Regularity results for optimal patterns in the branched transportation problem

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Outline

Monge's Problem

Problem (Monge's Problem, 1781)

- *μ^+, μ^- probability measures on $\mathbb{R}^N$;*
- *minimize*

$$M(t) := \int_{\mathbb{R}^N} |x - t(x)|^p \, \mathrm{d}\mu^+(x)$$

among measurable maps $t : \mathbb{R}^N \to \mathbb{R}^N$ such that $\mu^-(B) = \mu^+(t^{-1}(B))$ (transport maps).

- the value of $M(t)$ is the average transportation cost of moving the mass in x to its final position $t(x)$.
- the value $t(x)$ of the transport map in x tells only the final position of the mass in x; there is no information about the path done by the mass during the transportation.
- the mass implicitly moves on straight lines.

Kantorovich's Problem

Problem (Kantorovich's Problem, 1940)

- μ^+, μ^- *probability measures on* $\mathbb{R}^N$*;*
- *minimize*

$$K(\pi) := \int_{\mathbb{R}^N \times \mathbb{R}^N} |x - y|^p \, \mathrm{d}\pi(x, y)$$

among positive Borel measures on $\mathbb{R}^N \times \mathbb{R}^N$ *such that* $\mu^+(A) = \pi(A \times \mathbb{R}^N)$, $\mu^-(B) = \pi(\mathbb{R}^N \times B)$ *(transport plans).*

- If t is transport map, $\pi_t(A \times B) := \mu^+(A \cap t^{-1}(B))$ is a transport plan and $M(t) = K(\pi_t)$;
- $\pi(A \times B)$ is the amount of mass in A (w.r.t. μ^+) moved to B (w.r.t. μ^-);
- no information on the path followed by the mass;
- one can assume that the transportation is on straight lines.

Branched transportation problems

- Many natural systems show a distinctive tree-shaped structure: plants, trees, drainage networks, root systems, bronchial and cardiovascular systems.
- These systems could be described in terms of mass transportation, but Monge-Kantorovich theory is not suitable since the mass is carried from the initial to the final point on a straight line.

Branched transportation problems

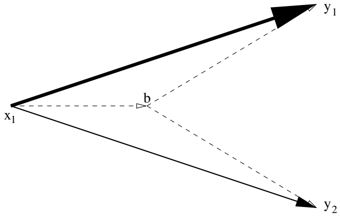
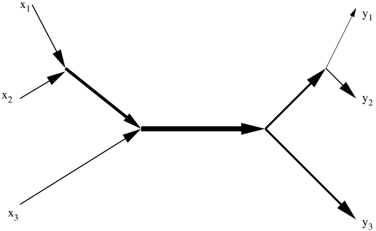


Figure: V-shaped versus Y-shaped transport.

The functional (discrete case)

- $\Omega \subset \mathbb{R}^N$ compact, convex;
- $\mu^+ = \sum_{i=1}^m a_i \delta_{x_i}, \mu^- = \sum_{j=1}^n b_j \delta_{y_j}$ convex combinations of Dirac masses;
- G weighted directed graph; $\operatorname{spt} \mu^+, \operatorname{spt} \mu^- \subseteq V(G)$;
- the mass flows from the initial measure μ^+ to the final measure μ^- "inside" the edges of the graph G.



The functional (discrete case)

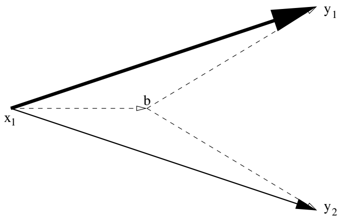
- The point is now to provide to each transport path G a suitable cost that makes keeping the mass together cheaper. The right cost function is

$$J_\alpha(G) := \sum_{e \in E(G)} [m(e)]^\alpha l(e),$$

$l(e)$ length of edge e, $0 \leq \alpha < 1$ fixed;

- this cost takes advantage of the subadditivity of the function $t \mapsto t^\alpha$ in order to make the tree-shaped graphs cheaper.

Branched transportation problems



$$m_1^\alpha |x_1 - y_1| + m_2^\alpha |x_1 - y_2| \geq |x - b| + m_1^\alpha |b - y_1| + m_2^\alpha |b - y_2|$$

The functional (continuous case)

- e (oriented edge) $\mapsto \mu_e = (\mathcal{H}^1_{|e})\hat{e}$ (vector measure);
- $G \mapsto T_G = \sum_{e \in E(G)} m(e)\mu_e$;
- div $T_G = \mu^+ - \mu^-$ sums up all conditions;
- a general irrigation pattern is defined by density and the cost as a lower semicontinuous envelope:

$$J_\alpha(T) = \inf_{T_{G_i} \to T} \liminf_{i \to +\infty} J_\alpha(T_{G_i}).$$

The Irrigation Problem

Problem (Irrigation problem)

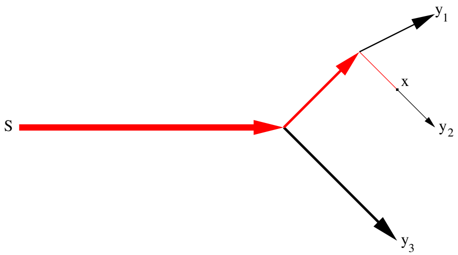
- *μ^+, μ^- probability measures on $\mathbb{R}^N$;*
- *minimize $J_\alpha(T)$ among irrigation patterns T such that* $\operatorname{div} T = \mu^+ - \mu^-$.

A pattern minimizing J_α is the best branched structure between the source μ^+ and the irrigated measure μ^-.

The landscape function

$\mu^+ = \delta_S$. For optimal graphs the landscape function Z is given by

$$Z(x) = \sum_{\text{path from } S \text{ to } x} [m(e)]^{\alpha-1} l(e).$$

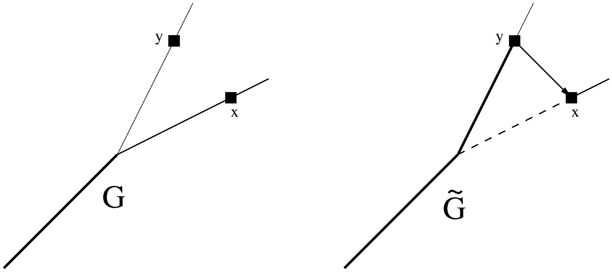


The landscape function can be defined also in the continuous setting.

Why to consider the landscape function?

- In a discrete form, the landscape function was already introduced in geophysics and is related to the problem of erosion and landscape equilibrium;
- the landscape function is related to first order variations of the functional J_α;
- the Hölder regularity of landscape function is related to the decay of the mass on the paths of the graph.

First order variations

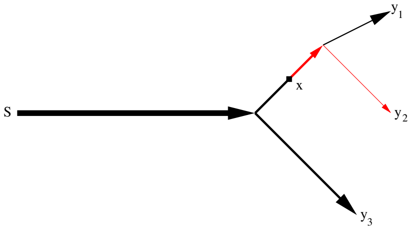


Theorem (First order gain formula)

The pattern $\tilde{G}$ satisfies

$$J_\alpha(\tilde{G}) - J_\alpha(G) \leq \alpha m(Z(y) - Z(x)) + m^\alpha |x - y|.$$

Mass decay on the graph



Theorem (Mass decay)

Suppose that G is optimal. If Z is Hölder continuous of exponent β, then the mass decay is exponent is $(1-\beta)/(1-\alpha)$:

$$m(x) \gtrsim l(x)^{\frac{1-\beta}{1-\alpha}},$$

and vice versa.

Hölder continuity when the irrigated measure is LAR

Definition

A measure μ is lower Ahlfors regular in dimension h if there exist $r_0 > 0$ and $c_A > 0$ such that:

$$\mu(B_r(x)) \geq c_A r^h \text{ for all } x \in \operatorname{spt} \mu \text{ and } 0 \leq r < r_0.$$

Theorem

Suppose that the irrigated measure is LAR in dimension h. Then, the landscape function Z is Hölder with exponent $\beta = 1 + h(\alpha - 1)$.

Some example shows that the landscape regularity may be better and may depend on the source of irrigation.

Best estimate on the Hölder exponent

Definition

A measure μ is upper Ahlfors regular in dimension h if there exists $C_A > 0$ such that:

$$\mu(B_r(x)) \leq C_A r^h \text{ for all } r > 0.$$

Theorem

Suppose that the irrigated measure is UAR above in dimension h and the landscape function Z is Hölder with exponent β. Then, $\beta \leq 1 + h(\alpha - 1)$.

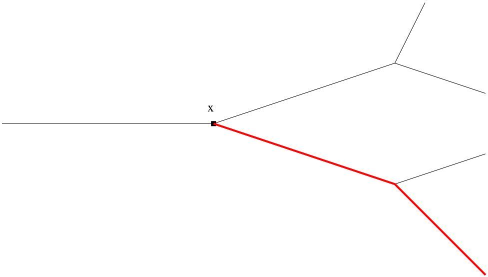
If the irrigated measure is Ahlfors regular in dimension h (both LAR and UAR), the best Hölder exponent is $1 + h(\alpha - 1)$.

Main branches from a point

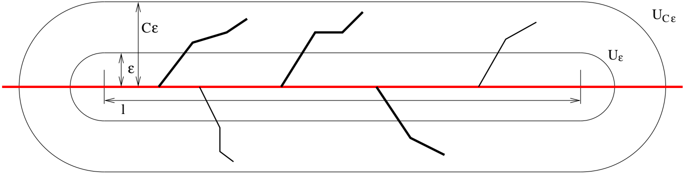
In the next the irrigated measure will always be *Ahlfors regular* in dimension h.

Definition (Main branches from a point x)

A *main branch* starting from a point x is the branch maximizing the residual length.

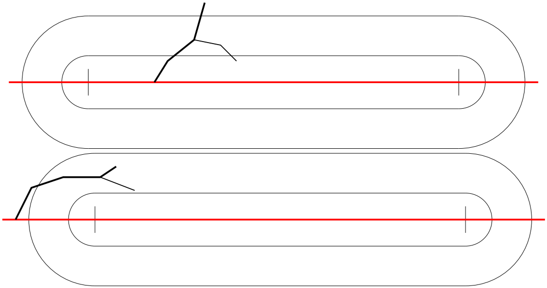


Fractal regularity



- N = number of branches bifurcating of residual length between ε and $C\varepsilon$.
- mass carried by one of such branches $\gtrsim \varepsilon^h$,
- mass of the tubolar neighbourhood of radius $C\varepsilon \sim l\varepsilon^{h-1}$,
- mass balance: $\varepsilon^h N \lesssim l\varepsilon^{h-1}$,
- $N \lesssim \frac{l}{\varepsilon}$.

Fractal regularity



- For small ε and a suitable choice of C the measure irrigated by "long branches" and by "far away branches" is a fraction of the measure of $U_{C\varepsilon} \setminus U_\varepsilon$;
- the mass carried by a branch of residual length between ε and $C\varepsilon$ is $\lesssim \varepsilon^h$;
- mass balance condition: $l\varepsilon^{h-1} \lesssim N\varepsilon^h$;
- $N \gtrsim \frac{l}{\varepsilon}$.

Bibliography

- A. Brancolini, S. Solimini.
  On the Hölder regularity of the landscape function.
  To appear on IFBj.

- A. Brancolini, S. Solimini.
  Fractal regularity results on optimal patterns.
  In preparation.

- F. Santambrogio.
  Optimal channel networks, landscape function and branched transport.
  IFBj (9), 2007.