Elastic deformations on the plane and approximations

(lecture III)

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Plan of the course

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• Lecture IV: *The approximation result*.

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- Lecture V: *Bi-Lipschits extension Theorem (part 1)*.

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• Lecture VI: *Bi-Lipschits extension Theorem (part 2)*.

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- if u is L bi-Lipschitz, then v is $CL^{7/3}$ bi-Lipschitz.

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- Region 3: $v\big(\gamma(\sigma,t)\big) = \tilde\gamma\big(\xi_{[x_0(t),x_1(t),\tilde x_0,\tilde x_1,\alpha,\beta]}(\sigma)\,,\,\tilde t\big)$.

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- Region 4: $v = u$.