Weak and Measure-Valued Solutions of the Incompressible Euler Equations

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Outline

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Incompressible Euler Equations

The Cauchy problem for the incompressible Euler equations of inviscid fluid motion reads

$$\begin{aligned} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p &= 0 \\ \operatorname{div} v &= 0 \\ v(\cdot, 0) &= v_0 \end{aligned}$$

where $v : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ and $p : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$ are sought for and $v_0 : \mathbb{R}^d \to \mathbb{R}^d$ is a given initial velocity field with $\operatorname{div} v_0 = 0$.

Weak Formulation

We say that $v \in L^2_{loc}(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R}^d)$ is a weak solution with initial data $v_0 \in L^2(\mathbb{R}^d)$ if

$$\int_0^\infty \int_{\mathbb{R}^d} (v \cdot \partial_t \phi + v \otimes v : \nabla \phi) dx dt + \int_{\mathbb{R}^d} v_0(x) \phi(x, 0) dx = 0$$

for every $\phi \in C_c^\infty(\mathbb{R}^d \times [0, \infty); \mathbb{R}^d)$ with $\operatorname{div} \phi = 0$ and

$$\int_{\mathbb{R}^d} v(x, t) \cdot \nabla \psi(x) dx = 0$$

for a.e. $t \in \mathbb{R}^+$ and every $\psi \in C_c^\infty(\mathbb{R}^d)$.

Motivation for Measure-Valued Solutions

Vanishing viscosity method:

- Solve the Cauchy problem for Navier-Stokes with viscosity $\epsilon > 0$
- Send $\epsilon \to 0$
- Show that the corresponding solutions v^ϵ converge to v, and that v is a weak solution of Euler.

This approach fails! The problem is that we can not pass to the limit in the nonlinearity due to conceivable oscillation and concentration effects. Measure-valued solutions (mvs) are designed to capture complex oscillation and concentration phenomena and thus to overcome this problem.

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Young Measures

A (generalised) Young measure on $\mathbb{R}^d$ with parameters in $\mathbb{R}^m$ is a triple $(\nu_x, \lambda, \nu_x^\infty)$, where

- $\nu_x \in \mathcal{P}(\mathbb{R}^d)$ for a.e. $x \in \mathbb{R}^m$ (oscillation measure)
- $\lambda \in \mathcal{M}^+(\mathbb{R}^m)$ (concentration measure)
- $\nu_x^\infty \in \mathcal{P}(S^{d-1})$ for λ-a.e. $x \in \mathbb{R}^m$ (concentration-angle measure)

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Generation

Let (v_n) be a sequence of maps $\mathbb{R}^m \to \mathbb{R}^d$ which is bounded in $L^2(\mathbb{R}^m)$. We say that (v_n) generates the Young measure $(\nu_x, \lambda, \nu_x^\infty)$ if

$$f(v_n)dx \overset{*}{\rightharpoonup} \left(\int_{\mathbb{R}^d} f(z) d\nu_x(z) \right) dx + \left(\int_{S^{d-1}} f^\infty(\theta) d\nu_x^\infty(\theta) \right) \lambda$$

in the sense of measures for every suitable $f : \mathbb{R}^d \to \mathbb{R}$. Here,

$$f^\infty(\theta) = \lim_{s \to \infty} \frac{f(s\theta)}{s^2}$$

is the recession function of f.

The Fundamental Theorem

Fundamental Theorem of Young Measures (DiPerna-Majda '87, Alibert-Bouchitté '97)

If (v_n) is a bounded sequence in $L^2(\mathbb{R}^m;\mathbb{R}^d)$, then there exists a subsequence which generates some Young measure $(\nu_x,\lambda,\nu_x^\infty)$, i.e.

$$f(v_n)dx \overset{*}{\rightharpoonup} \left(\int_{\mathbb{R}^d} f(z)d\nu_x(z)\right)dx + \left(\int_{S^{d-1}} f^\infty(\theta)d\nu_x^\infty(\theta)\right)\lambda.$$

Basic Examples ($m = d = 1$)

Example 1. (Oscillation)

$$v(x) = \begin{cases} +1 & \text{if } x \in [k, k+\frac{1}{2}), \\ -1 & \text{if } x \in [k+\frac{1}{2}, k+1) \end{cases}$$

and $v_n(x) = v(nx)$. Then clearly

$$f(v_n) \overset{*}{\rightharpoonup} \frac{1}{2}f(+1) + \frac{1}{2}f(-1) = \int_{\mathbb{R}} f(z)d\nu(z)$$

with $\nu = \frac{1}{2}\delta_{+1} + \frac{1}{2}\delta_{-1}$. Moreover $\lambda = 0$.

Example 2. (Concentration)

$$v_n(x) = \sqrt{n}\chi_{\left[-\frac{1}{2n};\frac{1}{2n}\right]}.$$

Then $\nu_x = \delta_0$ for a.e. x, $\lambda = \delta_0$, $\nu_0^\infty = \delta_{+1}$.

Example 3. (Concentration in various directions)

$$v_n(x) = \sqrt{n}\left(\chi_{\left[-\frac{1}{2n};0\right]} - \chi_{\left[0;\frac{1}{2n}\right]}\right).$$

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Example 4. (Diffuse concentration)

$$v_n(x) = \sqrt{n}\sum_{k=0}^{n-1}\chi_{\left[\frac{k}{n};\frac{k}{n}+\frac{1}{n^2}\right]}.$$

Then $\nu_x = \delta_0$ for a.e. x, $\lambda = \chi_{[0;1]}dx$, $\nu_x^\infty = \delta_{+1}$ for a.e. $x \in [0;1]$.

Example 5. (Strong convergence) If $v_n \to v$ strongly in L^2, then the v_n generate the Young measure $\nu_x = \delta_{v(x)}$, $\lambda = 0$.

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Measure-Valued Solutions for Euler

Let now, as before, (v^ϵ) be a sequence of weak (Hopf-Leray) solutions for Navier-Stokes with $\epsilon \to 0$ and $v^\epsilon(t=0) = v_0$. Since

$$\sup_{\epsilon>0} \sup_{t\geq 0} \int_{\mathbb{R}^d} |v^\epsilon(x,t)|^2 dx < \infty,$$

we can apply the Fundamental Theorem of Young measures to obtain $(\nu_{x,t}, \lambda, \nu^\infty_{x,t})$ such that

$$\begin{aligned} \partial_t \langle \nu_{x,t}, z \rangle + \operatorname{div} \left(\langle \nu_{x,t}, z \otimes z \rangle + \langle \nu^\infty_{x,t}, \theta \otimes \theta \rangle \lambda \right) + \nabla p(x,t) = 0 \\ \operatorname{div} \langle \nu_{x,t}, z \rangle = 0 \end{aligned}$$

in the sense of distributions. Here, we wrote $\langle \nu, f(z) \rangle := \int f(z) d\nu(z)$ and similarly for ν^∞.

Definition

A Young measure $(\nu_{x,t}, \lambda, \nu^{\infty}_{x,t})$ is called a *measure-valued solution (mvs)* of the Euler equations if

$$\partial_t \langle \nu_{x,t}, z\rangle + \operatorname{div}\left(\langle \nu_{x,t}, z\otimes z\rangle + \langle \nu^{\infty}_{x,t}, \theta\otimes\theta\rangle\lambda\right) + \nabla p(x,t) = 0$$
$$\operatorname{div}\langle \nu_{x,t}, z\rangle = 0$$

in the sense of distributions.

It is known that a mvs can be altered on a t-set of measure zero such that $t \mapsto \langle \nu_{x,t}, z\rangle$ is in $C([0,\infty); L^2_w(\mathbb{R}^d))$. Therefore it makes sense to speak about the *initial barycentre*

$$\bar{v}(x,0) = \langle \nu_{x,0}, z\rangle.$$

If $\bar{v}(t=0) = v_0$, we say that $(\nu, \lambda, \nu^{\infty})$ is a mvs with initial data v_0.

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Admissibility

A mvs obtained by a sequence of Hopf-Leray solutions of Navier-Stokes admits a disintegration of the concentration measure:

$$\lambda(dxdt) = \lambda_t(dx) \otimes dt$$

for some $\lambda_t \in \mathcal{M}^+(\mathbb{R}^d)$. We can then define the energy of the mvs by

$$E(t) := \frac{1}{2}\int_{\mathbb{R}^d} \langle \nu_{x,t}, |z|^2 \rangle dx + \frac{1}{2}\lambda_t(\mathbb{R}^d).$$

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A mvs with initial data v_0 is called *admissible* if $\lambda = \lambda_t \otimes dt$ and

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Facts about Admissible Measure-Valued Solutions

- Any weak solution v defines a mvs through $\nu_{x,t} = \delta_{v(x,t)}$, $\lambda = 0$
- There exists at least one admissible mvs for each $v_0 \in L^2(\mathbb{R}^d)$ with $\operatorname{div} v_0 = 0$
- Weak-strong uniqueness: If there exists a sufficiently smooth solution for a certain initial data, then every admissible mvs with the same initial data coincides with it (Brenier-De Lellis-Székelyhidi '09)

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The Main Result

Theorem (Székelyhidi-E.W. '11)

Let $(\nu_{x,t}, \lambda, \nu^\infty_{x,t})$ *be an admissible mvs with initial data* v_0. *Then there exists a sequence* (v_n) *of weak solutions that generates* $(\nu, \lambda, \nu^\infty)$ *as a Young measure. In addition,*

$$\|v_n(t=0) - v_0\|_{L^2(\mathbb{R}^d)} < \frac{1}{n}$$

and

$$\sup_{t\geq 0} \frac{1}{2}\int_{\mathbb{R}^d} |v_n(x,t)|^2 dx \leq \frac{1}{2}\int_{\mathbb{R}^d} |v_n(x,0)|^2 dx.$$

Discussion of the Result

- A priori, mvs seem to be a much weaker concept than weak solutions. The Theorem shows however that they are in a sense the same.
- DiPerna and Majda constructed explicit examples for the development of oscillations and concentrations in sequences of weak solutions. The Theorem shows that in fact *any* conceivable oscillation/concentration behaviour can be realised by a sequence of weak solutions.
- The result gives an example of a characterisation of Young measures generated by constrained sequences where the constant rank property does not hold.
- As a corollary, we obtain the following existence result:

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An Existence Assertion

Corollary

Let $H = \{v \in L^2(\mathbb{R}^d) : \operatorname{div} v = 0\}$*. There exists a dense subset* $\mathcal{E} \in H$ *such that for all* $v_0 \in \mathcal{E}$ *there exists a weak solution with initial data* v_0 *such that*

$$\sup_{t \geq 0} \frac{1}{2} \int_{\mathbb{R}^d} |v(x,t)|^2 dx \leq \frac{1}{2} \int_{\mathbb{R}^d} |v_0(x)|^2 dx.$$

Ingredients of the Proof

1. Owing to a result of De Lellis and Székelyhidi, it suffices to construct so-called subsolutions, i.e. pairs (v, u) with $v(x, t) \in \mathbb{R}^d$ and $u(x, t) \in \mathbb{R}^{d\times d}_{sym}$ such that

$$\begin{aligned} \partial_t v + \operatorname{div} u + \nabla p &= 0 \\ \operatorname{div} v &= 0 \end{aligned} \tag{1}$$

2. Use more or less standard Young measure techniques to reduce to the case that ν, ν^∞ are discrete and independent of x and t and λ is a constant multiple of Lebesgue measure
3. In order to generate this discrete homogeneous measure, construct a laminate consistent with (1), again relying on tools developed by De Lellis and Székelyhidi.

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Open Questions

- Can we say anything more about the set $\mathcal{E}$ of "wild initial data"?
- Do admissible weak solutions exist for all $v_0 \in H$? Without the admissibility condition, this is known to be true (E.W. '11)
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