Inverse problems for a Sellers type climate model

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Part of the work in collaboration with

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Contents

I. The Budyko-Sellers climate model (1969)

The model

- Climate : u= Earth surface temperature.
- Evolution of ice covering.
- Long time scale or seasonal model.
- Impact of pollution.

The model equation

$$\begin{cases} u_t - div_{\mathcal{M}}(k(x)\nabla u) = R_a(t,x,u) - R_e(t,x,u), \\ u(0) = u_0. \end{cases}$$

$\mathcal{M}$ = Earth surface (riemannian manifold)

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I. The Budyko-Sellers climate model (1969)

Non linear terms R_a and R_e

- $R_a(t,x,u) = Q(t,x)\beta(u)$.
  - Q = insolation function.
  - β = albedo.
- $R_e(u) = \epsilon u|u|^3$: emitted energy (greenhouse gazes,...).

Expressions of the albedo

- Sellers model : Lipschitz continuous function
- Budyko model : Maximal monotone graph

More complicated model (North, 1972)

k replaced by $|\nabla u|^{p-2} \Rightarrow$ p-Laplace operator

I. The Budyko-Sellers climate model (1969)

Mathematical study of the model : Well-posedness, stationary solutions, free boundary problems, approximate controllability, numerics...

- Hetzer(1989,...)
- Diaz(1993,...)
- Diaz-Hernandez-Tello (1997), Hetzer-Tello (2001)
- Diaz-Hetzer-Tello (2005), Bermejo-Carpio-Diaz-Tello (2009)

I. The Budyko-Sellers climate model (1969)

Exemple : $\mathcal{M} := \mathbb{S}^2$

- Local formula for Laplace operator in spherical coordinates (longitude, latitude)
- longitude= constant, x=sin(latitude) $\Rightarrow$ 1-D degenerate non linear parabolic equation

$$\begin{cases} u_t - (k(x)(1-x^2)u_x)_x = R_a(t,x,u) - R_e(t,x,u), \\ ((1-x^2)u_x)(t,-1) = ((1-x^2)u_x)(t,1) = 0, \\ u(0,x) = u_0(x). \end{cases}$$

-1 and 1 represent the North/South poles.

Inverse problems for the 1-D Sellers model

- Coefficients in the model : empirical data.

II. An overview of Lipschitz stability results

Model problem

Example of the heat equation

$$\begin{cases} u_t - \Delta u = g & \Omega \times (0, T) \\ u(t, .) = h & \partial\Omega \times (0, T) \\ u(t = 0) = u_0 & \Omega \end{cases}$$

Goal : determine source term g from measurements of u

Usual issues

- Local/global uniqueness
- Stability results (Logarithmic,Hölder or Lipschitz stability)
- Numerical reconstructions

Idea to get Lipschitz stability = use of global Carleman estimate

Introduced for the wave equation by Puel-Yamamoto (1996-1997)

Founding paper for parabolic equations : Imanuvilov-Yamamoto (1998)

Other Lipschitz stability results for parabolic equations

- Yamamoto-Zou (2001)
- Benabdallah-Dermenjian-Le Rousseau (2007), Benabdallah-Gaitan-Le Rousseau (2009)
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II. An overview of Lipschitz stability results

Method for Lipschitz stability results

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For parabolic systems

- Cristofol-Gaitan-Ramoul (2006)
- Benabdallah-Cristofol-Gaitan-Yamamoto (2009)
- Benabdallah-Cristofol-Gaitan-De Teresa (2010)

For other equations

- Hyperbolic equations : Imanuvilov-Yamamoto (2001), Bellassoued-Yamamoto (2006)...
- Schrödinger equation : Baudouin-Puel (2002), Mercado-Osses-Rosier (2008), Baudouin-Mercado (2008), Cardoulis-Gaitan (2010),...

Consider

$$u_t - (au_x)_x = 0 \qquad (0,T) \times (0,1)$$

with

$$a(x) = x^\alpha \qquad \alpha \in [0,2)$$

Classical theory for parabolic equations fails since $a(0) = 0$

Adapted Sobolev spaces

Meyer (1967), Adams (1975), Campiti-Metafune-Pallara (1998)

$$H_a^1(0,1) := \left\{ u \in L^2(0,1) \,\middle|\, \sqrt{a}u_x \in L^2(0,1) \right\}$$

Boundary value : two cases

- $0 \leq \alpha < 1$

$$H^1_{a,0}(0,1) := \left\{ u \in H^1_a(0,1) \,|\, u(0) = u(1) = 0 \right\}$$

- $1 \leq \alpha < 2$

$$H^1_{a,0}(0,1) := \left\{ u \in H^1_a(0,1) \,|\, u(1) = 0 \right\}$$

Degenerate unbounded operator

$$D(A) := \left\{ u \in H^1_{a,0}(0,1) \,\middle|\, au_x \in H^1(0,1) \right\}$$

$$\forall u \in D(A), \quad Au := -(au_x)_x$$

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Initial-boundary value problem

$$
\begin{cases}
u_t - (a u_x)_x = g & (0,T) \times (0,1) \\
u(t,1) = 0 \text{ and } \begin{cases} u(t,0) = 0 \text{ for } 0 \le \alpha < 1 \\ a u_x(t,0) = 0 \text{ for } 1 \le \alpha < 2 \end{cases} & (0,T) \\
u(t=0) = u_0 & (0,1)
\end{cases}
$$

Well-posedness

- Analytic semigroup.
- Standard well-posedness results

Initial-boundary value problem

$$\begin{cases} u_t - (au_x)_x = g & (0,T)\times(0,1) \\ u(t,1) = 0 \text{ and } \begin{cases} u(t,0) = 0 \text{ for } 0 \le \alpha < 1 \\ au_x(t,0) = 0 \text{ for } 1 \le \alpha < 2 \end{cases} & (0,T) \\ u(t=0) = u_0 & (0,1) \end{cases}$$

Well-posedness

- Analytic semigroup.
- Standard well-posedness results

Theorem (Cannarsa-Martinez-Vancostenoble, 2005-2008)

Let $0 \leq \alpha < 2$ and $\omega \subset (0,1)$. For all $u_0 \in L^2(0,1)$, there exists $g \in L^2((0,T) \times \omega)$ such that $u_g(T) = 0$.

Carleman estimate with weights adapted to the degeneracy

Cannarsa-Martinez-Vancostenoble, 2005-2008

Some extensions

- Degeneracy at both extreme points : Martinez-Vancostenoble (2006)
- Semilinear equations : Alabau-Cannarsa-Fragnelli (2006)
- Degenerate operator in non divergence form : Cannarsa-Fragnelli-Rochetti (2008)

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IV. Inverse Problems for the linear 1-D degenerate equation

An inverse source problem to solve

The initial-boundary value problem

$$\begin{cases} u_t - (au_x)_x = g & Q_T \\ u(t,1) = 0 \text{ and } \begin{cases} u(t,0) = 0 \text{ for } 0 \leq \alpha < 1 \\ au_x(t,0) = 0 \text{ for } 1 \leq \alpha < 2 \end{cases} & (0,T) \\ u(t=0) = u_0 & (0,1) \end{cases}$$

$$0 < t_0 < T,\ T' = \frac{T + t_0}{2},\ \omega := (a,b) \text{ with } 0 < a < b < 1$$

Goal

Determine source term g in two different cases :

$$\begin{array}{lll} (au_x)_x(T',.) & \text{and} & u_{tx}(.,1)|_{(t_0,T)} \text{ (boundary observation)} \\ (au_x)_x(T',.) & \text{and} & u_t|_{(t_0,T)\times\omega} \text{ (loc. distributed observation)} \end{array}$$

Remark

Non-uniqueness of g in $L^2(0,T;L^2(0,1))$ (controllability arguments)

Introduce $C_0 > 0$ and admissible subset

$$\mathcal{G}(C_0) := \left\{ g \in H^1(0,T;L^2(0,1)) \,\middle|\, |g_t(t,x)| \le C_0 \left|g(T',x)\right| \text{ a.e. in } Q_T^{t_0} \right\}$$

Example : $g(t,x) = r(t,x)f(x)$

- r known function, $r \in C^1([0,T] \times [0,1])$, r positive at time $t = T'$.
- $f \in L^2(0,1)$ is the unknown coefficient to recover

Theorem 1 (Cannarsa-Tort-Yamamoto, 2010)

Let $u_0 \in L^2(0,1)$. $\exists C = C(T, t_0, C_0, \alpha) > 0$ s.t. $\forall g_1 = rf_1$, $\forall g_2 = rf_2$,

$$\|f_1 - f_2\|^2_{L^2(0,1)} \leq C \left(\|(a(u_1 - u_2)_x)_x (T', .)\|^2_{L^2(0,1)} + \|((u_1 - u_2)_t)_x (., 1)\|^2_{L^2(t_0,T)} \right) \quad (1)$$

Theorem 2 (Cannarsa-Tort-Yamamoto, 2010)

Let $u_0 \in L^2(0,1)$. $\exists C = C(T, t_0, \omega, C_0, \alpha) > 0$ s.t. $\forall g_1 = rf_1$, $\forall g_2 = rf_2$,

$$\|f_1 - f_2\|^2_{L^2(0,1)} \leq C \left(\|(a(u_1 - u_2)_x)_x (T', .)\|^2_{L^2(0,1)} + \|(u_1 - u_2)_t\|^2_{L^2(t_0,T;L^2(\omega))} \right) \quad (2)$$

IV. Inverse Problems for the linear 1-D degenerate equation

Main tool : a global Carleman estimate for degenerate operators

Weight functions

- R a positive parameter
- θ a time weight function $\theta(t) := \dfrac{1}{(t(T-t))^4}$
- p a space weight function adapted to the degeneracy and the observation
- $\sigma = \theta p$
- $\beta(t) := T + t_0 - 2t$

Let z be a solution of

$$\begin{cases} z_t - (az_x)_x = h & Q_T \\ z(t,1) = 0 \text{ and } \begin{cases} z(t,0) = 0 \text{ for } 0 \leq \alpha < 1 \\ az_x(t,0) = 0 \text{ for } 1 \leq \alpha < 2 \end{cases} & (0,T) \end{cases}$$

IV. Inverse Problems for the linear 1-D degenerate equation

Main tool : a global Carleman estimate for degenerate operators

Proposition (Cannarsa-Tort-Yamamoto, 2010)

$\exists C_1 = C_1(T, t_0, \alpha) > 0$, $\exists R_0 = R_0\,(T, t_0, \alpha) > 0$ s. t. $\forall R \geq R_0$,

$$\iint_{Q_T^{t_0}} \left(R^3\theta^3 x^{2-\alpha} z^2 + R\theta x^{\alpha} z_x^2 + \frac{1}{R\theta} z_t^2 + R\theta^{\frac{3}{2}} |\beta| p z^2 \right) e^{-2R\sigma} dxdt$$

$$\leq C_1 \left(\iint_{Q_T^{t_0}} h^2 e^{-2R\sigma} dxdt + \int_{t_0}^{T} R\theta z_x^2\,(t,1)\, e^{-2R\sigma(t,1)} dt \right) \quad (3)$$

In Cannarsa-Martinez-Vancostenoble (2008), estimate of the two first terms

Take f_1 and f_2 in $L^2(0,1)$; u_1 and u_2 corresponding solutions

$w := u_1 - u_2$ satisfies

$$\begin{cases} w_t - (aw_x)_x = r(f_1 - f_2) & Q_T \\ w(t,1) = 0 \text{ and } \begin{cases} w(t,0) = 0 \text{ for } 0 \leq \alpha < 1 \\ aw_x(t,0) = 0 \text{ for } 1 \leq \alpha < 2 \end{cases} & (0,\,T) \\ w(0,x) = 0 & (0,1) \end{cases}$$

Classical estimates in the wrong direction...

- Apply Carleman estimate to $z := w_t$.

Lemma (Hardy-type inequality)

Let $1 < \alpha^\star < 2$ and let $a^\star(x) = x^{\alpha^\star}$.

$$\int_0^1 x^{\alpha^\star - 2} f^2 \leq \frac{4}{(1-\alpha^\star)^2} \int_0^1 x^{\alpha^\star} f_x^2 \quad \forall f \in H^1_{a^\star,0}\,(0,1)\,.$$

The initial-boundary value problem

$$\begin{cases} u_t - ((1-x^2)u_x)_x = Q(t,x)\beta(u) - \epsilon(u)u|u|^3, \\ ((1-x^2)u_x)(t,-1) = ((1-x^2)u_x)(t,1) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{4}$$

$$x \in I := (-1,1), \quad t \in (0,T).$$

Set $a(x) := 1 - x^2 = (1+x)(1-x)$

- Degeneracy at both extreme points -1 and 1.
- Each degeneracy equivalent to x^α at 0 (section II) with $\alpha = 1$.

Same notations and results as in section II

- Analytic semigroup !

Assumptions on the albedo β

- β smooth ($\approx C^2(\mathbb{R})$) and bounded.
- $\beta(u) \geq \beta_1 > 0$.

Assumptions on ϵ in R_e

- ϵ smooth ($\approx C^2(\mathbb{R})$) and bounded.
- $\epsilon(u) \geq \epsilon_1 > 0$.

Assumptions on the insolation function Q

- $Q(t,x) = r(t)q(x)$
- $r \in C^1(\mathbb{R})$ τ-periodic.
- $q \in L^\infty(-1,1)$, $q \geq 0$.

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- $q \in L^\infty(-1, 1)$, $q \geq 0$.

A space of initial conditions

$$\mathcal{U} := \{u_0 \in D(A) \cap L^\infty(I) : Au_0 \in L^\infty(I)\}.$$

Theorem (Diaz, 1993 ; Tort-Vancostenoble (2011))

Let $u_0 \in \mathcal{U}$ and $T > 0$. Then problem (4) has a unique classical solution

$$u \in C([0,T]; D(A)) \cap C^1([0,T]; L^2(0,1)).$$

Moreover u and u_t belong to $L^\infty((0,T) \times (0,1))$ (maximum principle).

Global regular solution of the 1-D Sellers model

- Important for inverse coefficient problems.

VI. A uniqueness and stability result for the insolation function

The inverse problem to solve

The initial-boundary value problem

$$\begin{cases} u_t - ((1-x^2)u_x)_x = r(t)q(x)\beta(u) - \epsilon(u)u|u|^3, \\ ((1-x^2)u_x)(t,-1) = ((1-x^2)u_x)(t,1) = 0, \\ u(0,x) = u_0(x). \end{cases}$$

Determine insolation space distribution q

Measurements of u

$(au_x)_x(T', .)$ and $u_t|_{(t_0,T)\times\omega}$ (loc. distributed observation)

$$0 < t_0 < T,\ T' = \frac{T+t_0}{2},\ \omega := (a,b) \text{ with } -1 < a < b < 1$$

Admissible set of unknown coefficient

$$\mathcal{Q}_D := \left\{ q \in L^\infty(I) : \|q\|_{L^\infty(I)} \leq D \right\}, \quad D > 0.$$

- Uniform bound for corresponding solutions

Theorem (Tort-Vancostenoble, 2011)

Let $u_0 \in \mathcal{U}$ and $\omega := (L_1, L_2)$ with $-1 < L_1 < L_2 < 1$. There exists $C > 0$ s. t. for all q_1 and q_2 belonging to $\mathcal{Q}_D$, the corresponding solutions u_1 and u_2 of (4) satisfy :

$$\|q_1 - q_2\|^2_{L^\infty(I)} \leq C\big(\big\|(u_1 - u_2)(T', .)\big\|^2_{D(A)} + \|u_1 - u_2\|^2_{L^2((t_0,T)\times\omega)}\big).$$

$w := u_1 - u_2$ solution of the linear equation

$$\begin{cases} w_t - ((1-x^2)w_x)_x = r(q_1 - q_2)\beta(u_1) + g_2 + g_3 \\ ((1-x^2)w_x)(t,-1) = ((1-x^2)w_x)(t,1) = 0, \\ w(0,x) = 0. \end{cases}$$

Determination of a source term in a linear degenerate parabolic equation

Skeleton of the proof

- Adapt the result by Cannarsa-Tort-Yamamoto to degeneracies at both extreme points.
- $r(q_1 - q_2)\beta(u_1)$ admissible source term.
- Absorb g_2 and g_3 thanks to Carleman estimate.

The initial-boundary value problem

$$\begin{cases} u_t - (\lambda a u_x)_x = h & Q_T \\ u(t,1) = 0 \text{ and } \begin{cases} u(t,0) = 0 \text{ for } 0 \leq \alpha < 1 \\ a u_x(t,0) = 0 \text{ for } 1 \leq \alpha < 2 \end{cases} & (0,T) \\ u(t=0) = u_0 & (0,1) \end{cases}$$

Goal

Determine diffusion constant λ in two different cases :

$$(au_x)_x(T',.) \quad \text{and} \quad u_{tx}(.,1)|_{(t_0,T)} \text{ (boundary observation)}$$

$$(au_x)_x(T',.) \quad \text{and} \quad u_t|_{(t_0,T)\times\omega} \text{ (loc. distributed observation)}$$

Goal : use previous results for source terms

Fix $0 < \Lambda_0 < \Lambda_1$ and take λ, μ in (Λ_0, Λ_1)

- u_λ and u_μ corresponding solutions.
- Set $w := u_\lambda - u_\mu$.

w is the solution of

$$\begin{cases} w_t - (\lambda x^\alpha w_x)_x = (\lambda - \mu)(x^\alpha u_{\mu,x})_x & Q_T \\ w(t,1) = 0 \text{ and } \begin{cases} w(t,0) = 0 \text{ for } 0 \le \alpha < 1 \\ a w_x(t,0) = 0 \text{ for } 1 \le \alpha < 2 \end{cases} & (0,T) \\ w(t=0) = 0 & (0,1) \end{cases}$$

$(\lambda - \mu)(x^\alpha u_{\mu,x})_x$: admissible source term

- Regularity of $(x^\alpha u_{\mu,x})_x$?
- Positivity of $(x^\alpha u_{\mu,x})_x(T', .)$?

Adapted space of initial data $\mathcal{U}$ and of source term $\mathcal{H}$

Theorem (Tort, 2011)

Let $\alpha \in [0,2)$, $u_0 \in \mathcal{U}$ and $h \in \mathcal{H}$. There exists $C > 0$, such that for all λ and μ in (Λ_0, Λ_1), u_λ and u_μ satisfy

$$|\lambda - \mu|^2 \leq C \left(\left\| \left(x^\alpha \left(u_\lambda - u_\mu \right)_x \right)_x \left(T', . \right) \right\|^2_{L^2(0,1)} + \left\| \left(u_\lambda - u_\mu \right)_{tx} (., 1) \right\|^2_{L^2(t_0, T)} \right).$$

VII. Some related works and perspectives

An inverse problem for the 2-D Sellers climate model, in preparation

2-D model on a Riemannian manifold $\mathcal{M}$

$$\begin{cases} u_t - \Delta_{\mathcal{M}} u = Q(t,x)\beta(u) - u|u|^3 & (0,T) \times \mathcal{M} \\ u(0,x) = u_0(x) & \mathcal{M} \end{cases}$$

Recover insolation function Q

Difficulties

- Derivations and operators on a riemannian manifold.
- Regularity of solutions (Sobolev spaces on riemannian manifolds,...)

1-D model

$$\begin{cases} u_t - ((1-x^2)|u_x|^{p-2}u_x)_x = Q(t,x)\beta(u) - \epsilon(u)u|u|^3, \\ ((1-x^2)u_x)(t,-1) = ((1-x^2)u_x)(t,1) = 0, \\ u(0,x) = u_0(x). \end{cases}$$

Model on a riemannian manifold $\mathcal{M}$

$$\begin{cases} u_t - div_{\mathcal{M}}(|\nabla u|^{p-2}\nabla u) = Q(t,x)\beta(u) - u|u|^3 & (0,T)\times\mathcal{M}, \\ u(0,x) = u_0(x) & \mathcal{M}. \end{cases}$$

Some results on controllability exist (Diaz, 2000)