Elastic deformations on the plane and approximations

(lecture II)

Aldo Pratelli

Department of Mathematics, University of Pavia (Italy)

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Plan of the course

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• Lecture VI: *Bi-Lipschits extension Theorem (part 2)*.

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BAD NEWS: Convolution does *not* work! (unless u, $u^{-1} \in W^{2,\infty}$)
(Example by Seregin and Shilkin)

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BAD NEWS: Even taking "randomly" arbitrarily many points does *not* work!

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All this works with the distance

$$d(u,v) = d^*_{L^\infty}(u,v) = \left\|u - v\right\|_{L^\infty} + \left\|u^{-1} - v^{-1}\right\|_{L^\infty}.$$

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So our *dream result* is to take $u,\ u^{-1} \in W^{1,p}$, and approximate with $d = d^*_{W^{1,p}}$.

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- Why doesn't it work for the inverse?

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Theorem (Daneri, P.): Let $u : \partial D \to \mathbb{R}^2$ be L bi-Lipschitz. Then there exists an extension $u : \mathcal{D} \to \mathbb{R}^2$ which is CL^4 bi-Lipschitz.