots$)
- Decay of electromagnetic waves in Schwarzschild geometry (Blue, $\cdots$)
- Decay of gravitational waves in Schwarschild geometry (Holzegel, $\cdots$)

Wave Equations

Define

$$Z_{+,\lambda} = R_{-,\lambda} + R_{+,\lambda}, \;\; Z_{-,\lambda} = R_{-,\lambda} - R_{+,\lambda}. \tag{4}$$

Then

$$\frac{\partial^2 Z_{\pm,\lambda}}{\partial t^2} - \frac{\partial^2 Z_{\pm,\lambda}}{\partial x^2} = V_{\pm,\lambda} Z_{\pm,\lambda}, \tag{5}$$

where

$$V_{\pm,\lambda}(x) = \pm \frac{\partial}{\partial x}\left(\frac{\sqrt{r(r-2M)}}{r^2}\lambda \right) + \frac{r-2M}{r^3}\lambda^2$$

and $\lambda = 1, 2, \cdots$.

Our first goal is to study asymptotic behavior for solutions of (5) with the initial data

$$Z_{\pm,\lambda}(0,x) = u_{\pm}(x), \quad \partial_t Z_{\pm,\lambda}(0,x) = v_{\pm}(x). \tag{6}$$

Asymptotic Behavior

Theorem As $t \to \infty$, the solution of (5) and (6) satisfies

- for any integer α in the range $1 \le \alpha \le 2\lambda + 1$,

$$\begin{aligned}&\|\langle\cdot\rangle^{-\alpha-\lambda} Z_{+,\lambda}(t,\cdot)\|_{L^\infty(\mathbb{R})}\\ \le & Ct^{-\alpha+1}\|\langle\cdot\rangle^{\alpha+\lambda}(|u_{+,\lambda}| + |u'_{+,\lambda}| + |v_{+,\lambda}|)(\cdot)\|_{L^1(\mathbb{R})}.\end{aligned}$$

- $Z_{-,\lambda}$ satisfies

$$\begin{aligned}&\|\langle\cdot\rangle^{-3\lambda-1}(Z_{-,\lambda}(t,\cdot) - D\hat{\phi}_{-,\lambda,1}(\cdot))\|_{L^\infty(\mathbb{R})}\\ \le & Ct^{-2\lambda}\left\|\langle\cdot\rangle^{3\lambda+1}\left(\sum_{i=0}^{1}\left|\frac{d^i u_{-,\lambda}}{dx^i}\right| + |v_{-,\lambda}|\right)(\cdot)\right\|_{L^1(\mathbb{R})},\end{aligned}$$

where

$$\hat{\phi}_{-,\lambda,1}(x) = e^{-\lambda\int_{-\infty}^{x}\frac{\sqrt{\Delta(x)}}{r^2(x)}dx}, \quad \text{and} \quad D = \int_{\mathbb{R}}\hat{\phi}_{-,\lambda,1}(x)v_{-,\lambda}(x)dx.$$

Proof of Decay Rate for Dirac Waves

For the Dirac equations, we can compute D explicitly

$$\begin{aligned} D &= \int_{\mathbb{R}} \left(\frac{\sqrt{\Delta}}{r^2}(g_{+,\lambda} + g_{-,\lambda}) - \frac{\partial}{\partial x}(g_{-,\lambda} + g_{+,\lambda}) \right) \hat{\phi}_{-,\lambda,1} dx \\ &= \int_{\mathbb{R}} \left(\frac{\sqrt{\Delta}}{r^2} + \frac{\partial}{\partial x} \right) \hat{\phi}_{-,\lambda,1}(g_{+,\lambda} + g_{-,\lambda}) dx \\ &= 0, \end{aligned} \tag{7}$$

where we use $\left(\frac{\sqrt{\Delta}}{r^2} + \frac{\partial}{\partial x} \right) \hat{\phi}_{-,\lambda,1} = 0$.

Schrödinger Operators and Their Potentials

Proposition *The asymptotic behavior of $V_{\pm,\lambda}$ is given by*

$$V_{\pm,\lambda} = O(\frac{\sqrt{2}\lambda}{16M^3} e^{1/4} e^{\frac{x}{4M}}), \quad \text{as } x \to -\infty,$$
$$V_{\pm,\lambda} = \frac{\lambda(\lambda \mp 1)}{x^2}(1 + O(x^{-1+\varepsilon})), \quad \text{as } x \to \infty, \tag{8}$$

where the $O-$terms are of the symbol type. In addition, we have

$$V_{+,1} = -\frac{M}{x^3}(1 + O(x^{-1+\varepsilon})), \tag{9}$$

The operator

$$\mathscr{H}_{\pm,\lambda} := -\frac{d^2}{dx^2} + V_{\pm,\lambda}, \tag{10}$$

is a self adjoint operator on $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$. Moreover, $\mathscr{H}_{\pm,\lambda}$ is a compact perturbation of the self-adjoint operator $-\Delta$. Therefore,

$$\sigma_{ess}(\mathscr{H}_{\pm,\lambda}) = \sigma_{ac}(\mathscr{H}_{\pm,\lambda}) = [0, \infty). \tag{11}$$

Spectrum of Schördinger Operators

- $V_{+,\lambda} > 0$ for $r \in (2M, \infty)$ when $\lambda \in \mathbb{N}$. Thus $\mathscr{H}_{+,\lambda}$ has no eigenvalues.
- Claim: $\mathscr{H}_{-,\lambda}$ have no eigenvalues.

  Proof: If Z_- is an eigenfunction of $\mathscr{H}_{-,\lambda}$ associated with an eigenvalue $\sigma < 0$. Then $Z = \lambda^2 \frac{\sqrt{\Delta}}{r^2} Z_- + \lambda Z'_-$ satisfies

$$\mathscr{H}_{+,\lambda} Z = \sigma Z. \tag{12}$$

  This implies that $\mathscr{H}_{+,\lambda}$ has a negative eigenvalue. Contradiction!
- $\sigma(\mathscr{H}_{-,\lambda}) = \sigma(\mathscr{H}_{+,\lambda}) = [0, \infty)$.

Spectral Representation

The solutions of (5) and (6) can be represented by

$$Z_{\pm,\lambda}(t) = \cos(t\sqrt{\mathscr{H}_{\pm,\lambda}})u_{\pm} + \frac{\sin(t\sqrt{\mathscr{H}_{\pm,\lambda}})}{\sqrt{\mathscr{H}_{\pm,\lambda}}}v_{\pm}, \tag{13}$$

where

$$\cos(t\sqrt{\mathscr{H}_{\pm,\lambda}})u_{\pm} = \frac{2}{\pi}\int_0^{\infty}\int_{\mathbb{R}} \sigma\cos(t\sigma)\mathrm{Im}[G_{\pm,\lambda}(x,x',\sigma)]u_{\pm}(x')dx'd\sigma$$

and

$$\frac{\sin(t\sqrt{\mathscr{H}_{\pm,\lambda}})}{\sqrt{\mathscr{H}_{\pm,\lambda}}}v_{\pm} = \frac{2}{\pi}\int_0^{\infty}\int_{\mathbb{R}} \sin(t\sigma)\mathrm{Im}[G_{\pm,\lambda}(x,x',\sigma)]v_{\pm}(x')dx'd\sigma,$$

and $G_{\pm,\lambda}(x,x',\sigma)$ is the Green's function of the equation

$$\mathscr{H}_{\pm,\lambda}Z = \sigma^2 Z.$$

Green's function

Jost solutions:

$$f_\lambda^\pm(x,\sigma) \sim e^{\pm i\sigma x} \quad \text{for} \quad x \to \pm\infty$$

Wronskian:

$$W(f_\lambda^+, f_\lambda^-) = f_\lambda^+ \frac{df_\lambda^-}{dx} - \frac{df_\lambda^+}{dx} f_\lambda^-$$

Green's function:

$$G(x,x',\sigma) = \frac{f_\lambda^-(x',\sigma)f_\lambda^+(x,\sigma)\Theta(x-x') + f_\lambda^-(x,\sigma)f_\lambda^+(x',\sigma)\Theta(x'-x)}{W(f_\lambda^-(\cdot,\sigma), f_\lambda^+(\cdot,\sigma))}$$

Lemma The Wronskian of the Jost solutions, $W(\sigma)$, satisfies

$$|W(\sigma)| \geq 2\sigma$$

for $\sigma > 0$.

Zero Energy Solutions for $\mathscr{H}_{+,\lambda}$

For $x \to \infty$:

$$\hat{\phi}_{+,\lambda,0}(x) \sim \frac{1}{2\lambda - 1} x^{\lambda}(1 + O(|x|^{-1+\varepsilon})),$$
$$\hat{\phi}_{+,\lambda,1}(x) \sim x^{1-\lambda}(1 + O(|x|^{-1+\varepsilon})),$$

For $x \to -\infty$:

$$\hat{\phi}_{+,\lambda,i}(x) = C_1 x + C_2 + O(|x|^{-1})$$
$$\hat{\phi}'_{+,\lambda,i}(x) = C_1 + O(|x|^{-1})$$

For any bounded solution for $\mathscr{H}_{+,\lambda} Z = 0$, we have

$$0 = \int_{\mathbb{R}} Z \mathscr{H}_{+,\lambda} Z dx = \int_{\mathbb{R}} |Z'|^2 + V_{+,\lambda} Z^2 dx.$$

Hence, any bounded solution of $\mathscr{H}_{+,\lambda} Z = 0$ on $\mathbb{R}$ must be zero.

Zero Energy Solutions for $\mathscr{H}_{-,\lambda}$

Solutions of $\mathscr{H}_{-,\lambda}\phi = 0$:

$$\hat{\phi}_{-,\lambda,0}(x) = \hat{\phi}_{-,\lambda,1}(x) \int_0^x \frac{1}{\hat{\phi}^2_{-,\lambda,1}(y)} dy,$$

and

$$\hat{\phi}_{-,\lambda,1}(x) = e^{-\lambda \int_{-\infty}^x \frac{\sqrt{\Delta(x)}}{r^2(x)} dx}.$$

It is easy to see that $\hat{\phi}_{-,\lambda,1}$ is bounded. Furthermore, we have

$$\begin{aligned} \hat{\phi}_{-,\lambda,0}(x) &= \frac{1}{(2\lambda+1)B_0} x^{\lambda+1}(1 + O(x^{-1+\varepsilon})), \\ \hat{\phi}_{-,\lambda,1}(x) &= B_0 x^{-\lambda}(1 + O(x^{-1+\varepsilon})), \end{aligned} \tag{14}$$

for $x > x_0 > 0$ for some positive constant $B_0 > 0$.

Schrödinger Equations with Small Energy

$|\sigma x|$ small: for $x \in [-\delta\sigma^{-1}, \delta\sigma^{-1}]$ with a fixed $\delta > 0$,

$$\phi_{-,\lambda,i}(x,\sigma) = \hat{\phi}_{-,\lambda,i}(1 + h_{-,\lambda,i}(x,\sigma)).$$

$|\sigma x|$ big: $\tilde{\phi}(z,\sigma) = \phi_{-,\lambda}(\sigma^{-1}z,\sigma)$ satisfies

$$\tilde{\phi}'' + \left(1 - \frac{\lambda(\lambda+1)}{z^2}\right)\tilde{\phi} = \sigma^{-2}\left[V_{-,\lambda}\left(\frac{z}{\sigma}\right) - \frac{\sigma^2\lambda(\lambda+1)}{z^2}\right]\tilde{\phi}$$

Therefore,

$$\phi_{-,\lambda}(x,\sigma) = \beta_\lambda\sqrt{\sigma x}H_{\lambda+1/2}(\sigma x)(1 + \mathfrak{h}_\lambda(\sigma x,\sigma)) \qquad (15)$$

for $x \in [\sigma^{\epsilon-1}, \infty)$, where $H_{\lambda+1/2} = J_{\lambda+1/2} + iY_{\lambda+1/2}$ is the Hankel function and $\beta_\lambda = i\sqrt{\pi/2}e^{i\pi\lambda/2}$.

Representation of Jost Solutions

Jost solutions:

$$f_{+,\lambda} = \tilde{\phi}(\sigma x, \sigma)$$

and $f_\lambda^-(x,\sigma) = e^{-i\sigma x}(1 + q_\lambda^-(x,\sigma))$ with q_λ^- satisfying

$$|\partial_\sigma^m \partial_x^k q_\lambda^-(x,\sigma)| \le C(m,k,a)(e^{\frac{x}{2M}} + e^{\frac{x}{8M}}) \quad (16)$$

for all $x \in (-\infty, a]$ and all $\sigma \in [-\sigma_0, \sigma_0]$.

Representation of Jost solutions:

$$\begin{aligned} f_\lambda^+ \sim & \alpha_{\lambda,1} B_0 \beta_\lambda \sigma^{\lambda+1}(1 + O(\sigma^\varepsilon) + iO(\sigma^{-2\lambda\varepsilon}))\phi_{-,\lambda,0} \\ & + i\frac{\alpha_{\lambda,0}\beta_\lambda}{B_0}\sigma^{-\lambda}(1 + O(\sigma^\varepsilon) + iO(\sigma^{(2\lambda+2)\varepsilon}))\phi_{-,\lambda,1}, \\ f_\lambda^- \sim & -i(1 + O(\sigma^\varepsilon))\sigma\phi_{-,\lambda,0} + (1 + O(\sigma^\varepsilon))\phi_{-,\lambda,1} \end{aligned}$$

Green's function:

$$\begin{aligned} G(x,x',\sigma) \sim & \frac{i}{\sigma}\phi_{-,\lambda,1}(x',\sigma)\phi_{-,\lambda,1}(x,\sigma) + O(\sigma^{2\lambda-1}) \\ & + \text{terms smooth up to order } 2\lambda - 1 \end{aligned}$$

Sine Evolution

Let $\chi_\delta(x) = \chi(x/\delta)$ and $\delta \in C_0^\infty$ satisfy

$$\chi(x) = \begin{cases} 1 & \text{if } |x| \le 1/2, \\ 0 & \text{if } |x| \ge 1. \end{cases}$$

Then

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{R}} \sin(t\sigma) \mathrm{Im}[G_{-,\lambda}(x, x', \sigma)] w(x') \\ &\quad \chi_\delta(\sigma x)\chi_\delta(\sigma x')\chi_{\sigma_0}(\sigma)\langle x\rangle^{-\alpha}\langle x'\rangle^{-\alpha} dx' d\sigma \\ \sim &\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\sin(t\sigma)}{\sigma} \phi_{-,\lambda,1}(x,\sigma)\phi_{-,\lambda,1}(x',\sigma) w(x') \\ &\quad \chi_\delta(\sigma x)\chi_\delta(\sigma x')\chi_{\sigma_0}(\sigma)\langle x\rangle^{-\alpha}\langle x'\rangle^{-\alpha} dx' d\sigma \\ \sim &\int_0^\infty \frac{\sin\sigma}{\sigma} d\sigma \int_{\mathbb{R}} \hat{\phi}_{-,\lambda,1}(x') w(x') \langle x'\rangle^{-\alpha} dx' \hat{\phi}_{-,\lambda,1}(x) \langle x\rangle^{-\alpha} \end{aligned}$$

Summary and Some Problems

Summary

- Decay rate of massless Dirac waves in Schwarzschild geometry
- Asymptotic behavior of two nonlinear wave equations with potentials

Some Problems

- Decay rate of Dirac waves in Schwarzschild geometry with general data
- Improved decay rate in Kerr geometry
- Other waves (electromagnetic waves, gravitational waves) in black hole geometry

Thanks!