Elastic deformations on the plane and approximations

(lecture I)

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• Lecture I: *Mappings of finite distortion and orientation-preserving homeomorphisms*.

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• Lecture VI: *Bi-Lipschits extension Theorem (part 2)*.

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A Jordan curve can be oriented clockwise or counterclockwise.

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Important consequence: to determine whether u is orientation preserving, it is enough to check $u_{|\partial\Omega}$.

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Ok, but *how much* is "smooth enough"?

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If $u : \mathbb{R}^n \supseteq \Omega \to \mathbb{R}^n$ admits a differential at x and $\det Du(x) > 0$, then we call distorsion of u at x the number

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$$K_u(x) = \frac{\big|Du(x)\big|^n}{n^{n/2}\det Du(x)} \,.$$

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There is a huge bibliography on this (e.g. Ball, Csorney, Hencl, Iwaniec, Koskela, Maly, Sbordone. . .).

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- $e^{\lambda K} \in L^1_{\rm loc}$,

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- For an homeomorphism u, u Sobolev $\not\Longrightarrow$ u^{-1} Sobolev
- Can we say that orientation preserving imply $\det Du > 0$ a.e.?

Thank you