Some Topics in Compressible Navier-Stokes System

Zhouping Xin

The Institute of Mathematical Sciences

The Chinese University of Hong Kong

Program on Nonlinear Hyperbolic PDEs

July 4-8, 2011, Trieste, Italy

Contents

Introduction

The full compressible Navier-Stokes equations are:

$$\begin{cases} \partial_t \rho + div(\rho u) = 0, \\ \partial_t(\rho u) + div(\rho u \otimes u) + \nabla P = div(T), \\ \partial_t(\rho E) + div(\rho u E + uP) = div(uT) + k\triangle\theta, \end{cases} \qquad (0)$$

ρ: density,
θ: temperature,
u: velocity,
e: internal energy,
$P = P(e, \rho)$: pressure,
$E = \frac{1}{2}|u|^2 + e$: total energy,
$T = \mu(\nabla u + (\nabla u)^t) + \lambda(div\, u)I$: stress tensor (1)

μ and λ are viscosity coefficients satisfying

$$\mu > 0, \quad \lambda + \frac{2}{N}\mu \geq 0, \quad N : \text{space dimension} \tag{2}$$

$k \geq 0$; heat conduction coefficient.

The Compressible Isentropic Navier-Stokes system (CNS) reads:

$$\begin{cases} \partial_t \rho + div \rho = 0 \\ \partial_t(\rho u) + div(\rho u \otimes u) + \nabla P = div\, T, \end{cases} \tag{3}$$

$$P = P(\rho) = A\rho^\gamma$$

Some KEY Issues:

- Global well-posedness theory of smooth or weak solutions for various boundary conditions
- Asymptotic behavior of solutions for physically relevant parameter regimes
  - Large Reynolds number limits (which leads to internal layer and boundary layer theory etc.)
  - Small Mach number limits (incompressible limits which leads to various incompressible fluid models)

Introduction

- Dynamical stability of basic waves (steady flows, nonlinear and linear waves, etc.).
- Numerical Methods for computing physically relevant flows.

Overall Pictures:

- Significant progresses have been achieved in 1D.
- Almost completely open to Multi-D!

Introduction

Major Difficulties:

- Mixed-type system: hyperbolic+parabolic for non-vacuum regions.
- Strong degeneracies near vacuum.
- Strong nonlinearities: inertial+pressure difference+their interactions.
- Strong nonlinearity in the energy equations

Main progress:

- **Local Well-posedness of Classical Solutions away from vacuum:**

  Nash (1962): Existence

  Serrin (1959): Uniqueness

- **Local Well-posedness of Classical (or strong) Solutions containing vacuum states:**

  Cho Y., Choe H. J., Kim H. 2003, 2004, 2006

- Local Existence of classical Solutions (Cho-Kim (2006)):

Assumption

If (ρ_0, u_0) *satisfies*

$$\begin{cases} 0 \le \rho_0, \quad \rho_0 - \tilde{\rho}, P - P(\tilde{\rho}) \in H^3, \quad u_0 \in D_0^1 \cap D^3 \\ -\mu \triangle u_0 - (\lambda + \mu)\nabla \mathit{div} u_0 + \nabla P(\rho_0) = \rho_0 g, \end{cases} \tag{4}$$

for $\rho_0^{1/2} g, \nabla g \in L^2$.

Conclusion

$\exists\ T_1 \in (0, \infty)$ *and a unique classical solution* (ρ, u) *in* $\Omega \times (0, T_1]$.

Introduction

- **Global Existence of Classical Solutions away from vacuum:**

  Kazhikhov & Shelukhin (1977): 1D, large initial data

  Weigant & Kazhikhov (1995): 2D, large initial data, for very special μ, λ.

Introduction

Theorem (Matsumura-Nishida (1980))

If the initial data (ρ_0, u_0, θ_0) *satisfies*

$$\|\rho_0 - 1, u_0, \theta_0 - 1\|_{H^3(R^3)} \ll 1,$$

THEN $\exists!$ *global classical solution* (ρ, u, θ) *such that*

$$\sup_{0 \le t < \infty} \|\rho - 1, u, \theta - 1\|_{H^3(R^3)}(t) \ll 1.$$

Furthermore, the solution behaves diffusively.

- **Basic Idea of Analysis:** Energy Method+Spectrum Analysis
- Generalizations to weak solutions by Hoff (1995).

Introduction

- Matsumura-Nishida's theory requires that the solution has SMALL oscillations from a uniform non-vacuum state so that the density is strictly AWAY from the vacuum and the gradient of the density remains bounded uniformly in time.

Open Problem 1

Does there exist a global classical solution for large oscillations and vacuum with constant state as far field which could be either vacuum or non-vacuum? Can the classical CNS be well behaved near vacuum?

- **Global existence of weak solutions containing vacuum states:**
  - The density vanishes at far fields, or even has compact support. Lions (1993, 1998): 3D, large initial data, when $\gamma \geq 9/5$, Feireisl (2001): 3D, large initial data, when $\gamma > 3/2$. Jiang-Zhang (2001): $\gamma > 1$, for spherically symmetric solutions.

Theorem (Lions-Feireisl (1993, 1998, 2001))

If $\gamma > 3/2$ and the initial data (ρ_0, u_0) satisfies

$$C_0 \triangleq \frac{1}{2}\int \rho_0 |u_0|^2 dx + \frac{1}{\gamma - 1}\int P(\rho_0)dx < \infty. \tag{5}$$

THEN $\exists$ a global weak solution (ρ, u).

Introduction

- **Basic Ideas of Analysis:** Energy Method + Weak Convergence Method
- Some partial results on the asymptotic behavior of solutions such as small Mach number limit, etc. have been established for such weak solutions.

Open Problem 2

The regularity and uniqueness of Lions-Feireisl's weak solutions. In particular, can one define the particle paths for such solutions?

Introduction

- Desjardins (1997): For 2D periodic case,

$$\sqrt{\rho}u_t \in L^2(0,T;L^2), \nabla u \in L^\infty(0,T;L^2),$$

as long as the density is bounded.

- Hoff (2005): A new type of global weak solutions with **small energy**, which have extra regularity information compared with Lions-Feireisl, for general $P(\rho)$ and far field density away from vacuum, provided

$$\mu > \max\{4\lambda, -\lambda\}.$$

Blowup phenomena and blowup criteria

- **Blowup of Smooth Solutions containing vacuum states:**

Theorem (Xin, 1998)

If

$$(\rho_0, u_0, \theta_0) \in H^s(R^d)(s > [d/2] + 2), \text{ for Full NS },$$

$$(\rho_0, u_0) \in H^s(R^1)(s > 2), \text{ for CNS},$$

and $\rho_0(x)$ *has compact support.*

THEN smooth solutions in $C^1([0, T]; H^s)$ *have to blow up in finite time.*

Blowup phenomena and blowup criteria

- Idea: total pressure behaves dispersively:

$$\int_{R^d} P dx \leq \begin{cases} C(1+t)^{-(\gamma-1)d} & \text{for } \gamma \in (1, 1+2/d) \\ C(1+t)^{-2} & \text{for } \gamma > 1+2/d. \end{cases}$$

where $\gamma > 1$ is the ratio of specific heat.

Theorem (Huang-Li-Luo-Xin (2010))

If (ρ_0, u_0) is spherically symmetry and satisfies

$$(\rho_0, u_0) \in H^s(R^2)(s > 2), \tag{6}$$

and $\rho_0(x)$ has compact support.

THEN smooth solutions $(\rho, u) \in C^1([0,T]; H^s)$ have to blow up in finite time.

Blowup phenomena and blowup criteria

These theorems raise the question of the mechanism of blowup and structure of possible singularities:

- **Blow Up Criteria for strong solutions:**

Cho-Choe-Kim (2006),

Fan-Jiang (2007),

Huang-Xin (2009),

Fan-Jiang-Ou (2009),

Huang-Li-Xin (2009),

Sun-Wang-Zhang (2010)

. . . . . .

Blowup phenomena and blowup criteria

In particular, if T^* is the maximal existence time of the local strong solution (ρ, u), THEN

Theorem (Huang-Li-Xin (2010))

For 3D CNS where initial density may contain vacuum states,

$$\lim_{T \to T^*} \left(\|divu\|_{L^1(0,T;L^\infty)} + \|\rho^{\frac{1}{2}} u\|_{L^s(0,T;L^r)} \right) = \infty, \tag{7}$$

$$\lim_{T \to T^*} \left(\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\rho^{\frac{1}{2}} u\|_{L^s(0,T;L^r)} \right) = \infty, \tag{8}$$

with r and s satisfying

$$\frac{2}{s} + \frac{3}{r} \le 1, \quad 3 < r \le \infty.$$

Remark

If $\mathrm{div} u \equiv 0$, *(7) and (8) reduce to the well-known Serrin's blowup criterion for 3D incompressible Navier-Stokes equations. Therefore, This results can be regarded as the Serrin type blowup criterion on 3D compressible Navier-Stokes equations.*

Blowup phenomena and blowup criteria

- Main ideas: Estimates on material derivatives of velocity+

Lemma (Beale-Kato-Majda type inequality)

For $3 < q < \infty$, there is a constant $C(q)$ such that the following estimate holds for all $\nabla u \in L^2 \cap D^{1,q}$,

$$\begin{aligned} \|\nabla u\|_{L^\infty} \le C\left(\|\mathrm{div} u\|_{L^\infty} + \|\mathrm{rot} u\|_{L^\infty}\right)\log(e + \|\nabla^2 u\|_{L^q}) \\ + C\|\nabla u\|_{L^2} + C. \end{aligned} \tag{9}$$

Blowup phenomena and blowup criteria

Theorem

If $\mu > \lambda/7$, *then*

$$\lim_{T \to T^*} \|\mathit{div} u\|_{L^1(0,T;L^\infty)} = \infty,$$
$$\lim_{T \to T^*} \|\rho\|_{L^\infty(0,T;L^\infty)} = \infty, \tag{10}$$

where initial density may contain vacuum states.

Theorem

If $\inf \rho_0 > 0$, *then*

$$\lim_{T \to T^*} \|\mathit{div} u\|_{L^1(0,T;L^\infty)} = \infty. \tag{11}$$

Global classical solutions with large oscillations and vacuum

Consider the Cauchy problem to isentropic compressible Navier-Stokes equations:

$$
\begin{cases}
\rho_t + \mathrm{div}(\rho u) = 0, \\
(\rho u)_t + \mathrm{div}(\rho u \otimes u) - \mu \Delta u - (\mu + \lambda)\nabla(\mathrm{div} u) + \nabla P(\rho) = 0, \\
u(x,t) \to 0, \quad \rho(x,t) \to \tilde{\rho} \geq 0, \quad \text{as } |x| \to \infty, \\
(\rho, u)|_{t=0} = (\rho_0, u_0), \quad x \in R^3
\end{cases}
$$

Assumption

For given $M > 0$ *(not necessarily small),* $\tilde{\rho} \geq 0$, $\beta \in (1/2, 1]$, *and* $\bar{\rho} \geq \tilde{\rho} + 1$, *suppose that the initial data* (ρ_0, u_0) *satisfy*

$$\begin{aligned} &0 \leq \inf \rho_0 \leq \sup \rho_0 \leq \bar{\rho}, \quad \|u_0\|_{\dot{H}^\beta}^2 \leq M, \\ &u_0 \in \dot{H}^\beta \cap D^1 \cap D^3, \quad (\rho_0 - \tilde{\rho}, P(\rho_0) - P(\tilde{\rho})) \in H^3, \end{aligned} \tag{12}$$

and the compatibility condition

$$-\mu \triangle u_0 - (\mu + \lambda) \nabla \mathit{div} u_0 + \nabla P(\rho_0) = \rho_0 g, \tag{13}$$

for some $g \in D^1$ *with* $\rho_0^{1/2} g \in L^2$.

Global classical solutions with large oscillations and vacuum

Conclusion (Huang-Li-Xin (2010))

$\exists\varepsilon(\bar\rho, M)$ *s.t. if initial energy* C_0 *satisfies* $C_0 \le \varepsilon$, *the Cauchy problem has a unique global classical solution* (ρ, u) *satisfying for any* $0 < \tau < T < \infty$,

$$\begin{cases} 0 \le \rho(x,t) \le 2\bar\rho, \quad x \in R^3, t \ge 0, \\ (\rho - \tilde\rho, P - P(\tilde\rho)) \in C([0,T]; H^3), \\ u \in C([0,T]; D^1 \cap D^3) \cap L^2(0,T; D^4) \cap L^\infty(\tau, T; D^4), \\ u_t \in L^\infty(0,T; D^1) \cap L^2(0,T; D^2) \cap L^\infty(\tau, T; D^2) \cap H^1(\tau, T; D^1), \\ \sqrt{\rho} u_t \in L^\infty(0,T; L^2), \end{cases}$$

Conclusion (Continued)

and the following large-time behavior:

$$\lim_{t\to\infty}\int(|\rho-\tilde\rho|^q+\rho^{1/2}|u|^4+|\nabla u|^2)(x,t)dx=0,$$

$$\forall q\in\begin{cases}(2,\infty), & \text{for } \tilde\rho>0,\\ (\gamma,\infty), & \text{for } \tilde\rho=0.\end{cases}$$

where

$$C_0\triangleq\int\left(\frac{1}{2}\rho_0|u_0|^2+\rho_0\int_{\tilde\rho}^{\rho_0}\frac{P(s)-P(\tilde\rho)}{s^2}ds\right)dx.$$

Theorem (blowup behavior)

Assume that $\exists x_0 \in R^3$ *such that* $\rho_0(x_0) = 0$. *Then if* $\tilde{\rho} > 0$,

$$\lim_{t \to \infty} \|\nabla \rho(\cdot, t)\|_{L^r} = \infty,$$

for any $r > 3$.

Global classical solutions with large oscillations and vacuum

Remark

The solution obtained above becomes a classical one for positive time. Although it has small energy, yet whose oscillations could be arbitrarily large. In particular, both interior and far field vacuum states are allowed.

Remark

If $\tilde{\rho} > 0$, the requirement of small energy, is equivalent to smallness of the mean-square norm of $(\rho_0 - \tilde{\rho}, u_0)$. Therefore, our conclusions generalize the classical theory of Matsumura-Nishida (1980) to the case of large oscillations and far field density being either vacuum or non-vacuum. However, our solution may contain vacuum states, whose appearance leads to the large time blowup behavior, this is in sharp contrast to that in Matsumura-Nishida (1980) and Hoff (2005, 2008) where the gradients of the density are suitably small uniformly for all time.

Remark

When $\tilde{\rho} = 0,$ the small energy assumption is equivalent to that both the kinetic energy and the total pressure are suitably small, and there is no requirement on the size of the set of vacuum states. In particular, the initial density may have compact support. Thus, our results can be regarded as uniqueness and regularity theory of Lions-Feireisl's weak solutions with small initial energy.

Remark

We have given a positive answer to the Open Problems 1, 2 provided initial energy is suitably small.

Remark

For the incompressible Navier-Stokes system, Fujita-Kato (1964) and Kato (1984) proved that the system is globally wellposed for small initial data in the homogeneous Sobolev spaces $\dot{H}^{1/2}$ or in L^3. In our case, since the initial energy is small, we need the boundedness assumptions on the $\dot{H}^\beta$-norm of the initial velocity. It should be noted here that $\dot{H}^\beta \hookrightarrow L^{6/(3-2\beta)}$ and $6/(3-2\beta) > 3$ for $\beta > 1/2$, which implies that, compared with the results of Fujita-Kato (1964) and Kato (1984), our conditions on the initial velocity may be optimal under the smallness conditions on the initial energy.

Remark

It is very surprising that the above theory holds only in 3-dimensional. Indeed, in the case $\tilde{\rho} = 0$, one would not expect the same global existence result as already showing by the symmetric solutions with compact density. On the other hand, for the far fields away from vacuum, the corresponding results can be generalized to 2-dimensional, and furthermore, the result can be even improved by relaxing the requirement $u_0 \in H^s$ from $1/2 < s < 1$ to $0 < s < 1$.

Remark

Similar theory holds for bounded domains and periodic problems.

Remark

Similar results hold for the full compressible Navier-Stokes system in the case $\tilde{\rho} > 0$, although the theory fails for $\tilde{\rho} = 0$.

Remark

The main results fail for the compressible Navier-Stokes system with viscosity coefficients degenerate at vacuum. This settles a longstanding question on the validity of the classical CNS near vacuum.

(1) The Classical Theory away from Vacuum

(a) Local theory: The local theory can be established by

- well-posedness theory of linear symmetric system with variable coefficients by using Kato's theory.
- iteration scheme and contraction mapping principle based on energy estimate in a similar way as symmetric-hyperbolic system.

Remark: No special structure of the coupled hyperbolic-parabolic system of CNS is used for this theory.

(b) Global theory: based on energy estimates. However, the global in time high order regularity estimates are depending crucially on the dissipative structure of the CNS system as follows. Set

$$U = (\rho, u, \theta), \quad \bar{U} = (1, 0, \bar{\theta}), \quad \bar{\theta} > 0$$

Then the CNS (0) can be written as

$$A^0(U)\partial_t U + \sum_{j=1}^{N} A^j(U)\,\partial_{x_j} U - \sum_{j,k=1}^{N} B^{jk}(U)\partial^2_{x_j x_k} U = g(U, D_x U)$$

Some Basic Ideas of Analysis for CNS

where

$$A^0(U) = \begin{pmatrix} \frac{P_\rho}{\rho} & & \\ & \rho I & \\ & & \frac{\rho e_\theta}{\theta} \end{pmatrix}, \sum_{j=1}^{N} A^j(U)\xi_j = \begin{bmatrix} (\frac{P_\rho}{\rho})u\cdot\xi & P_\rho\xi & 0 \\ P_\rho\xi^t & \rho(u\cdot\xi)I & P_\theta\xi^t \\ 0 & P_\theta\xi & \frac{\rho e_\theta}{\theta}(u\cdot\xi) \end{bmatrix}$$

$$\sum_{j,k} B^{jk}(U)\xi_j\xi_k = \begin{bmatrix} 0 & & \\ & \mu|\xi|^2 I + (\mu+\lambda)\xi^t\xi & \\ & & (\frac{k}{\theta})|\xi|^2 \end{bmatrix}$$

$$g(U, D_xU) = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\theta}\Psi \end{bmatrix}$$

$$\Psi = \frac{\mu}{2}\sum_{i,j=1}^{N}(\partial_{x_j}u^i + \partial_{x_i}u^j)^2 + \lambda(\mathrm{div}u)^2$$

Some Basic Ideas of Analysis for CNS

Then the following conditions are satisfied:

(i) $A^0(\bar{U})$ is real, symmetric, and positive;

(ii) $A^j(\bar{U})$ are real symmetric $(j = 1, \cdots, N)$;

(iii) $B^{ij}(\bar{U})$ are real symmetric, $B^{jk} = B^{kj}$, and
$$\sum B^{jk}(U) w_j w_k \geq 0, \quad \forall w \in S^{N-1}$$

(iv) $\exists$ real constant $(N+2) \times (N+2)$ matrices K^j
$(j = 1, \cdots, N) \ni$

(a) $K^j A^0(\bar{U})$ are real and anti=symmetric, i.e.,
$(K^j A^0)^t = -K^j A^0, \quad j = 1, \cdots, N$

(b) $\sum_{j,k=1}^{N} \{\frac{1}{2}[K^j A^k(\bar{U}) + (K^j A^k(\bar{U}))^t] + B^{jk}(\bar{U})\} w_j w_k >$
$0, \quad \forall w \in S^{N-1}$

Some Basic Ideas of Analysis for CNS

In fact, K^j are given by

$$\sum_{j=1}^{N} K^j \xi_j = \alpha \begin{bmatrix} 0 & P_\rho(\bar{U})\xi & 0 \\ -P_\rho(\bar{U})\xi^t & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with $\alpha > 0$ begin properly chosen!

Remark: The conditions (a) and (b) above are called Kawashima dissipative condition, which makes sure that this solutions to the linearized problem decays to zero, i.e.,

$$A^0\partial_t U + \sum_j A^j(U)\,\partial_x U - \sum_{j,k} B^{jk}(U)\partial^2_{x_j D_h} U = 0$$

$$||D_x^l U(t)||^2 \le C\{e^{-c_1 t}||D_x^l U(0)||^2 + (1+t)^{-(2\gamma+l)}||U(0)||^2_{L^\rho}\}$$

$$\gamma = N(\frac{1}{2p} - \frac{1}{4}), \quad \gamma' = n(\frac{1}{2q} - \frac{1}{4})$$

In fact, K^j are proper multiplies in the energy estimates.

Some Basic Ideas of Analysis for CNS

(2) On the local well-posedness of classical or strong solutions with vacuum

- (i) compatibility of the initial data.
- (ii) standard iteration scheme, regularization, and cut-off arguments.
- (iii) The a priori estimates depend crucially on the uniform elliptic regularity of the momentum equation even at vacuum.

Some Basic Ideas of Analysis for CNS

(3) On blow-up of the smooth solutions to CNS

For the full CNS, the blow-up of smooth solution with compactly supported initial density is proved by

- (key) dispersive of the total pressure

$$\int_{\mathbb{R}} P dx \leq \begin{cases} C(1+t)^{-(\gamma-1)N} & \forall \gamma \in (1, 1+\frac{2}{N}) \\ C(1+t)^{-2} & \forall t \geq 1+\frac{2}{N} \end{cases}$$

  which follows from studying the functional

Some Basic Ideas of Analysis for CNS

$$I_\gamma(t) = \begin{cases} \int_{\mathbb{R}N} |x - u(x,t)(t+1)|^2 \rho(x,t)dx \\ \quad + \frac{2}{\gamma - 1}(t+1)^2 \int_{\mathbb{R}N} p(x,t)dx & \gamma \in (1, 1 + \frac{2}{N}) \\ \int_{\mathbb{R}N} |x - u(x,t)t|^2 p(x,t)dx \\ \quad + \frac{2}{\gamma - 1} t^2 \int_{\mathbb{R}^d} p(x,t)dx & \gamma \geq 1 + \frac{2}{N} \end{cases}$$

Then

$$\frac{d}{dt} I_j(t) = \begin{cases} \leq \frac{2 - N(\gamma - 1)}{t+1} I_\gamma(t), & \gamma \in (1, 1 + \frac{2}{N}) \\ \leq 0, & \forall \gamma \geq (1 + \frac{2}{N}) \end{cases}$$

Some Basic Ideas of Analysis for CNS

- Estimate of the support of the density: Let B_{c_1} be the minimal ball containing the support $\rho_0(x)$. Let

$$B_R(t) = \{(x,t)|x = x(t,x_0), \quad \frac{dx}{dt} = u, \quad x_0 \in B_{c_1}\}$$

Fact: $B_R(t) = B_{c_1} \times \{t\}$

which follows from the elliptic system

$$\begin{cases} \text{div}T = 0 \\ \text{div}(uT) + k\Delta\theta = 0 \end{cases} \quad \text{on } t \times \mathbb{R}^N \backslash S_R(t)$$

Some Basic Ideas of Analysis for CNS

- Since $\int_{\mathbb{R}^N} \rho(x,t)dx = \int_{\mathbb{R}^N} \rho_0(x)dx = m_0$. Thus $\forall \gamma \in (1, 1+\frac{2}{N})$

$$\begin{aligned} I_\gamma(0) &\geq \frac{2}{\gamma-1}(1+t)^{(\gamma-1)N}\int_{\mathbb{R}^N} p(x,t)dx \\ &\geq \frac{2}{\gamma-1}(1+t)^{(\gamma-1)N} e^{\frac{S_1}{c}} V_{B_R(t)} \frac{1}{V_{B_R(t)}} \int_{B_R(t)} (\rho(x,t))^\gamma dx \\ &\geq \frac{2}{\gamma-1}(1+t)^{(\gamma-1)N} e^{\frac{S_1}{c}} V_{B_R(t)}^{1-\gamma} m_0^\gamma. \end{aligned}$$

Some Basic Ideas of Analysis for CNS

For the isentropic CNS, it seems difficult to get steps above, indeed, it is not true in general. However, for 2-d symmetric flow, the momentum equation becomes

$$\rho(\partial_t u + u \cdot \partial_r u) + (P(\rho))_r = (2\mu + \lambda)(\partial_r u + r^{-1} u)_r$$

so on $\mathbb{R}^2 \times \{t\} \backslash S_R(t)$,

$$(2\mu + \lambda)(\partial_r u + r^{-1} u)_r = 0 \Rightarrow u(r,t) = c(t) r^{-1}$$

$$u(x,t) = u(r,t)\frac{x}{r} \in C^1([0,T] : H^s(\mathbb{R}^2)) \Rightarrow u \equiv 0 \text{ on } \mathbb{R}^2 \times \{t\} \backslash S_R(t)$$

Some Basic Ideas of Analysis for CNS

(4) On blow-up creteria: The key elements are estimates vorticity $w = \nabla \times u$, effective viscosity

$$F = (2\mu + \lambda) \text{ div } u - P(\rho)$$

and the material derivative of the velocity $\dot{u} \equiv \partial_t + u \cdot \nabla u$.

- Hodge decomposition: the momentum equation of CNS $\Leftrightarrow$

$$\Delta F = \text{div } (\rho \dot{u}), \mu \Delta w = \nabla \times (\rho \dot{u})$$
$$\Leftrightarrow \qquad \rho \dot{u} = \nabla G - \nabla \times w$$

- Transport equation for pressure

$$\partial_t P + \text{div } (Pu) + (\gamma - 1) P \text{ div} u = 0$$

Some Basic Ideas of Analysis for CNS

Step 1 : $\sup_{0\le t\le T} ||\rho^{\frac{1}{2}}u(t)||_{L^2}^2 + ||\rho||_{L^\gamma}^\gamma) + \int_0^T ||\nabla u||_{L^2}^2 dt \le C$

Step 2 : $\sup_{0\le t\le T} ||\nabla u||_{L^2}^2 + || \int_0^T \int \rho|\partial_t u|^2 dx\, dt \le C$

Step 3 :

$$\sup_{0\le t\le T} \int \rho|\dot{u}|^2 dx + \int_0^T (||\nabla u||_{L^2}^2 + ||\mathrm{div}\ u||_{L^\infty}^2 + ||w||_{L^\infty}^2) dt \le C$$

Step 4 : $\sup_{0\le t\le T} (||\rho||_{H^1\cap W^1 q} + ||\nabla u||_{H^1} \le C$

which is based on the Beale-Kato-Majda inequality

Step 5 : $\sup_{0\le t\le T} \int \rho|u|^q(x,t)dx \le C, \quad q > 3$

Some Basic Ideas of Analysis for CNS

These steps are based on the following elliptic estimates:

Lemma: $\exists$ positive constant C depending only on λ and μ such that for any $p \in [2, 6]$

$$\begin{cases} ||\nabla F||_{L^6} + ||\nabla w||_{L^p} \leq C||\rho\dot{u}||_{L^p}, \\ ||F||_{L^p} + ||w||_{L^p} \leq C||\rho\dot{u}||_{L^2}^{\frac{(3p-6)}{(2p)}} (||\nabla u||_{L^2} + ||p - p(\rho)||_{L^2})^{\frac{(6p)}{(2p)}} \\ ||\nabla u||_{L^p} \leq C(||F||_{L^p} + ||w||_{L^p}) + C||p - p(\tilde{\rho})||_{L^p} \\ ||\nabla u||_{L^p} \leq C||\nabla u||_{L^2}^{\frac{(6-p)}{(2p)}} (||\rho\dot{u}||_{L^2} + ||p - p(\tilde{\rho})||_{L^6})^{\frac{(3p-6)}{2p}} \end{cases}$$

These elliptic estimates are also used frequently in the analysis for global existence of smooth solutions below.

Some Basic Ideas of Analysis for CNS

(5) Analysis for the Global Well-Posedness of Smooth Solutions

Main difficulties:

- the appearance of vacuum
- no other constraints on the viscosity coefficients except the physical restrictions

KEY Issue:

- the time-independent upper bound for the density
- the time-depending higher norm estimates of the smooth solution

Some Basic Ideas of Analysis for CNS

Main ideas:

- basic estimates on the material derivatives of the velocity.
- weighted spatial mean estimates on the gradient and the material derivatives of the velocity.
- estimates on $L^1(0, \min\{1, T\}; L^\infty)$-norm and the time-independent ones on $L^{8/3}(\min\{1, T\}, T; L^\infty)$-norm of the effective viscous flux $F \triangleq (2\mu + \lambda)\text{div}u - P(\rho) + P(\tilde{\rho})$.

Some Basic Ideas of Analysis for CNS

- Zlotnik's inequality for time-uniform upper bound for the density (**KEY** estimates)
- Beale-Kato-Majda type inequality for time-depending higher order estimates on both the density and velocity

Some Basic Ideas of Analysis for CNS

Sketch of the main estimates: Let (ρ, u) be a classical solution to the barotropic CNS with initial data on $[0,T]\times\mathbb{R}^3$. Set

$$A_1(T) \triangleq \sup_{t\in[0,T]} \left(\sigma\|\nabla u\|_{L^2}^2\right) + \int_0^T\int \sigma\rho|\dot{u}|^2 dxdt,$$

$$A_2(T) \triangleq \sup_{t\in[0,T]} \sigma^3\int \rho|\dot{u}|^2 dx + \int_0^T\int \sigma^3|\nabla\dot{u}|^2 dxdt,$$

$$A_3(T) \triangleq \sup_{0\le t\le T}\int \rho|u|^3(x,t)dx.$$

Then the following Basic Energy Estimate holds

$$\sup_{0\le t\le T}\int\left(\frac{1}{2}\rho|u|^2+G(\rho)\right)dx+$$
$$\int_0^T\int\left(\mu|\nabla u|^2+(\lambda+\mu)(\text{div }u)^2\right)dxdt\le C_0.$$

The key a priori estimates on (ρ,u) are given in

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Proposition 1: Let the assumptions in Theorem 5 hold. Then for

$$\delta_0 \triangleq \frac{(2\beta - 1)}{(4\beta)} \in (0, \frac{1}{4}],$$

there exists $\varepsilon(\bar{\rho}, M) > 0$, $K(\bar{\rho}, M) > 0$ such that if (ρ, u) is a smooth solution satisfying $C_0 \leq \varepsilon$ and

$$\sup_{\mathbb{R}^3 \times [0,T]} \rho \leq 2\bar{\rho}, \qquad A_1(T)+A_2(T) \leq 2C_0^{\frac{1}{2}}, \qquad A_3(\sigma(T)) \leq 2C_0^{\delta_0},$$

the following estimates hold

$$\sup_{\mathbb{R}^3 \times [0,T]} \rho \leq \frac{7}{4}\bar{\rho}, \qquad A_1(T) + A_2(T) \leq C_0^{\frac{1}{2}}, \qquad A_3(\sigma(T)) \leq C_0^{\delta_0}.$$

The proof of this proposition can be done by several steps.

Step 1: Basic estimates on velocity field and its material derivatives.

The basic estimates are given

Lemma 1:

$$A_1(T) \leq C(\bar{\rho})C_0 + C(\bar{\rho}) \int_0^T \int \sigma |\nabla u|^3 \, dxdt,$$

$$A_2(T) \leq C(\bar{\rho})C_0 + C(\bar{\rho})A_1(T) + C(\bar{\rho}) \int_0^T \int \sigma^3 |\nabla u|^4 \, dxdt,$$

provided $0 \leq \rho \leq 2\bar{\rho}$.

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Lemma 1 is obtained by applying multiplier

$$\sigma^m \dot{u}(\partial_t + \mathrm{div}(u\cdot))^k, \quad m = 0, 1, 2, 3, \quad k = 0, 1$$

to the momentum system

$$\rho\dot{u} + \nabla p = \mu\Delta u + (\mu + \lambda)(\mathrm{div}\ u)$$

and estimating the resulting identities and using the transport equation for P.

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Step 2: Short time energy estimates

Lemma 2: It holds that

$$\sup_{0\le t\le\sigma(T)} t^{1-\beta}\|\nabla u\|_{L^2}^2 + \int_0^{\sigma(T)} t^{1-\beta}\int \rho|\dot{u}|^2\,dxdt \le K(\bar{\rho},M),$$

$$\sup_{0\le t\le\sigma(T)} t^{2-\beta}\int \rho|\dot{u}|^2\,dx + \int_0^{\sigma(T)} t^{2-\beta}\int |\nabla\dot{u}|^2\,dxdt \le K(\bar{\rho},M),$$

provided $C_0 \le \varepsilon_0$.

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Lemma 2 follows by splitting and interpolation. Fix (u, ρ), consider $u = w_1 + w_2$ with

$$\begin{aligned} \mathcal{L}w_1 &= 0, & w_1(x,0) &= u_0(x) \\ \mathcal{L}w_2 &= -\nabla p(\rho), & w_2(x,0) &= 0 \end{aligned}$$

where

$$\mathcal{L}w = \rho\dot{w} - (\mu\Delta w + (\mu+\lambda)\nabla(\text{div } w))$$

with

$$\dot{w} = \partial_t w + u\nabla \cdot w.$$

Applying standard estimates and interpolation to w_1, w_2 has better estimates!

Step 3: Short time high energy estimates

Lemma 3: It holds that

$$\sup_{0\le t\le \sigma(T)} \int \rho |u|^3 dx \le C_0^{\delta_0}$$

provided that $C_0 \le \varepsilon_1 \le \varepsilon_0$.

Lemma 3 follows from the energy estimate with multiplier $3|u|u$ to the momentum system and Lemma 2.

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Step 4: Estimates on the effective viscous flux

Define the effective viscous flux as

$$F \triangleq (2\mu + \lambda)\mathrm{div}\, u - (P(\rho) - P(\tilde{\rho})).$$

Then the following time independent bounds are essential to estimate the density.

Lemma 4: There exists constant $C = C(\bar{\rho}, M)$ such that

$$\int_0^{\sigma(T)} \|F\|_{L^\infty} dt \le C(\bar{\rho}, M) C_0^{\frac{3\delta_0}{8}},$$

$$\int_{\sigma(T)}^{T} \|F\|_{L^\infty}^{\frac{8}{3}} dt \le C(\bar{\rho}, M) C_0^{\frac{2}{3}}.$$

This follows from the following estimates:

$$
\begin{aligned}
& \int_0^{\sigma(T)} ||F(\cdot,t)||_{L^\infty} \\
\leq\ & C\int_0^{\sigma(T)} ||F(\cdot,t)||_{L^6}^{\frac{1}{2}} ||\nabla F(\cdot,t)||_{L^6}^{\frac{1}{2}}\, dt \\
\leq\ & C(\bar\rho)\int_0^{\sigma(T)} ||\rho^{\frac{1}{2}}\dot u||_{L^2}^{\frac{1}{2}} ||\nabla \dot u||_{L^2}^{\frac{1}{2}}\, dt \\
\leq\ & C(\bar\rho)\int_0^{\sigma(T)} t^{\frac{-(2-\beta)}{4}} ||\rho\dot u||_{L^2}^{\frac{1}{2}} (t^{2-\beta}||\nabla \dot u||_{L^2}^{2})^{\frac{1}{4}}\, dt \\
\leq\ & C(\bar\rho, M)\int_0^{\sigma(T)} (t^{\frac{-(2-\beta)}{3}} ||\rho\dot u||_{L^2}^{\frac{2}{3}}\, dt)^{\frac{3}{4}} \\
\leq\ & C(\bar\rho, M)(\int_0^{\sigma(T)} t^{-[(2-\beta)(-\delta_0+\frac{2}{3})+\delta_0]} (t^{2-\beta}||\rho^{\frac{1}{2}}\dot u||_{L^2}^{2})^{-\delta_0+\frac{1}{3}} (t||\rho^{\frac{1}{2}}\dot u||_{L^2}^{2})^{\delta_0}\, dt)^{\frac{3}{4}} \\
\leq\ & C(\bar\rho, M)(A_1(\sigma(T)))^{\frac{3\delta_0}{4}} \leq C(\bar\rho, M)C_0^{\frac{3\delta_0}{8}}.
\end{aligned}
$$

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$$
\begin{aligned}
& \int_{\sigma(T)}^{T} ||F(\cdot,t)||_{L^\infty}^{\frac{8}{3}} dt \\
\leq\ & C \int_{\sigma(T)}^{T} ||F(\cdot,T)||_{L^2}^{\frac{2}{3}} ||\nabla F(\cdot,t)||_{L^6}^{2} dt \\
\leq\ & C C_0^{\frac{1}{6}} \int_{\sigma(T)}^{T} ||\rho\dot{u}||_{L^6}^{2} dt \leq C_0^{\frac{1}{6}} \int_{\sigma(T)}^{T} ||\dot{u}||_{L^6}^{2} dt \\
\leq\ & C(\bar{\rho}) C_0^{\frac{1}{6}} \int_{\sigma(T)}^{T} ||\nabla\dot{u}||_{L^2}^{2} dt \leq C(\bar{\rho}) C_0^{\frac{1}{6}+\frac{1}{2}} \leq C(\bar{\rho}) C_0^{\frac{2}{3}}.
\end{aligned}
$$

Step 5: Super-norm estimate on the density

To apply this lemma to bound density, we recall a lemma in the theory of ordinary differential equation due to Zlotink.

Lemma 5 [Zlotnik]: Consider the problem

$$\begin{cases} y'(t) = g(y) + b'(t) \text{ on } [0,T], \quad y(0) = y^0, \\ g \in C(R), \quad y, b \in W^{1,1}(0,T), \quad g(\infty) = -\infty \\ b(t_2) - b(t_1) \le N_0 + N_1(t_2 - t_1) \text{ for all } 0 \le t_1 < t_2 \le T. \end{cases}$$

Then, $y(t) \le \max\left\{y^0, \bar{\zeta}\right\} + N_0 < \infty$ on $[0,T]$, where $\bar{\zeta}$ is a constant such that $g(\zeta) \le -N_1$ for $\zeta \ge \bar{\zeta}$.

Rewrite the continuity equation as

$$D_t\rho = g(\rho) + b'(t),$$

where

$$D_t\rho \triangleq \rho_t + u\cdot\nabla\rho, \qquad g(\rho) \triangleq -\frac{a\rho}{2\mu+\lambda}(\rho^\gamma - \tilde{\rho}^\gamma),$$

$$b(t) \triangleq -\frac{1}{2\mu+\lambda}\int_0^t \rho F dt.$$

For all $0 \le t_1 < t_2 \le \sigma(T)$,

$$\begin{aligned} |b(t_2)-b(t_1)| \le &\ C\int_0^{\sigma(T)} \|(\rho F)(\cdot,t)\|_{L^\infty} dt \\ \le &\ C(\bar{\rho}, M) C_0^{\frac{3\delta_0}{8}}. \end{aligned}$$

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Thus Lemma 5 implies that

$$\sup_{t\in[0,\sigma(T)]} \|\rho\|_{L^\infty} \le \frac{3\bar\rho}{2},$$

for C_0 suitably small.

For all $\sigma(T) \le t_1 \le t_2 \le T$,

$$\begin{aligned}
|b(t_2) - b(t_1)| \le &\ C(\bar\rho)\int_{t_1}^{t_2} \|F(\cdot,t)\|_{L^\infty} dt \\
\le &\ \frac{a}{2\mu+\lambda}(t_2 - t_1) + C(\bar\rho)\int_{\sigma(T)}^{T} \|F(\cdot,t)\|_{L^\infty}^{\frac{8}{3}} dt \\
\le &\ \frac{a}{2\mu+\lambda}(t_2 - t_1) + C(\bar\rho)C_0^{\frac{2}{3}}.
\end{aligned}$$

Applying Lemma 5 again leads to

$$\sup_{t\in[0,T]} \|\rho\|_{L^\infty} \leq \frac{7\bar{\rho}}{4},$$

for C_0 suitably small.

Collecting all these steps leads to the proof of Proposition 1.

The next key step is the following time-dependent estimates on the spatial gradient of the smooth solution (ρ, u).

Proposition 2: Under the assumptions of Proposition 1, the following estimates hold

$$\sup_{0\le t\le T}\int_{\mathbb{R}^3}\rho|\dot{u}|^2dx+\int_0^T\int_{\mathbb{R}^3}|\nabla\dot{u}|^2dxdt\le C,$$

$$\sup_{0\le t\le T}(||\nabla\rho||_{L^2\cap L^6}+||\nabla u||_{H^1})+\int_0^T||\nabla u||_{L^\infty}dt\le C,$$

where the positive constant C depends on T.

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To see this, one recall a Beal-Kato-Majda type inequality,

$$\begin{aligned} ||\nabla u||_{L^\infty(\mathbb{R}^3)} \leq\ & C\left(||\mathrm{div}\, u||_{L^\infty(\mathbb{R}^3)} + ||\mathrm{curl}\, u||_{L^\infty(\mathbb{R}^3)}\right) \\ & \log\left(e + ||\nabla^2 u||_{L^q(\mathbb{R}^3)}\right) + C||\nabla u||_{L^2(\mathbb{R}^3)} + C \end{aligned}$$

for all $\nabla u \in L^2(\mathbb{R}^3) \cap D^{1,q}(\mathbb{R}^3)$, $q \in (3, \infty)$. Note that

$$||\nabla^2 u||_{L^p} \leq C(||\rho \dot{u}||_{L^p} + ||\nabla p||_{L^p}), \quad p \in [2, 6]$$

which follows from the momentum equations regarded as an elliptic system.

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Thus

$$\begin{aligned} ||\nabla u||_{L^\infty(\mathbb{R}^3)} \leq \; & C(||\mathrm{div}\, u||_{L^\infty} + ||\mathrm{curl}\, u||_{L^\infty}) \log(e + ||\nabla \dot{u}||_{L^2}) \\ & + C(||\mathrm{div}\, u||_{L^\infty} + ||\mathrm{curl}\, u||_{L^\infty}) \log(e + ||\nabla \rho||_{L^6}) + C \end{aligned}$$

Since

$$\partial_t ||\nabla \rho||_{L^\rho} \leq C(1 + ||\nabla u||_{L^\infty}) ||\nabla \rho||_{L^p} + C||\nabla^2 u||_{L^p}$$

as it follows from the continuity equation, one gets

$$f'(t) \leq Cg(t)f(t) + Cg(t)f(t)\log f(t) + Cg(t)$$

where

$$\begin{aligned} f(t) & \triangleq e + ||\nabla \rho||_{L^6}, \\ g(t) & \triangleq 1 + (||\mathrm{div}\, u||_{L^\infty} + ||\mathrm{curl}\, u||_{L^\infty}) \log(e + ||\nabla \dot{u}||_{L^2}) + ||\nabla \dot{u}||_{L^2} \end{aligned}$$

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Note that

$$\int_0^T g(t)\,dt \le C\int_0^T ||\nabla \dot{u}||_{L^2}^2\,dt \le C.$$

Thus, the logarithmic Gronwall's inequality leads

$$\sup_{0\le t\le T} ||\nabla\rho||_{L^6(\mathbb{R}^3)} \le C,$$

and

$$\int_0^T ||\nabla u||_{L^\infty}\,dt \le C.$$

The rest of the Proposition 2 follows easily.

With Proposition 1 and Proposition 2 at hand, the high order estimates can be obtained in a similar way as in the analysis of blow-up criterions. Indeed, one has

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Time-dependent high norm estimates:

Proposition 3: There is a positive constant $C = C(T)$ such that

$$\sup_{0\le t\le T}\int \rho|\partial_t u|^2 dx + \int_0^T\int |\nabla\partial_t u|^2 dx\, dt \le C;$$

$$\sup_{t\in[0,T]}(||\rho - \rho^2||_{H^2} + ||p(\rho) - p(\tilde{\rho})||_{H^2}) \le C;$$

$$\sup_{t\in[0,T]}(||(\partial_t\rho, \partial_t P)||_{H^1} + \int_0^T ||(\partial_t^2\rho, \partial_t^2 P)||_{L^2}^2)dt \le C;$$

$$\sup_{t\in[0,T]}||(\rho - \bar{\rho}, P - P(\tilde{\rho}))||_{H^3} \le C;$$

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$$\sup_{t\in[0,T]} (||\nabla\partial_t u||_{L^2} + ||\nabla u||_{H^2})$$
$$+ \int_0^T (||\rho\partial_t^2 u||_{L^2}^2 + ||\nabla\partial_t u||_{H^1}^2 + ||\nabla u||_{H^3}^2)dt \leq C;$$

and $\forall\tau \in (0,T]$, $\exists C = C(\tau,T)$ such that

$$\sup_{t\in[\tau,T]} (||\nabla\partial_t u||_{H^1} + ||\nabla^4 u||_{L^2}) + \int_\tau^T ||\nabla\partial_t^2 u||_{L^2}^2 dt \leq C(\tau,T).$$

Thank You!