Existence of algebraic vortex spirals and ill-posedness of inviscid flow

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Compressible Navier-Stokes and Euler equations

$$
\begin{aligned}
\varrho_t + \nabla\cdot(\varrho\vec{v}) &= 0, && \text{[mass]}\\
(\varrho\vec{v})_t + \nabla\cdot(\varrho\vec{v}\otimes\vec{v}) \quad +\nabla p &= \nabla^T S, && \text{[momentum]}\\
(\varrho e)_t + \underbrace{\nabla\cdot(\varrho e\vec{v})}_{\text{convection}} \quad + \underbrace{\nabla\cdot(p\vec{v})}_{\text{pressure}} &= \underbrace{\nabla\cdot(S\vec{v})}_{\text{viscosity}} + \underbrace{\nabla\cdot(\kappa\nabla T)}_{\text{heat conduction}} && \text{[energy]}
\end{aligned}
$$

where ϱ density, $\vec{v}$ velocity, T temperature (functions of t, x)

$$
\begin{aligned}
S &= 2\mu\left(\frac{1}{2}(\nabla\vec{v} + \nabla\vec{v}^T) - \frac{1}{3}\nabla\cdot\vec{v}\right),\\
e &= q + \frac{1}{2}|\vec{v}|^2,\\
p, q, \kappa, \mu &= \text{functions of } \varrho, T.
\end{aligned}
$$

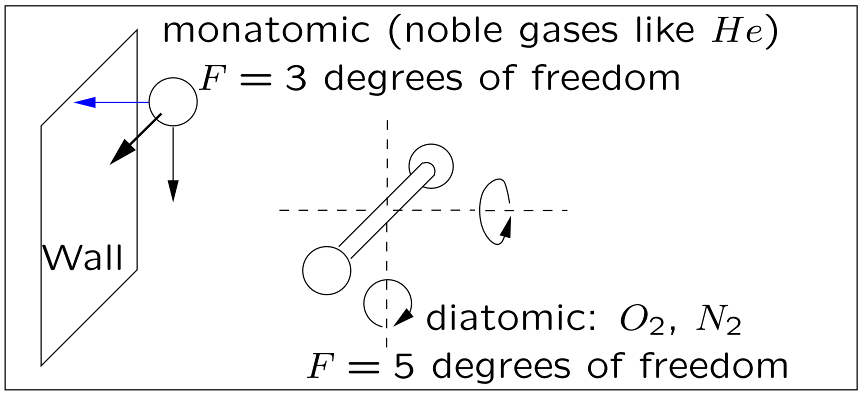
p pressure, q specific internal energy, e specific energy, S viscous stress. κ heat conductivity, μ viscosity coefficient.

Euler = Navier-Stokes without the blue terms.

Pressure law ("equation of state") (ϱ mass density, q heat per mass): Polytropic:

$$p(\varrho, q) = (\gamma - 1)\varrho q = \frac{2}{F}\varrho q$$

$\gamma = \frac{F+2}{F}$ where F is "number of degrees of freedom" per particle.
$\gamma = \frac{5}{3}$ for monatomic gas, $\gamma = \frac{7}{5}$ for diatomic gas, $\gamma = \frac{4}{3}$ otherwise (actual gas more complicated)



Boltzmann equipartition "theorem": equal time averages $\frac{1}{2}kT$ of kinetic energy $\frac{M}{2}v^2$ in each degree of freedom of each of N particles; only normal direction yields pressure on wall $\rightsquigarrow$ p formula

Entropy transport: consider smooth ϱ, v, q; $e = q + \frac{1}{2}|v|^2$.

$$
\begin{aligned}
& 0 = \varrho_t + \nabla \cdot (\varrho v) = \varrho_t + v \cdot \nabla \varrho + \varrho \nabla \cdot v \\
& 0 = (\varrho v)_t + \nabla \cdot (\varrho v \otimes v) + \nabla p \\
& \quad = \varrho v_t + v \varrho_t + \varrho v \cdot \nabla v + v \nabla \cdot (\varrho v) + \nabla p \\
\Rightarrow \quad & 0 = v_t + v \cdot \nabla v + \varrho^{-1} \nabla p \\
& 0 = (\varrho e)_t + \nabla \cdot (\varrho e v) + \nabla \cdot (p v) \\
& \quad = \varrho e_t + e \varrho_t + \varrho v \cdot \nabla e + e \nabla \cdot (\varrho v) + \nabla \cdot (p v) \\
\Rightarrow \quad & 0 = e_t + v \cdot \nabla e + \varrho^{-1} \nabla \cdot (p v) \\
& \quad = q_t + v_t \cdot v + v \cdot \nabla q + v \cdot \nabla v \cdot v + \varrho^{-1} p \nabla \cdot v + \varrho^{-1} \nabla p \cdot v \\
\Rightarrow \quad & 0 = q_t + v \cdot \nabla q + \varrho^{-1} p(\varrho, q) \nabla \cdot v
\end{aligned}
$$

$$
\begin{aligned}
s(\varrho, q)_t + v \cdot \nabla s(\varrho, q) &= s_\varrho(\varrho_t + v \cdot \nabla \varrho) + s_q(q_t + v \cdot \nabla q) \\
&= -\nabla \cdot v \Big(s_\varrho(\varrho, q)\varrho + s_q(\varrho, q)\varrho^{-1} p(\varrho, q) \Big)
\end{aligned}
$$

First-order PDE for $s(\varrho, q)$: method of characteristics. Example: most common choice $p = (\gamma - 1)\varrho q$ yields gas-dynamic entropy

$$
s = C_1 \Big(\log q + (1 - \gamma) \log \varrho \Big) + C_2 .
$$

Isentropic Euler: if s is constant in x at $t = 0$:

$$s_t + v \cdot \nabla s = 0,$$

hence same constant for all $t > 0$.
(False for non-smooth flow: shocks produce (physical) entropy.)

$$s = \text{constant} = C_1\Big(\log q + (1-\gamma)\log \varrho\Big) + C_2$$

$$\rightsquigarrow \qquad q = C(s)\varrho^{\gamma-1}, \qquad p(\varrho, q) = C\varrho q = C\varrho^{\gamma}$$

$$0 = \varrho_t + \nabla \cdot (\varrho v)$$
$$0 = (\varrho v)_t + \nabla \cdot (\varrho v \otimes v) + \nabla\Big(p(\varrho)\Big)$$

Smooth solutions are full (non-isentropic) Euler solutions.
Weak solutions are not; but close if shocks weak.

$$0 = v_t + v \cdot \nabla v + \varrho^{-1}\nabla(p(\varrho)) = v_t + v \cdot \nabla v + \nabla(\pi(\varrho))$$

$$\pi_\varrho = \frac{p_\varrho}{\varrho} \quad , \quad \pi(\varrho) = C'\varrho^{\gamma-1}$$

Potential flow (compressible)

Assume $\nabla \times v = 0$. Then $v = \nabla\phi$ (velocity potential ϕ). $\nabla^2 = \nabla\nabla^T$

$$0 = v_t + v \cdot \nabla v + \nabla(\pi(\varrho)) = \nabla\partial_t\phi + \nabla^2\phi\nabla\phi + \nabla(\pi(\varrho)) = \nabla(\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + \pi(\varrho))$$

$$\Rightarrow \partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + \pi(\varrho) = \text{const (Bernoulli)}$$

$$\varrho = \pi^{-1}(C - \partial_t\phi - \frac{1}{2}|\nabla\phi|^2), \qquad 0 = \varrho_t + \nabla \cdot (\varrho\nabla\phi)$$

$$0 = (\pi')^{-1}(-\phi_{tt} - \nabla\phi \cdot \nabla\phi_t) + (\pi')^{-1}\nabla\phi \cdot (-\nabla\phi_t - \nabla^2\phi\nabla\phi) + \varrho\Delta\phi$$

$$0 = -\phi_{tt} - 2\nabla\phi \cdot \nabla\phi_t - \nabla\phi^T\nabla^2\phi\nabla\phi + \underbrace{(\frac{d\pi}{d\varrho}\varrho)}_{c^2}\Delta\phi$$

$$0 = \begin{bmatrix} c^2 I - \nabla\phi\nabla\phi^T & -\nabla\phi \\ -\nabla\phi^T & -1 \end{bmatrix} : \hat{\nabla}^2\phi, \qquad \hat{\nabla} = (\nabla, \partial_t), \quad A : B = \text{tr}(A^T B)$$

Hyperbolic (if $c > 0$, true unless vacuum or strange pressure law):
Symmetric coefficient matrix, 1 negative, n positive eigenvalues

Symmetries

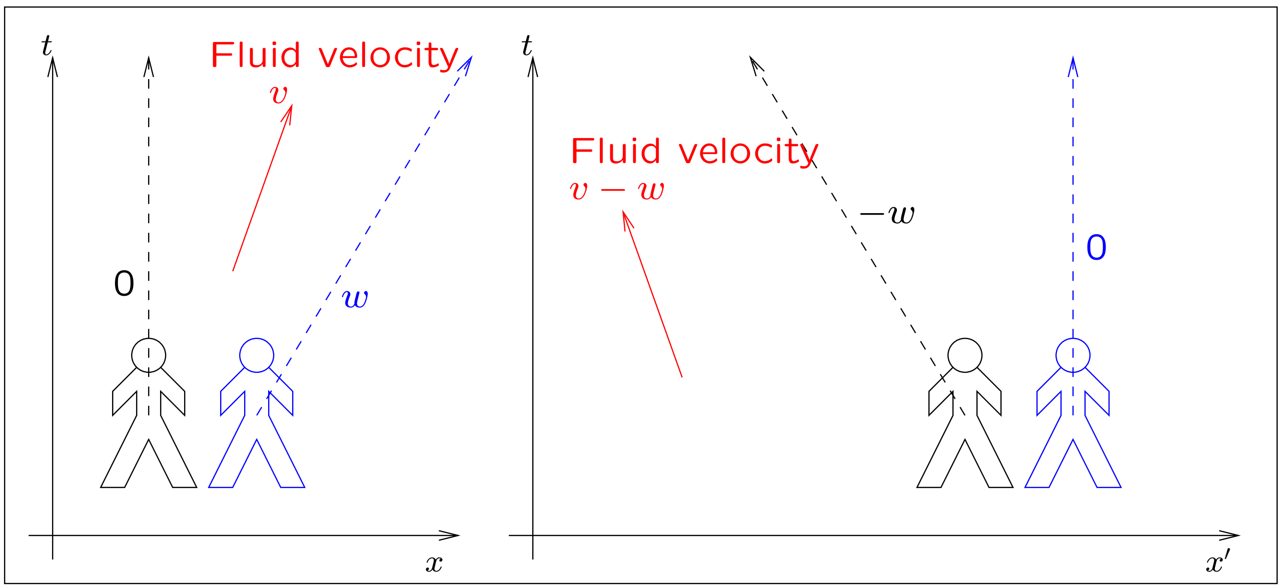
1. Rotation/reflection: Q orthogonal,

$$x' = Qx, \quad v'(x',t) = Qv(x,t), \quad \varrho'(x',t) = \varrho(x,t), \quad q'(x',t) = q(x,t)$$

Exercise: if v, ϱ, q solution, then v', ϱ', q' also.

2. Change of inertial frame: new origin at speed w relative to old,

$$x' = x - wt, \quad v' = v - w, \quad \varrho' = \varrho, \quad q' = q$$



Both combined: Galilean invariance (non-relativistic)

Navier-Stokes, Euler (compressible/not), potential flow ✓

(including weak/entropy solutions later).

For some p (polytropic): additional symmetries involving ρ, q.

Checking hyperbolic

$$0 = \begin{bmatrix} c^2 I - \nabla\phi\nabla\phi^T & -\nabla\phi \\ -\nabla\phi^T & -1 \end{bmatrix} : \hat{\nabla}^2\phi, \qquad \hat{\nabla} = (\nabla, \partial_t), \quad A : B = \mathrm{tr}(A^T B)$$

Change to coordinates of observer travelling with velocity $v = \nabla\phi$ $\rightsquigarrow$ his equation

$$0 = \begin{bmatrix} c^2 I & 0 \\ 0 & -1 \end{bmatrix} : \hat{\nabla}^2\phi, \qquad \hat{\nabla} = (\nabla, \partial_t), \quad A : B = \mathrm{tr}(A^T B)$$

Now obvious: n eigenvalues c^2, one eigenvalue -1.

Linear wave equation

$$0 = -\phi_{tt} - 2\nabla\phi \cdot \nabla\phi_t - \nabla\phi^T \nabla^2\phi\nabla\phi + c^2\nabla^2\phi$$

Linearize around $v = \nabla\phi \approx 0$: linear wave equation

$$0 = -\tilde{\phi}_{tt} - 0 - 0 + c^2\Delta\tilde{\phi}$$

Models sound waves (“acoustics”)

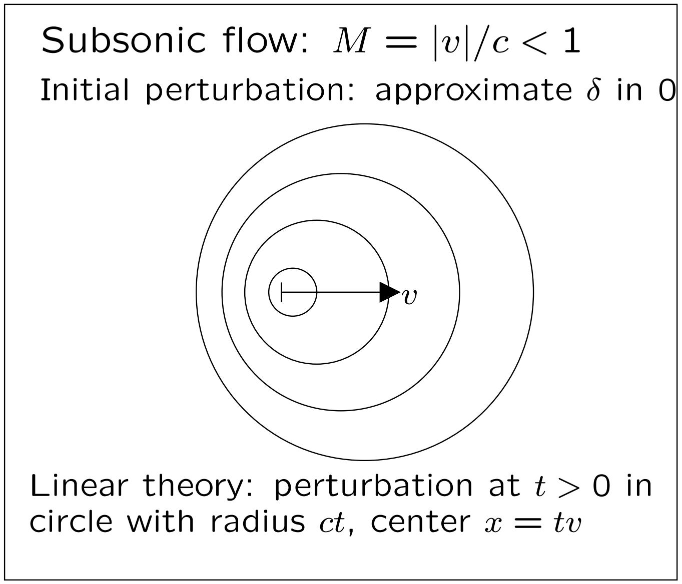
Linearize around $\nabla\phi \approx v = \text{const}$:

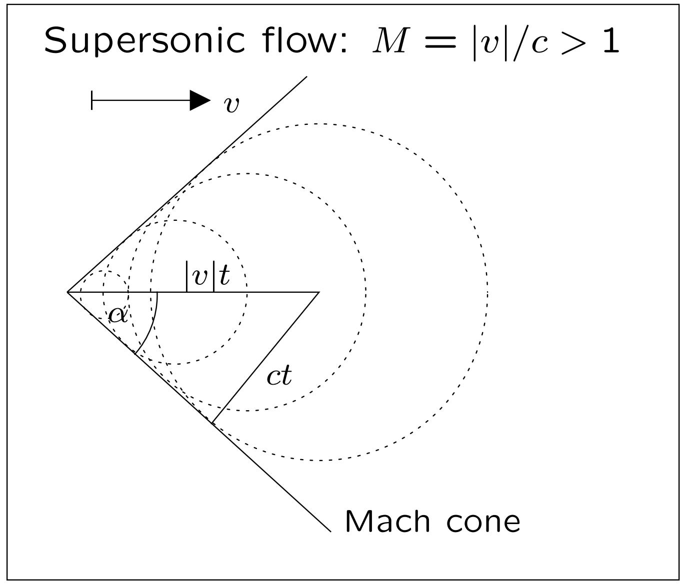
$$0 = -\tilde{\phi}_{tt} - 2v \cdot \nabla\tilde{\phi}_t + (c^2 - vv^T) : \nabla^2\tilde{\phi}$$

(Can obtain from $0 = c^2\Delta\tilde{\phi} - \tilde{\phi}_{tt}$ by “change of observer”.)

Subsonic/supersonic flow, Mach number

Given Euler solution, localized perturbation at $t = 0$, linearize:





Subsonic flow ($M < 1$): disturbances propagate in all directions

Supersonic: propagate (in linearization) only inside the Mach cone

$$\alpha = \arcsin \frac{ct}{|v|t} = \arcsin \frac{1}{M}$$

α Mach angle

$$\frac{y}{x} = \frac{\sin\alpha}{\cos\alpha} = \frac{1/M}{\sqrt{1-(1/M)^2}} = \frac{1}{\sqrt{M^2-1}}$$

Incompressible limit

$$p(\varrho) = \epsilon^{-1}\tilde{p}(\varrho) \qquad \epsilon \downarrow 0.$$

(Air: $c = 340\frac{m}{s}$, $\gg v$ in many applications)

$$c^2 = \frac{dp}{d\varrho}(\varrho) = \epsilon^{-1}\frac{d\tilde{p}}{d\varrho}(\varrho) \quad , \quad \pi(\varrho) = \epsilon^{-1}\tilde{\pi}(\varrho)$$

$$\varrho = \varrho_0 + \epsilon\varrho_1 + \ldots, \quad v = v_0 + \epsilon v_1 + \ldots \quad \rightsquigarrow \quad \tilde{\pi}(\varrho) = \pi_0 + \pi_1\epsilon + \ldots$$

$$0 = \rho_t + \nabla\cdot(\rho v) \quad , \quad 0 = v_t + \nabla\cdot(v\otimes v) + \epsilon^{-1}\nabla\tilde{\pi}$$

Order ϵ^{-1}: $\nabla\tilde{\pi}_0 = 0 \quad \Rightarrow \quad \varrho_0 = \text{const} > 0$

Order ϵ^0: $0 = \varrho_{0t} + \nabla\cdot(\varrho_0 v_0) \quad \Rightarrow \quad 0 = \nabla\cdot v_0$

$$0 = v_{0t} + \nabla\cdot(v_0\otimes v_0) + \nabla\pi_1$$

(requires smoothness; details: e.g. Klainerman/Majda, CPAM 1982)

Loosely speaking: Isentropic Euler = potential flow+ incompressible Euler

With viscosity: incompressible Navier-Stokes

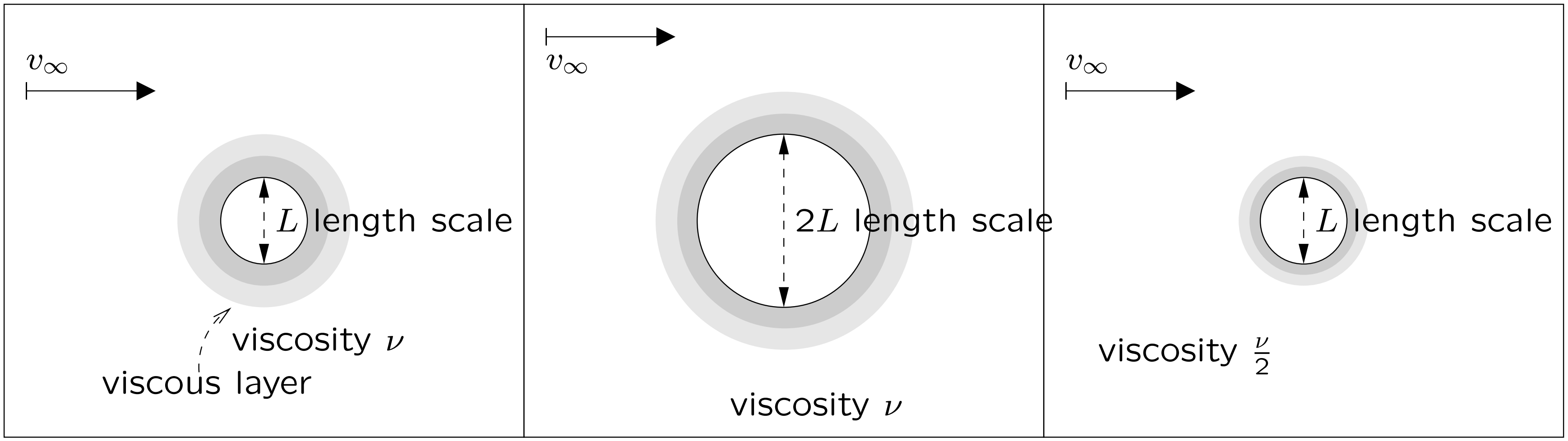
$$v_t + \nabla\cdot(v\otimes v) + \varrho^{-1}\nabla\pi = \nu\Delta v$$

Scaling

Consider steady incompressible Navier-Stokes:

$$\nabla \cdot (v \otimes v) + \nabla \pi = \nu \Delta v \quad , \quad \nabla \cdot v = 0$$

$$v = 0 \quad \text{on surface,} \qquad v \to v_\infty \quad \text{as } x \to \infty$$



Three parameters $(L, \nu, v_\infty > 0)$ reduced to one: Reynolds number:

$$\text{Re} = \frac{|v_\infty| L}{\nu} \quad \text{dimensionless}$$

Interesting limits: $|v_\infty| \to \infty$, or $L \to \infty$, or $\nu \downarrow 0$
all lead to incompressible Euler (formally)

Similar technique for compressible (more parameters)

Euler as a scaling limit

System of conservation laws for $U = (\varrho, \varrho\vec{v}, \varrho q)$:

$$\underbrace{\nabla \cdot \vec{f}(U)}_{\text{first-order}} = \nabla \cdot (A(U)\nabla U)$$

If U solution, then $U_\epsilon(\vec{x}) := U(\frac{\vec{x}}{\epsilon})$ [= considering large scale] solves

$$\epsilon\nabla \cdot \vec{f}(U_\epsilon) = \epsilon^2\nabla \cdot (A(U_\epsilon)\nabla U_\epsilon)$$

$$\nabla \cdot \vec{f}(U_\epsilon) = \epsilon\nabla \cdot (A(U_\epsilon)\nabla U_\epsilon)$$

Same principle for other higher-order terms (dispersive, ...).

At large scales, least-order terms “dominate”

Conservation laws:

$U = (\varrho, \varrho v^x, \varrho v^y, \varrho v^z, \varrho e)$ densities of mass, momentum, energy.

$$U_t + \nabla \cdot (f(U, \nabla U)) = 0$$

Formally: $\int dx \rightarrow$

$$0 = \frac{d}{dt} \int U(t,x) dx + \int \nabla \cdot (f(U)) dx = \frac{d}{dt} \int U(t,x) dx + 0$$

on compact boundary-less manifolds, e.g. $\mathbb{T}^d$ torus.

Complications:

1. boundaries (solid: no flow of mass, but flow of momentum; flow of energy if moving)
2. unbounded domains (mass infinite, must consider local conservation carefully)
3. source terms (gravitation in momentum/energy equation, ...):

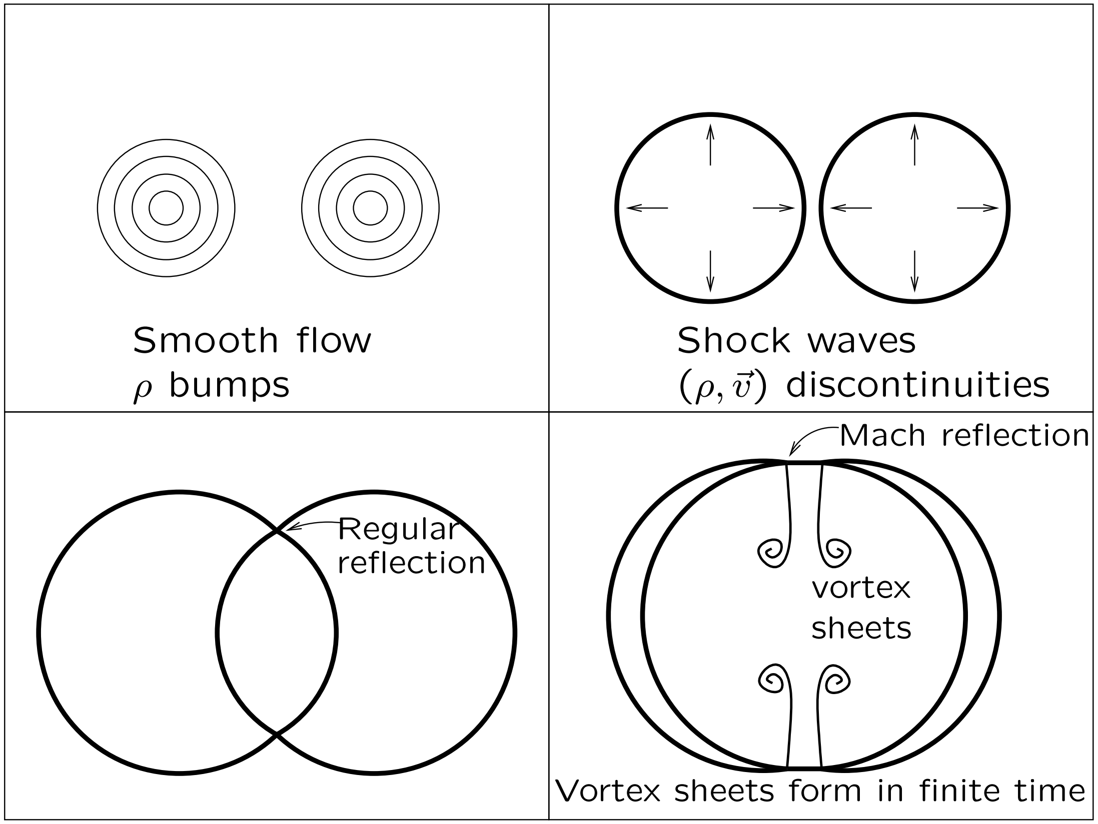
$$U_t + \nabla \cdot (f(U)) = g(U)$$

Balance laws

Discontinuity formation for compressible flow:

Shock waves: discontinuity in $\varrho, q, v \cdot n$.

Vortex sheets: discontinuity in $v \cdot t$.



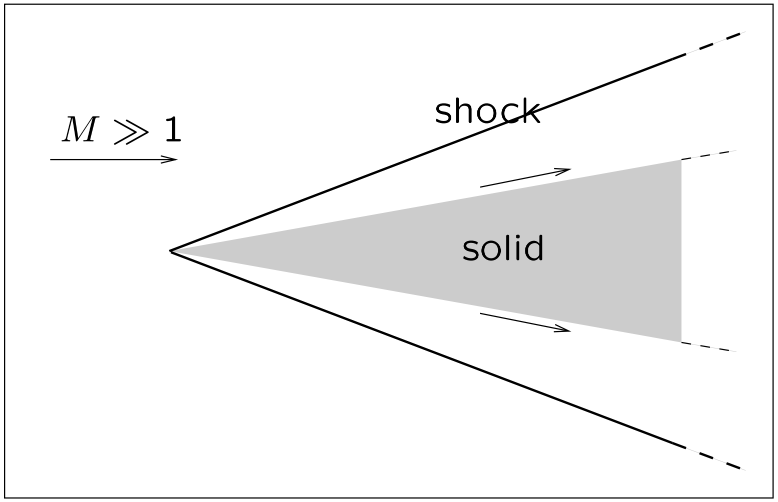
"It is not clear whether singularities form."

Not for incompressible Euler, but for compressible it is clear.

Long term goal: well-posedness theory for Euler and convergence theory for numerics.
$\longrightarrow$ Must deal with vortex sheets and shock waves.

Supersonic flow onto wedges

Concorde, military jets, space shuttle:



Challenge: find a notion of solution that includes non-differentiable and even discontinuous functions. Compressible Euler:

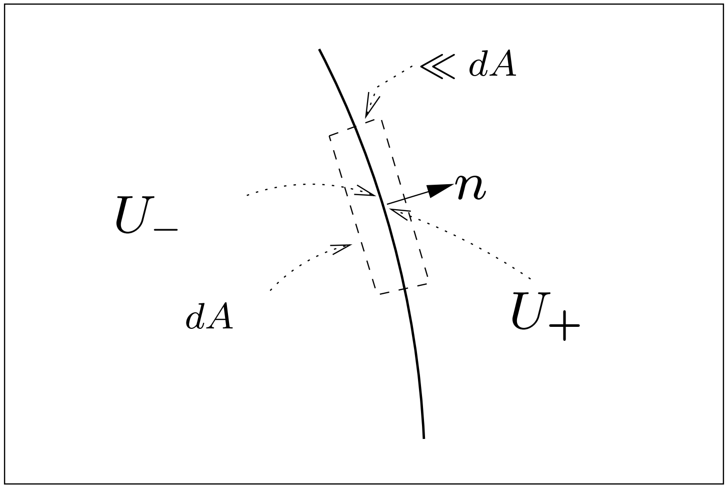
$$U_t + \nabla \cdot f(U) = 0 \qquad t \geq 0,\ x \in \mathbb{R}^d.$$

Multiply with smooth compactly supported ϕ, integrate:

$$0 = \int_0^\infty \int_{\mathbb{R}^d} \phi U_t + \phi \nabla \cdot f(U) dx\, dt = -\int_0^\infty \int_{\mathbb{R}^d} \phi_t U + f(U) \cdot \nabla \phi dx\, dt - \int_{\mathbb{R}^d} (U\phi)_{|t=0} dx$$

U "weak solution" if satisfied for all ϕ.

Discontinuities as weak solutions



Flux into (left): $f(U_-) \cdot n \; dS \; dt$.

Flux out (right): $f(U_+) \cdot n \; dS \; dt$.

$\ll |dA|$ side: neglect

Conservation $\Rightarrow$ must be equal:

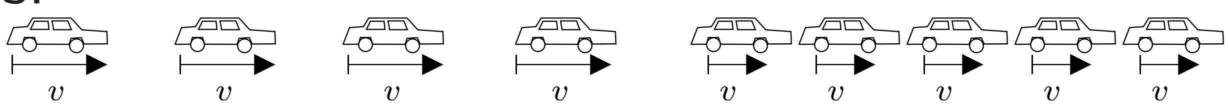
Rankine-Hugoniot condition:

$$\Big(f(U_+) - f(U_-)\Big) \cdot n = 0$$

For moving shocks (speed σ): $\Big(f(U_+) - f(U_-)\Big) \cdot n = \sigma(U_+ - U_-)$.

$$[f(U) \cdot n] = \sigma[U]$$

Traffic jams:



Whitham traffic flow model: car density $\varrho \geq 0$ (scalar), velocity $v(\varrho) = \max\{1 - \varrho, 0\}$, flux $f(\varrho) = \varrho v(\varrho)$

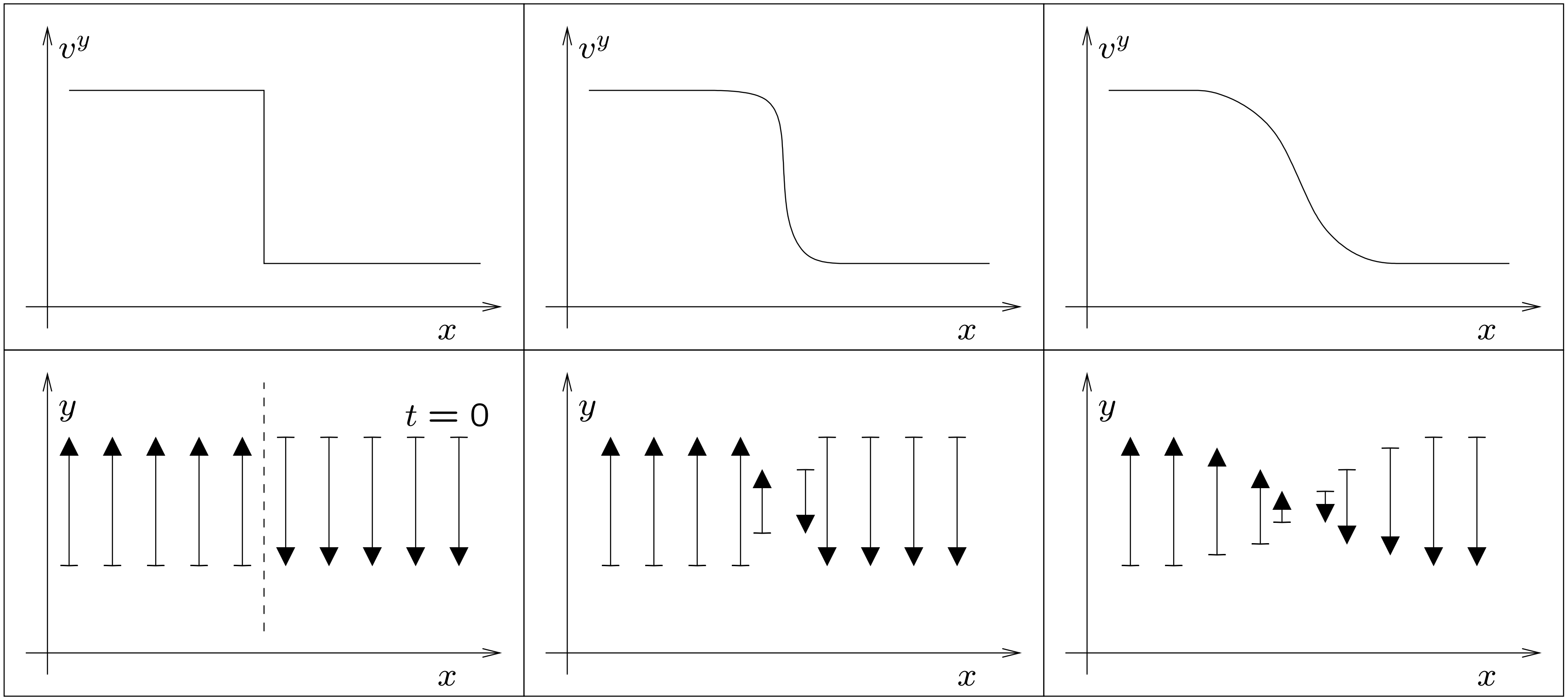
$$0 = \varrho_t + f(\varrho)_x = \varrho_t + f_\varrho(\varrho)\varrho_x$$

$\rightsquigarrow$ characteristics wave speed $f_\varrho(\varrho) = 1 - 2\varrho \qquad (\varrho \in [0, 1])$

Wave speed depends on state of medium $\to$ discontinuities may form

Compressible Euler (1d): wave speeds $v - c(\varrho), v, v + c(\varrho)$

Contact discontinuities. 2-d flow:



$t = 0$ $\qquad$ $t \gtrapprox 0$ $\qquad$ $t > 0$

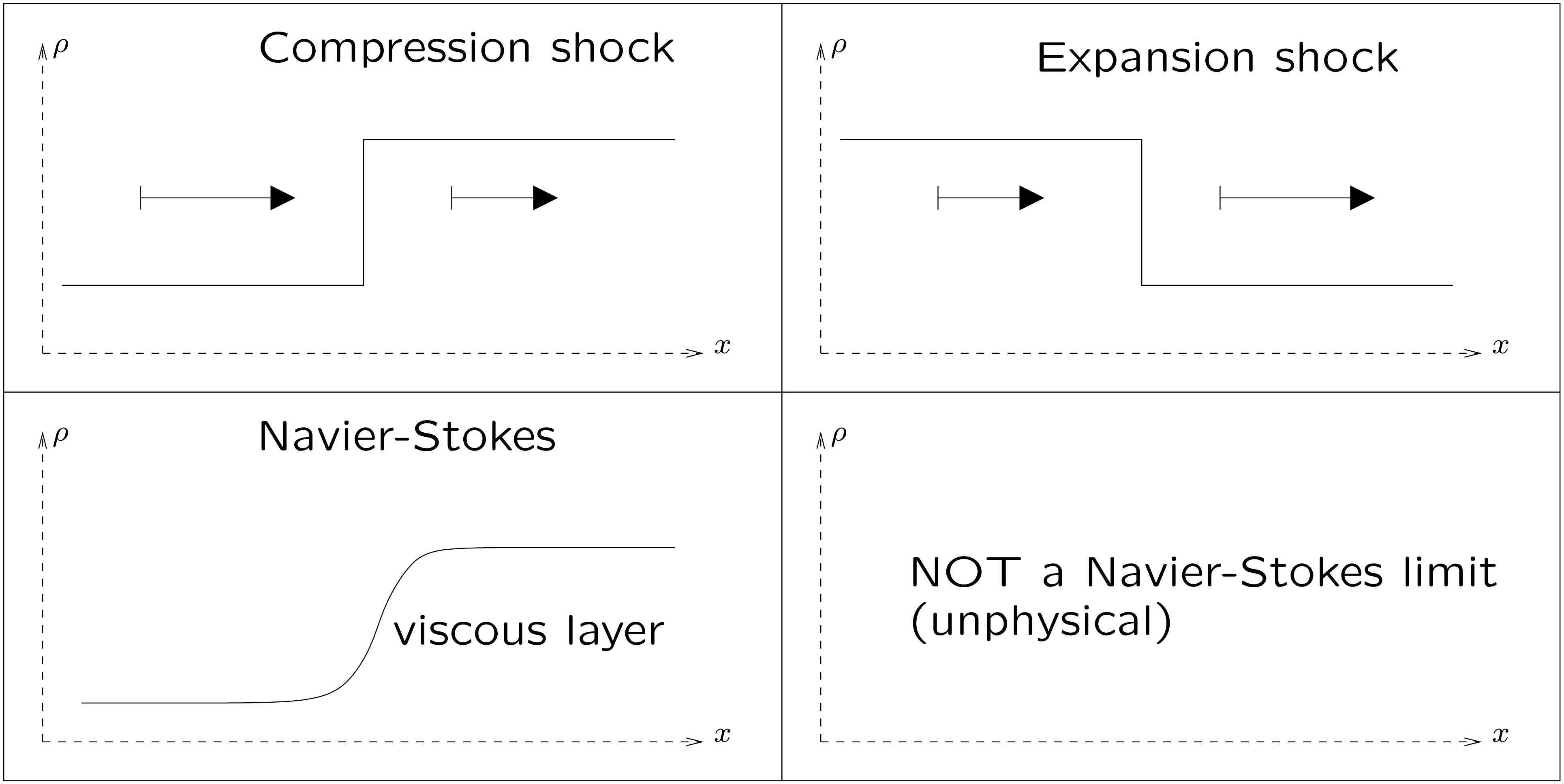
$v^x = v^z = 0$, $v^y = v^y(x)$ in incompressible Navier-Stokes:

$$v^y_t = \epsilon v^y_{xx} \quad \Rightarrow \quad v^y(t,x) = v^y\Big(\frac{1}{\sqrt{t\epsilon}}x\Big).$$

Compressible flow: analogous viscous profiles (more complicated)

Another type of contact: entropy jumps: $p \sim \varrho T$, $[p] = 0$, $[\varrho], [T] \neq 0$

Compression and expansion shocks



Shock wave: "width" scales like $\frac{1}{\epsilon}$.

Admissibility conditions

Fluid dynamics main/only source of justifications for definitions.

[Arnold: geodesics on Diff_0; Slemrod et al: link between Euler, isometric embedding]

Justification is informal, rigorous arguments only supporting role.

Vanishing viscosity condition: admissible $= \epsilon \downarrow 0$ limit (in some sense) of solutions of

$$\text{Euler} + \epsilon \cdot \text{perturbation (Navier-Stokes, Boltzmann, ...)}$$

Entropy condition: $\eta, \vec{\psi}$ entropy-entropy flux pair if

$$\frac{\partial \eta}{\partial U}(U)\frac{\partial \vec{f}}{\partial U}(U) = \frac{\partial \vec{\psi}}{\partial U}(U).$$

$\Rightarrow$ for smooth solutions U of $U_t + \nabla \cdot (f(U))$:

$$\eta(U)_t + \nabla \cdot (\vec{\psi}(U)) = 0$$

Weak solution U satisfies entropy condition if

$$\forall \text{ convex } \eta : \ \eta(U)_t + \nabla \cdot (\vec{\psi}(U)) \leq 0$$

Motivation: true for uniform viscosity ΔU, true for Navier-Stokes with $\eta = -\varrho s$, s entropy per mass (second law of thermodynamics).

Entropy condition for shock waves

For all smooth entropy-flux pairs $(\eta, \vec{\psi})$ with convex η:

$$\eta(U)_t + \nabla \cdot (\vec{\psi}(U)) \leq 0$$

For n pointing from $-$ to $+$ and for $[A] = A_+ - A_-$:

$$[\vec{\psi}(U) \cdot n] \leq \sigma[\eta(U)]$$

Check: satisfied ($<$) for compression shocks, violated ($>$) for expansion shocks.

Shock waves not truly "inviscid": a distributional "ghost" of the viscous/heat conduction terms remains in the zero viscosity/heat conduction coefficient limit

Known uniqueness results

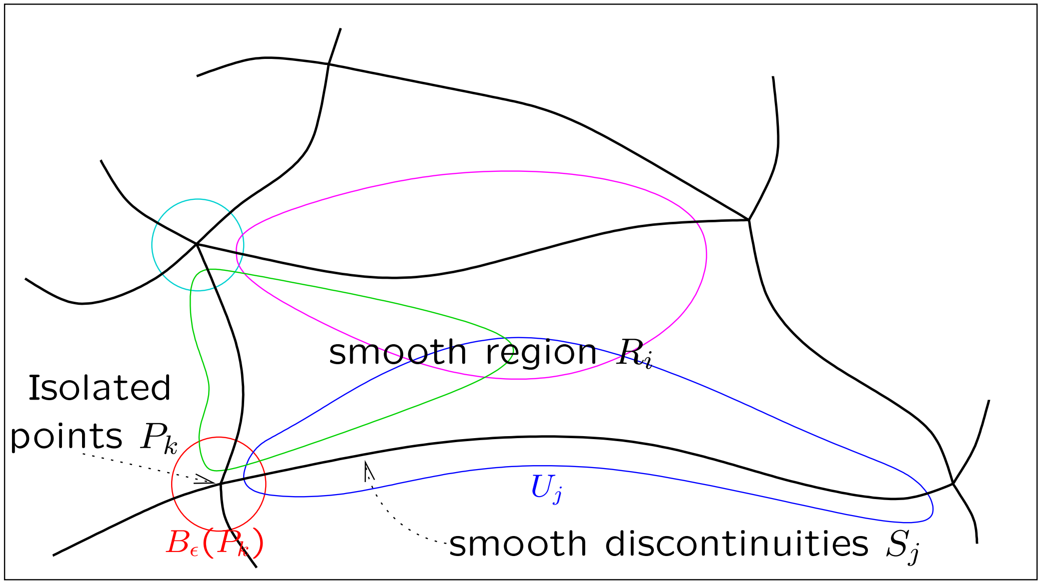
Scalar multi-dimensional conservation laws (..., Kružkov (1970)): uniqueness, vanishing viscosity $\Leftrightarrow$ entropy condition

1-d compressible Euler, small BV/closely related classes: uniqueness (Bressan/Crasta/Piccoli, Bressan/LeFloch, ...), vanishing uniform viscosity limit (Bianchini/Bressan 2005), vanishing Navier-Stokes viscosity limit (Chen/Perepelitsa 2010)

Dafermos/DiPerna: weak-strong uniqueness:

If $\exists$ classical ($\varrho, \vec{v}, T \in$ Lip) solution of multi-d compressible Euler, then no other weak entropy solutions for same initial data.

Piecewise smooth weak solutions



Regions R_i separated by C^1 hypersurfaces S_j, meeting in isolated points P_k.
$f \in C^1(R_i)$, $g \in C^0(R_i)$,
$\lim f \ \exists$ on each side in each point of S_j except P_k.

Fact: $\nabla \cdot f = g$ satisfied in weak sense

$$0 \stackrel{!}{=} \int_\Omega f \cdot \nabla\phi + g\phi \ dx$$

a. if satisfied in classical sense in R_i,

b. f satisfies Rankine-Hugoniot condition at S_j,

c. f, g not too singular in P_k: nearby, with $r = \text{dist}(x, P_k)$,

$$f(x) = o(r^{1-d}) \quad , \quad g(x) = O(r^{\delta-d}) \quad (\delta > 0)$$

Piecewise smooth weak solutions — isolated points

Consider one of the P_k. Assume $P_k = 0$ (coordinate change).

$$0 \overset{!}{=} \int_\Omega \nabla\phi \cdot f + \phi g \; dx$$

Choose $\theta^\epsilon(x) = \theta^\epsilon(|x|)$, $\theta^\epsilon \in C^\infty[0,\infty)$, $\theta^\epsilon(r) = \begin{cases} 1, & 0 \le r \le \frac{\epsilon}{2} \\ 0, & \epsilon \le r < \infty, \end{cases}$

$\theta^\epsilon = O(1)$, $\nabla\theta^\epsilon = O(\epsilon^{-1})$.

$$\phi(x) = \underbrace{\phi(x)\Big(1-\theta^\epsilon(x)\Big)}_{P_k \notin \text{supp}} + \phi(x)\theta^\epsilon(x)$$

$$\int_{B_\epsilon(0)} \nabla(\theta^\epsilon\phi)\cdot f \; dx = \int_0^\epsilon |\partial B_r| O(\epsilon^{-1}) o(r^{1-d}) dr = o(1) \quad \text{as } \epsilon \downarrow 0$$

$$\int_{B_\epsilon(0)} \theta^\epsilon \phi g \; dx = \int_0^\epsilon |\partial B_r| O(1) O(r^{\delta-d}) dr = O(\epsilon^\delta) \quad \text{as } \epsilon \downarrow 0$$

$\Rightarrow$ may remove $B_\epsilon(P_k)$ from $\text{supp}\,\phi$, at $o(1)_{\epsilon\downarrow 0}$ cost!

(Points have Hausdorff dimension $< d-1$, below hypersurfaces. Flux significant only through surface measure > 0, unless very singular.)

Proof (piecewise smooth weak solutions)
Given $\phi \in C^\infty(\Omega)$, $\operatorname{supp}\phi$ compact, $P_k \notin \operatorname{supp}\phi$.

Choose finite cover U_j of $\operatorname{supp}\phi$ so that
each U_j meets exactly one S_j and therefore exactly two R_i.

Smoothly partition $\phi = \sum_j \phi_j$ so that $\operatorname{supp}\phi_j \subset U_j$.

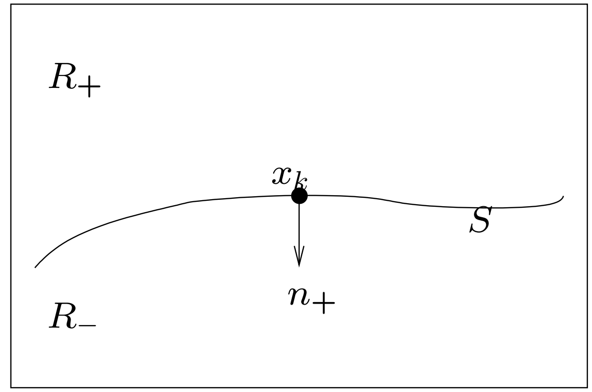
$$0 \stackrel{!}{=} \int_\Omega f \cdot \nabla\phi + g\phi \; dx = \sum_j \int_{U_j} f \cdot \nabla\phi_j + g\phi_j \; dx$$

Sufficient to check "weak solution" in each U_j separately.

Rankine-Hugoniot

$f_\pm$ limits on $R_\pm$ side.

$$0 \stackrel{!}{=} \int_{U_j} f \cdot \nabla\phi + g\ \phi\ dx = \sum_{\sigma=\pm} \int_{R_\sigma} f \cdot \nabla\phi + g\ \phi\ dx$$



$$\int_{R_\pm} f \cdot \nabla\phi + g\ \phi\ dx = \int_{R_\pm} \underbrace{(-\nabla \cdot f + g)}_{=0}\ \phi\ dx + \int_S \phi\ f_\pm \cdot n_\pm\ dS$$

$n_\pm$ unit normal to S in $x \in S$, outer to $R_\pm$. Note $n_- = -n_+$.

$$\sum_{\sigma=\pm} \int_S \phi\ f_\pm \cdot n_\pm dS = \int_S \phi\ \underbrace{(f_+ - f_-) \cdot n_+}_{=0}\ dS$$

if Rankine-Hugoniot condition

$$\boxed{(f_+ - f_-) \cdot n = 0}$$

Initial condition

$$e_t + \nabla \cdot f = g, \qquad e = e_0 \text{ given at } t = 0$$

Multiply with test function ϕ, $\int dx$, $\int dt$ by parts:

$$\int_0^\infty \int_{\mathbb{R}^d} e\ \phi_t + f \cdot \nabla\phi + g\ \phi\ dx\ dt + \int_{\mathbb{R}^d} e_0\ \phi_{|t=0}\ dx = 0$$

Fact: sufficient to check for $\operatorname{supp}\phi \Subset (0,\infty) \times \mathbb{R}^d$ and

$$e(t,\cdot) \to e_0 \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^d) \text{ as } t \downarrow 0.$$

as well as $f, g \in L^\infty_t([0,\infty); L^1_x(K))$ for compact K. (assumptions lazy)

$$\theta^\epsilon(t) \in C^\infty[0,\infty),\ \theta^\epsilon = \begin{cases} = 1, & 0 \le t \le \frac{\epsilon}{2}, \\ = 0, & \epsilon \le t < \infty, \end{cases} \ \theta^\epsilon = O(1),\ \theta^\epsilon_t = O(\epsilon^{-1}).$$

$$\phi = \underbrace{\phi(1 - \theta^\epsilon)}_{t=0 \notin \text{supp}} + \phi\theta^\epsilon.$$

Sufficient to check

$$\int_0^\infty \int_{\mathbb{R}^d} e\ (\theta^\epsilon\phi)_t + f \cdot \nabla(\theta^\epsilon\phi) + g\ \theta^\epsilon\ \phi\ dx\ dt + \int_{\mathbb{R}^d} e\ \phi_{|t=0}\ dx = 0$$

$(\theta^\epsilon \phi)_t = \theta^\epsilon_t \phi + O(1)_{\epsilon\downarrow 0}$, and $\mu_{(t,x)}\,\mathrm{supp}(\theta^\epsilon\phi) = O(\epsilon)$, so

$$\int_0^\infty \int_{\mathbb{R}^d} e\ \partial_t(\theta^\epsilon \phi)\ dx\ dt = O(\epsilon) + \int_0^\infty \int_{\mathbb{R}^d} \underbrace{e(t,x)}_{\overset{L^1_{\mathrm{loc}}}{\to} e_0}\ \theta^\epsilon_t(t)\ \underbrace{\phi(t,x)}_{\overset{L^\infty}{\to} \phi(0,x)}\ dx\ dt$$

$$\to \int_0^\infty \theta^\epsilon_t \cdot \int_{\mathbb{R}^n} e_0(x)\phi(0,x)dx\ dt = -\int_{\mathbb{R}^n} e_0\ \phi_{|t=0} dx$$

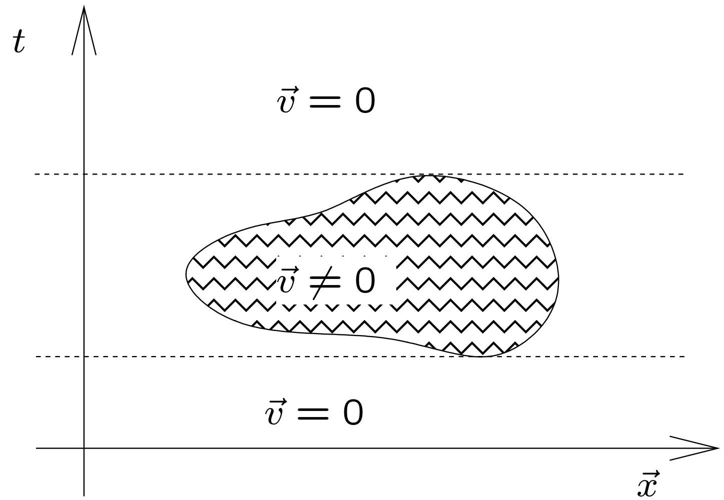
$$\int_0^\infty \int_{\mathbb{R}^d} \underbrace{f}_{=O(1)_{L^\infty_t L^1_x}} \cdot \underbrace{\nabla(\theta^\epsilon\phi)}_{=O(1)_{L^\infty_t L^\infty_x}} + \underbrace{g}_{=O(1)_{L^\infty_t L^1_x}} \underbrace{\theta^\epsilon\phi}_{=O(1)_{L^\infty_t L^\infty_x}} dx\ dt = O(\epsilon)$$

All estimates combined, get

$$\int_0^\infty \int_{\mathbb{R}^d} e\ \phi_t + f\cdot\nabla\phi + g\ \phi\ dx\ dt + \int_{\mathbb{R}^d} e\ \phi_{|t=0}\ dx = 0$$

Scheffer non-uniqueness

V. Scheffer (1993): $\exists$ incompressible Euler solutions $\vec{v} \in L^2(\mathbb{R}_t \times \mathbb{R}^3_x)$ with compact support in space-time:



A. Schnirelman (1996): Different, simpler proof for $\vec{v} \in L^2(\mathbb{R}_t \times \mathbb{T}^3_x)$.

External forces

Dafermos (1979), DiPerna (1979): cannot happen in compressible Euler flow (with entropy condition).

$\rightsquigarrow$ possible misinterpretations:

"No problem if we require conservation of energy."

"No problem if we consider compressibility."

De Lellis/Szekelyhidi (ARMA 2008) [MUST READ]: non-uniqueness example also for compressible Euler, with entropy and energy conserved.

De Lellis/Szekelyhidi solutions:

$\exists$ weak entropy solutions $U = (\varrho, \vec{v}, T) \in L^\infty(\mathbb{R}_t \times \mathbb{R}_x^n)$ with same initial data.

$$\text{Compact support in space:} \quad \bigcup_t \operatorname{supp} U(t,\cdot) \Subset \mathbb{R}^3$$

Entropy and energy conserved, can be considered "shock-free".
$\Rightarrow$ vorticity is the cause of non-uniqueness

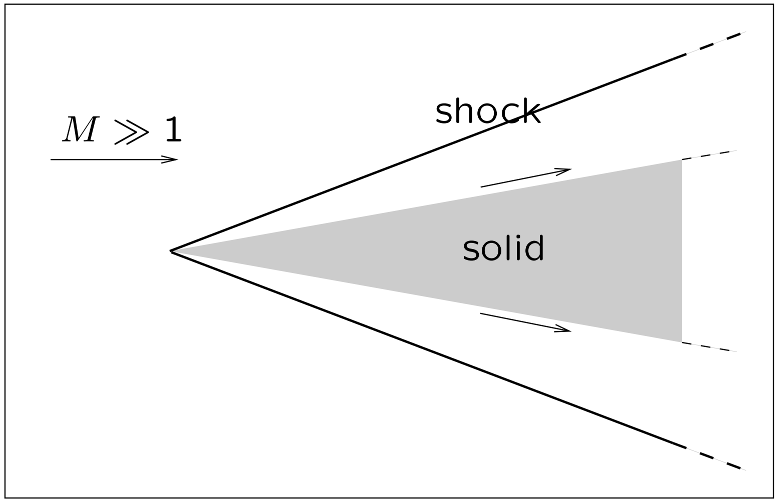
"Hope: problem absent for 'most' initial data."
De Lellis/Szekelyhidi: non-uniqueness for residual (complement countable union of nowhere dense sets in L^2) set of initial data.

"De Lellis/Szekelyhidi solutions are 'crazy'."
What else if not L^∞? Compressible Euler requires space with discontinuities; BV too narrow for multi-d (Rauch 1986).

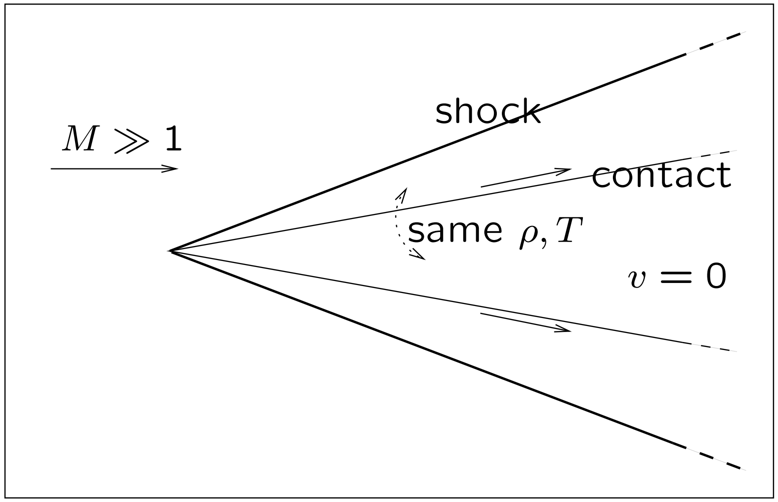
"Nuisance for theory, but no practical relevance."
Problem has shown up in numerics and even physics, but underestimated $\rightarrow$

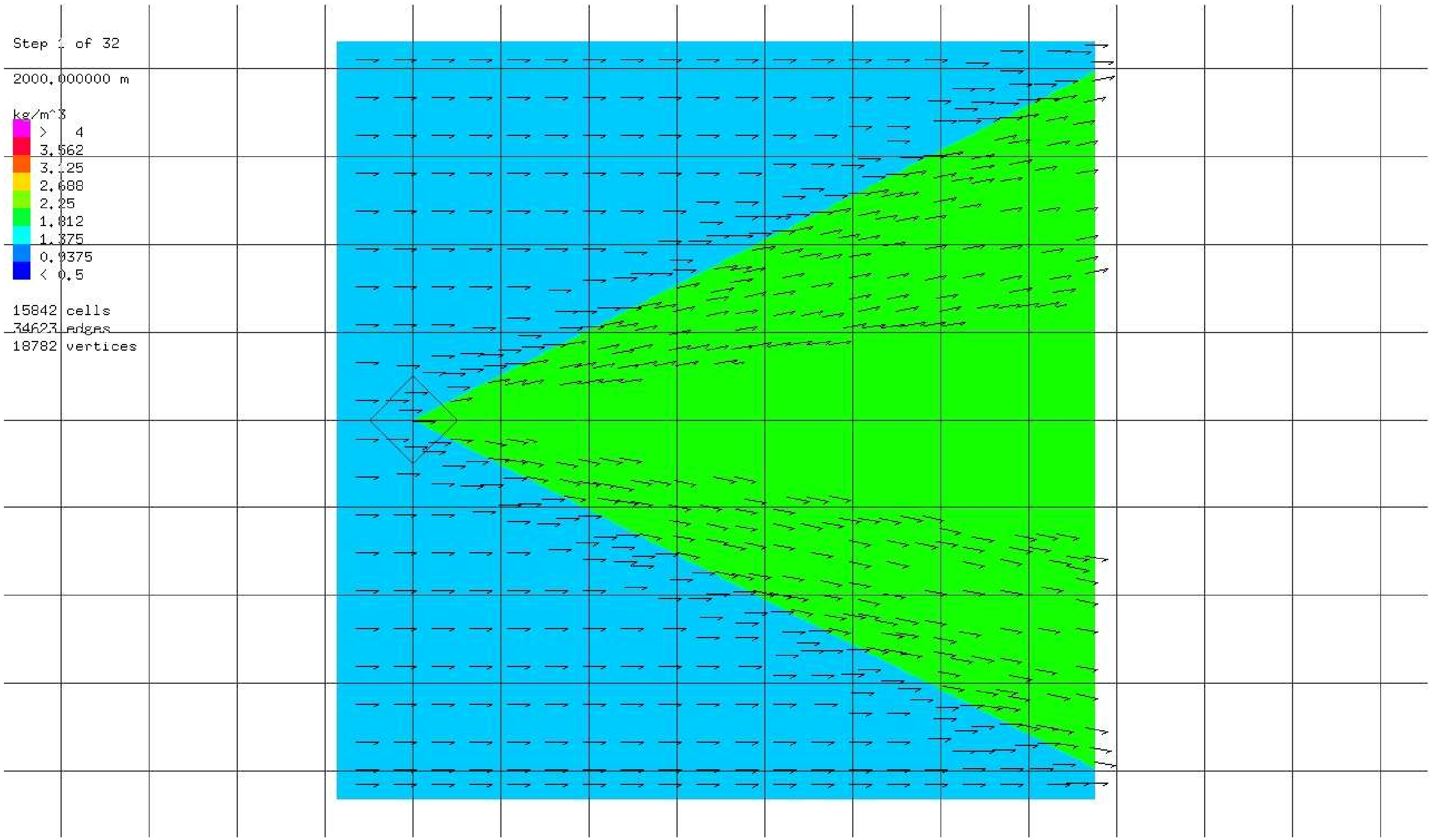
Initial data (and steady entropy solution)

September 2002:

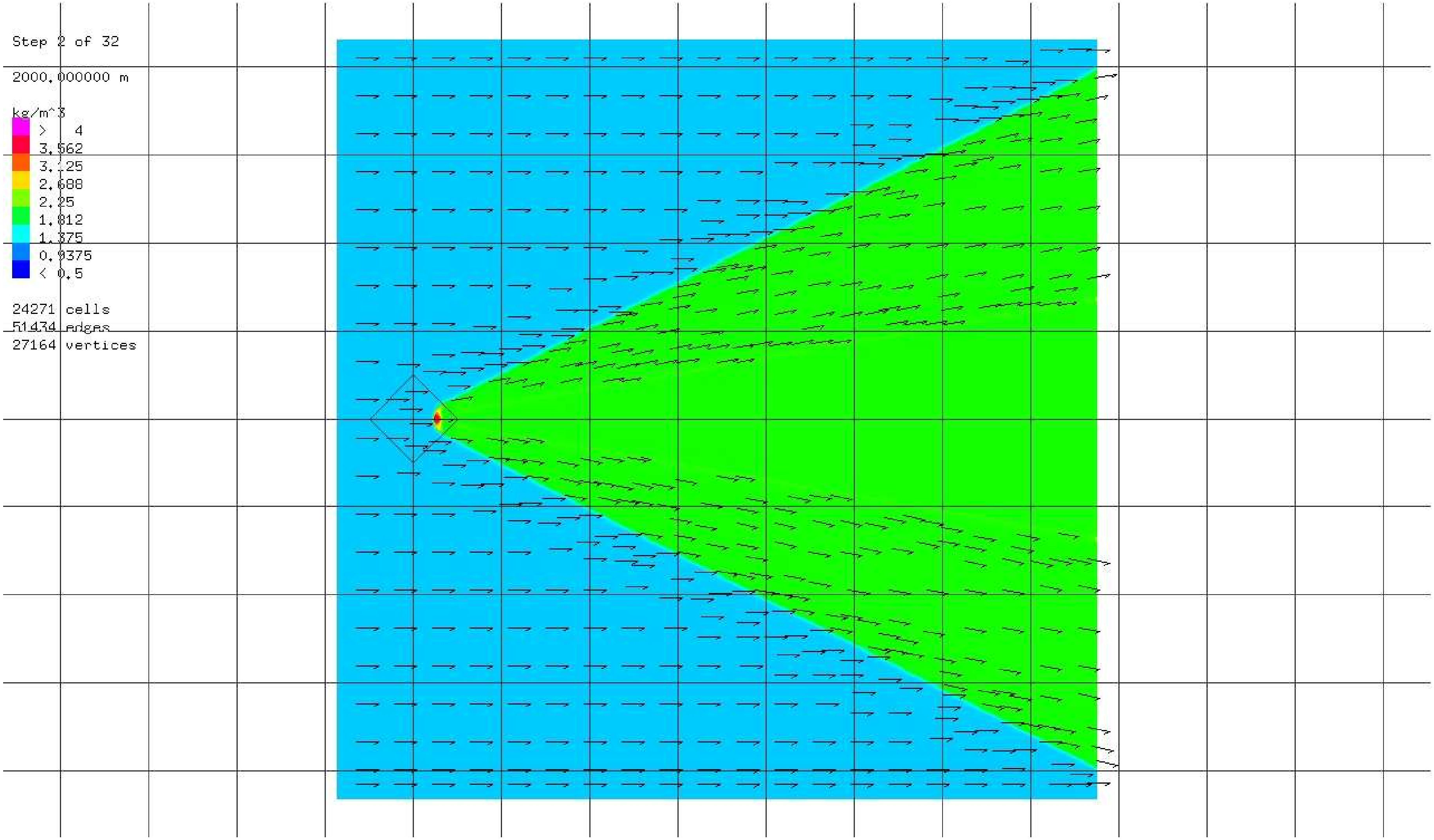


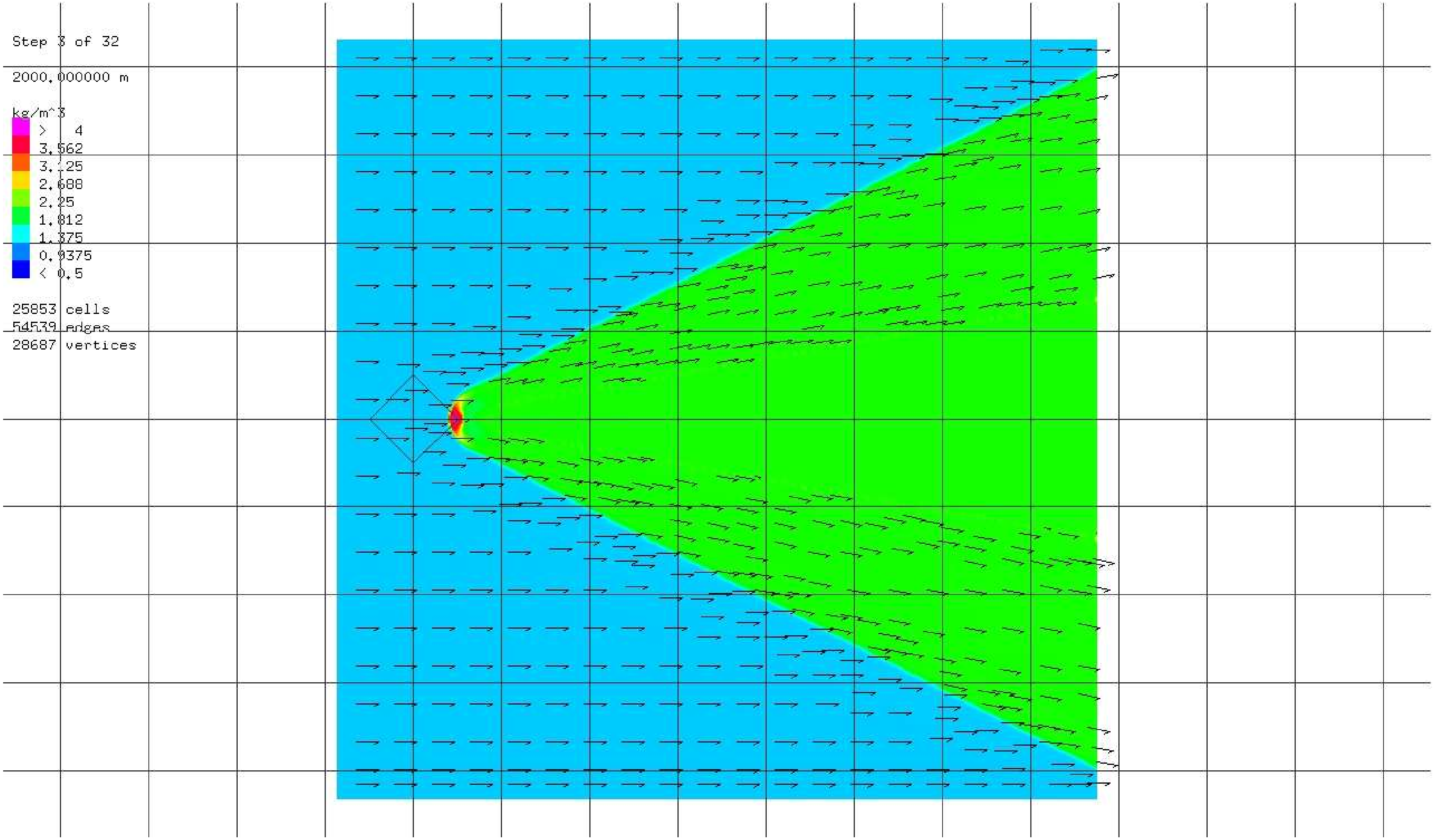
Experiment (easier due to Cartesian uniform grid):



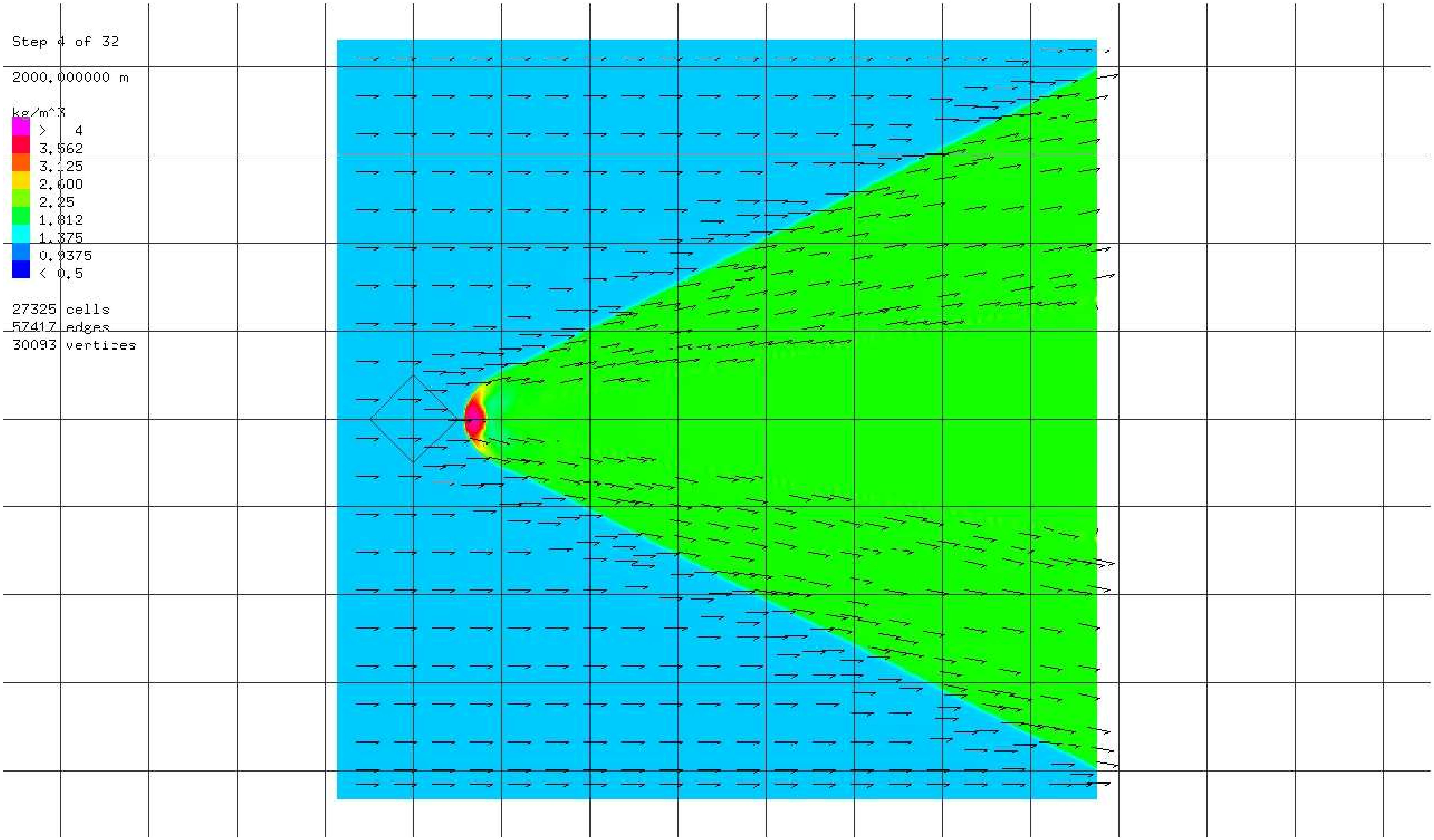


Step 1 of 32
2000.000000 m
kg/m^3
> 4
3.562
3.125
2.688
2.25
1.812
1.375
0.9375
< 0.5
15842 cells
34623 edges
18782 vertices

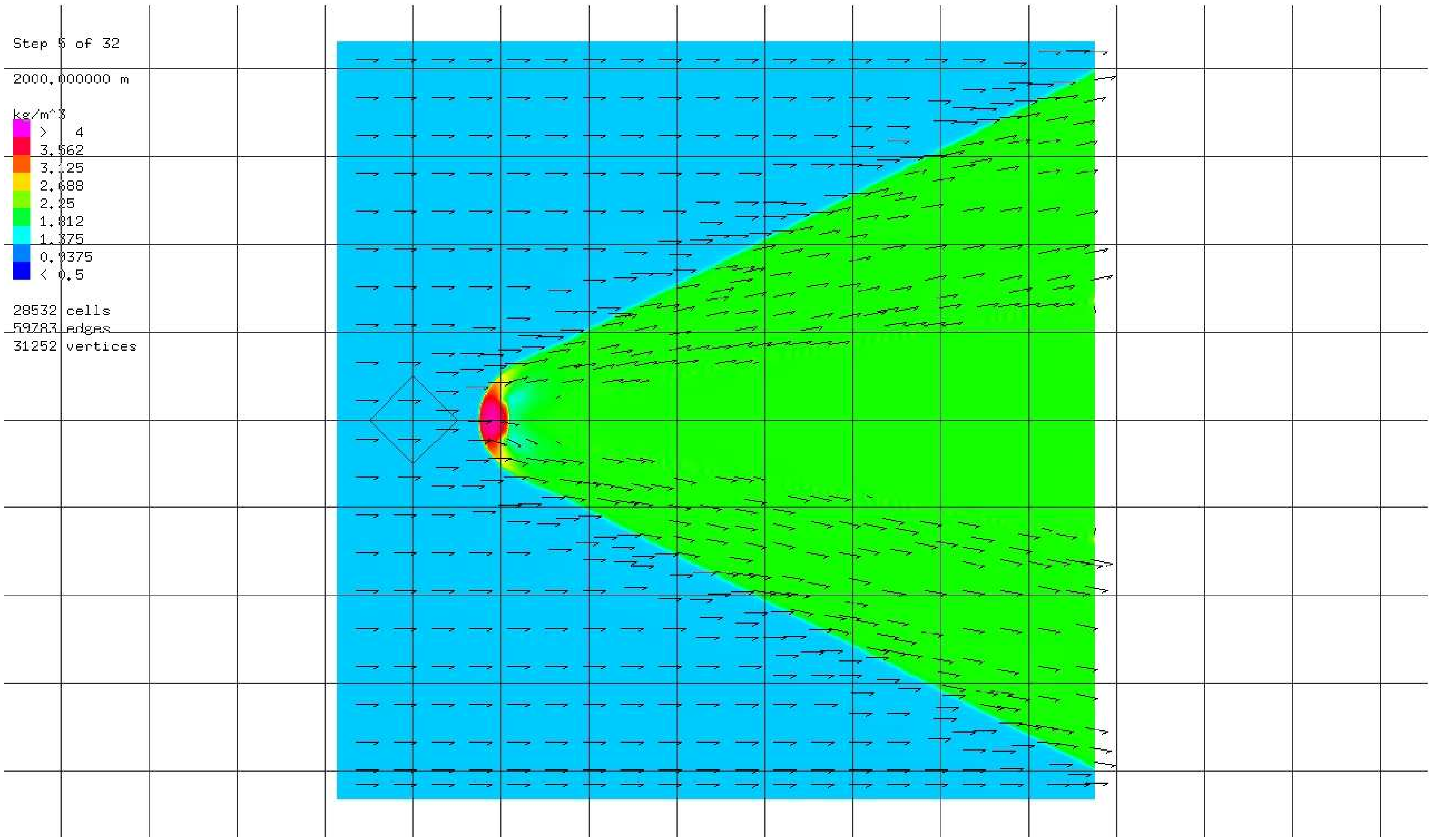

Step 2 of 32
2000.000000 m
kg/m^3
> 4
3.562
3.125
2.688
2.25
1.812
1.375
0.9375
< 0.5
24271 cells
51434 edges
27164 vertices

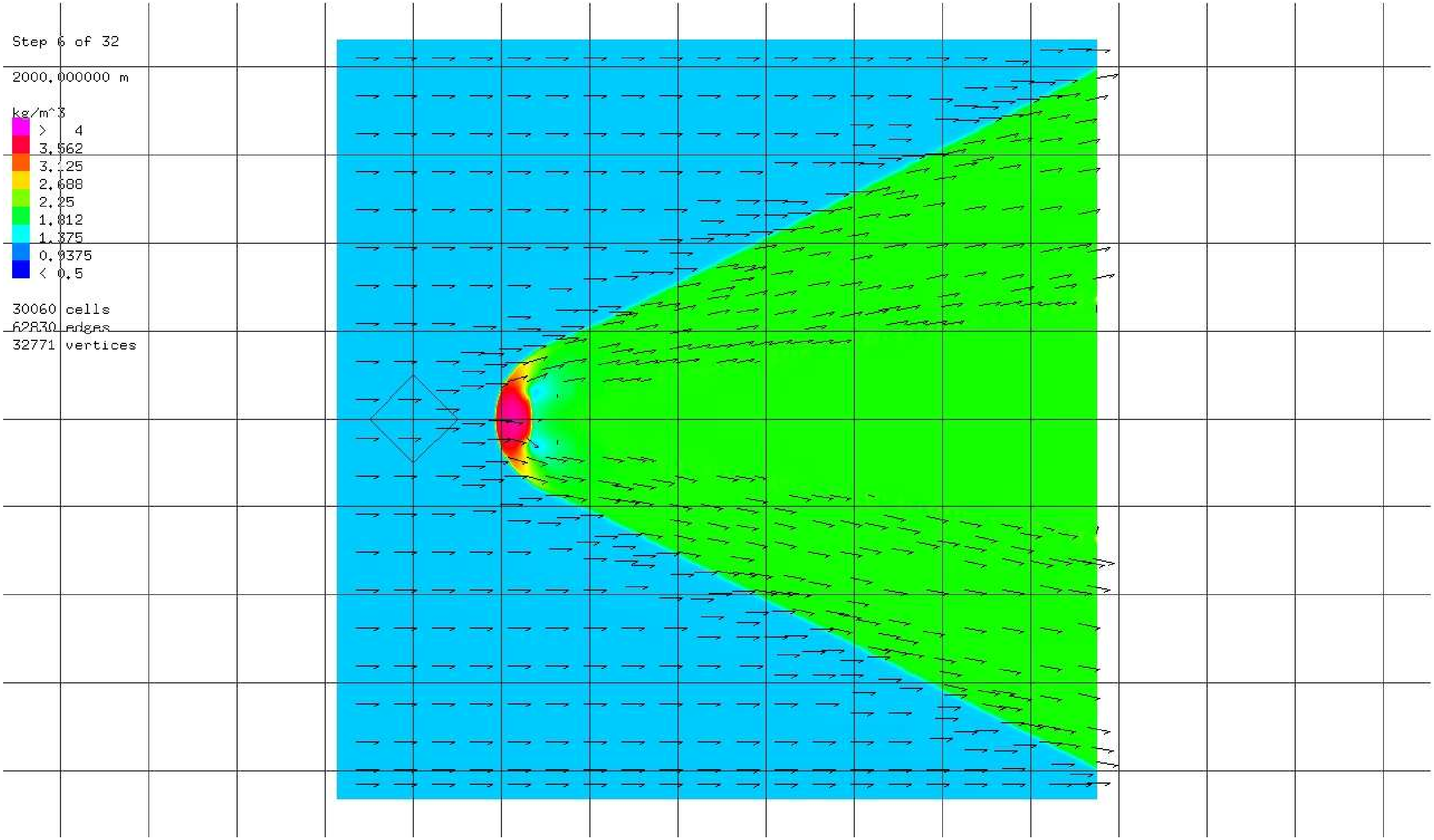


Step 3 of 32
2000.000000 m
kg/m^3
> 4
3.562
3.125
2.688
2.25
1.812
1.375
0.9375
< 0.5
25853 cells
54539 edges
28687 vertices

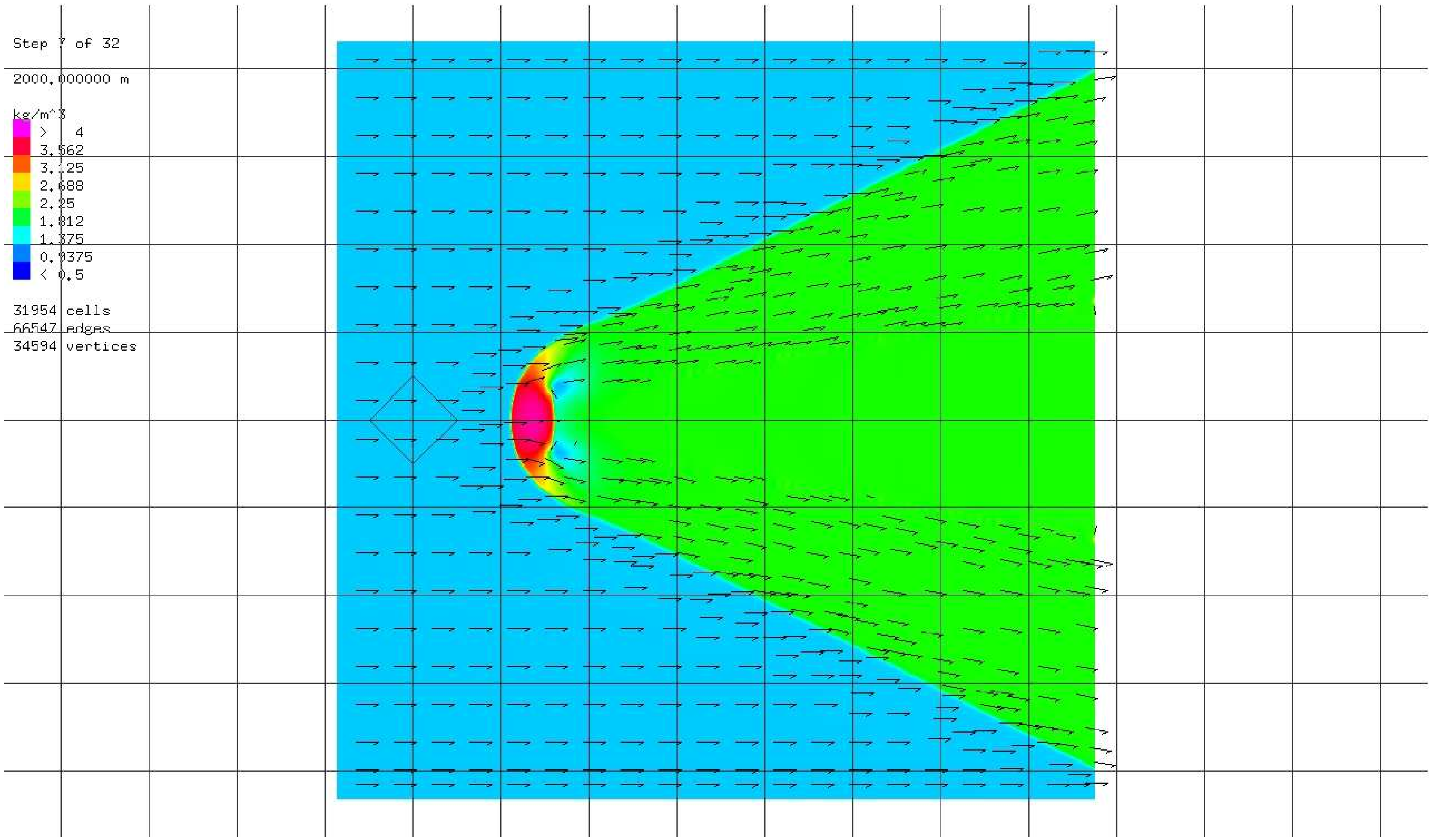


Step 4 of 32
2000.000000 m
kg/m^3
> 4
3.562
3.125
2.688
2.25
1.812
1.375
0.9375
< 0.5
27325 cells
57417 edges
30093 vertices

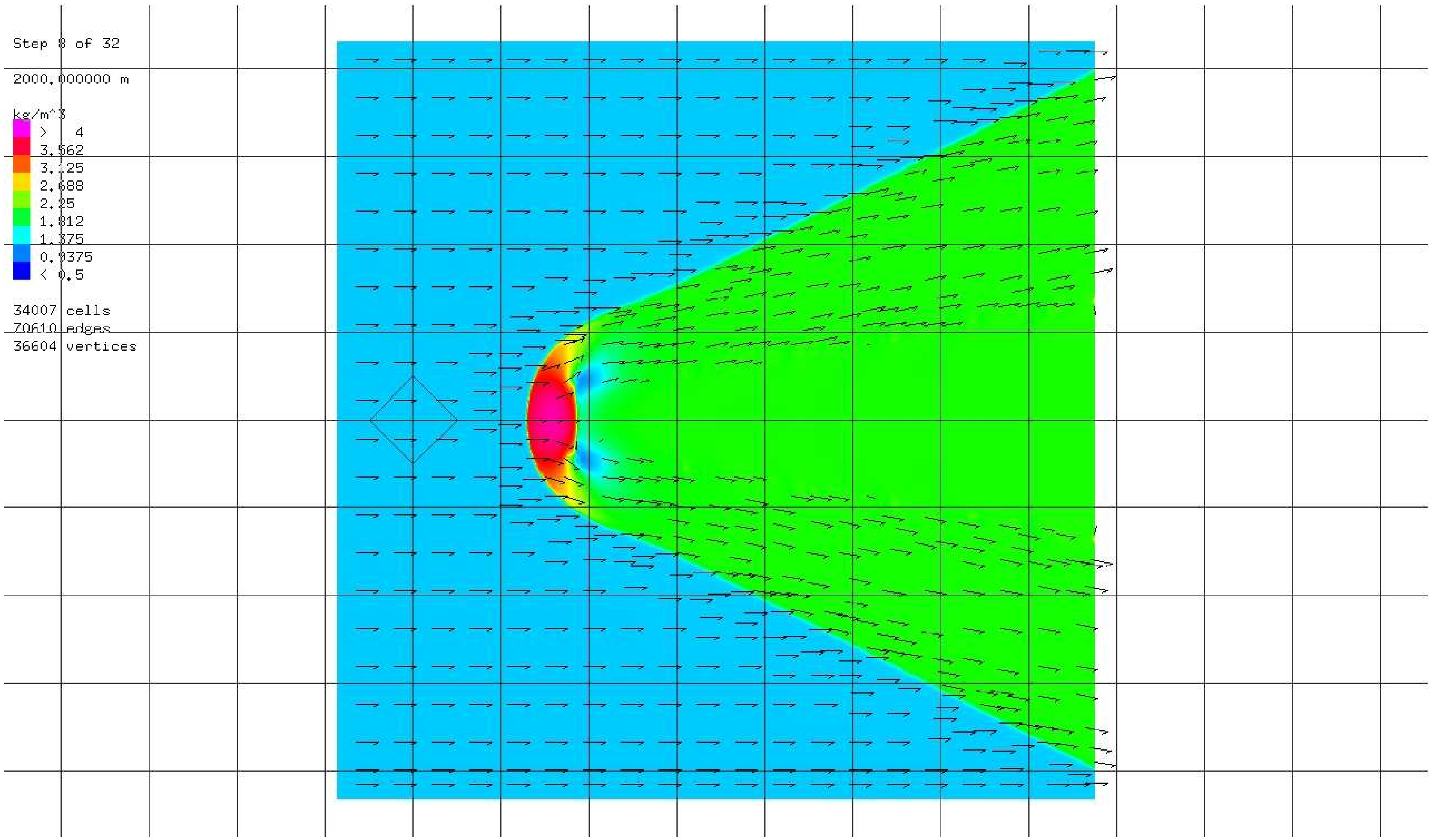

Step 5 of 32
2000.000000 m
kg/m^3
> 4
3.562
3.125
2.688
2.25
1.812
1.375
0.9375
< 0.5
28532 cells
59783 edges
31252 vertices

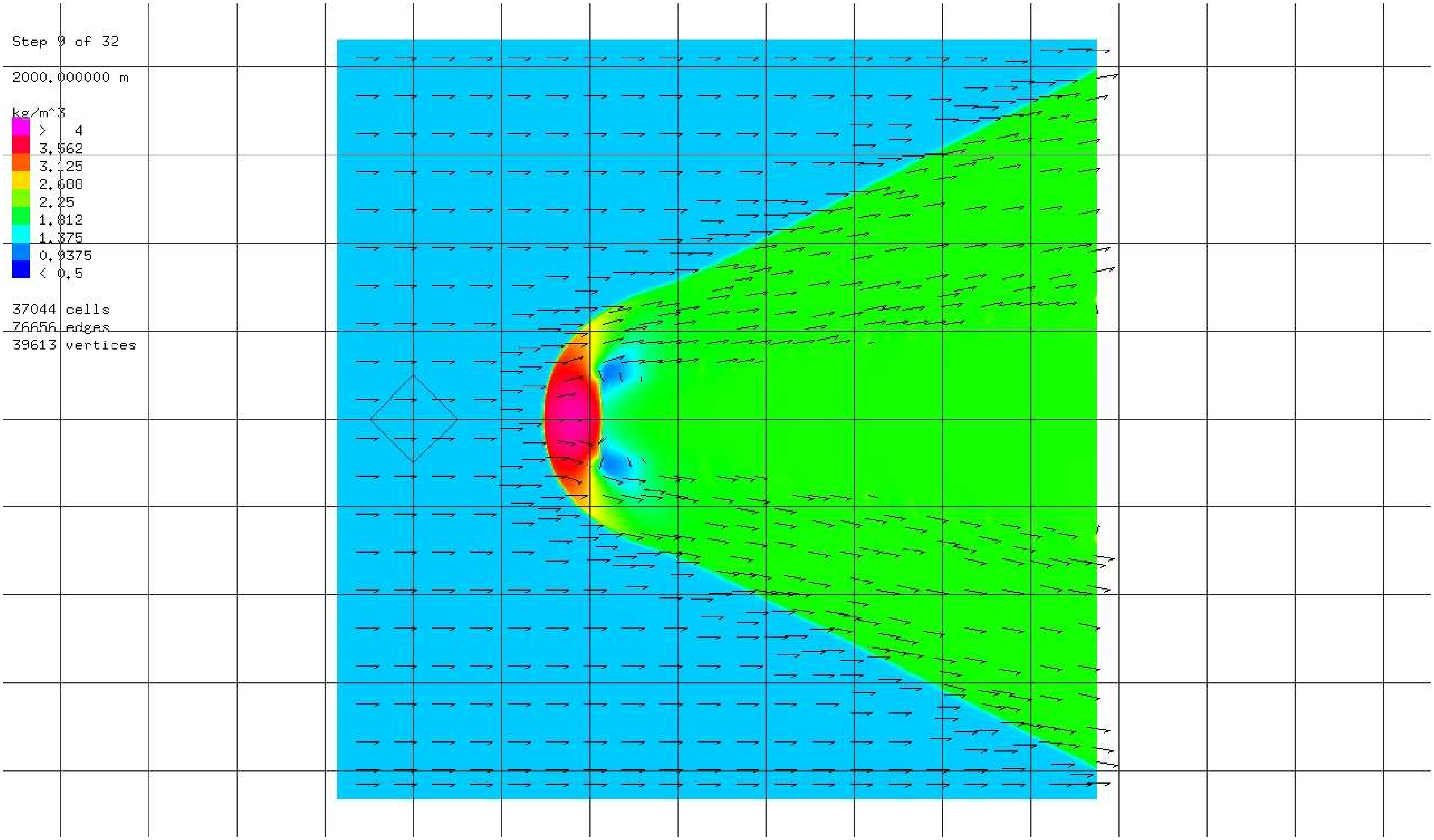


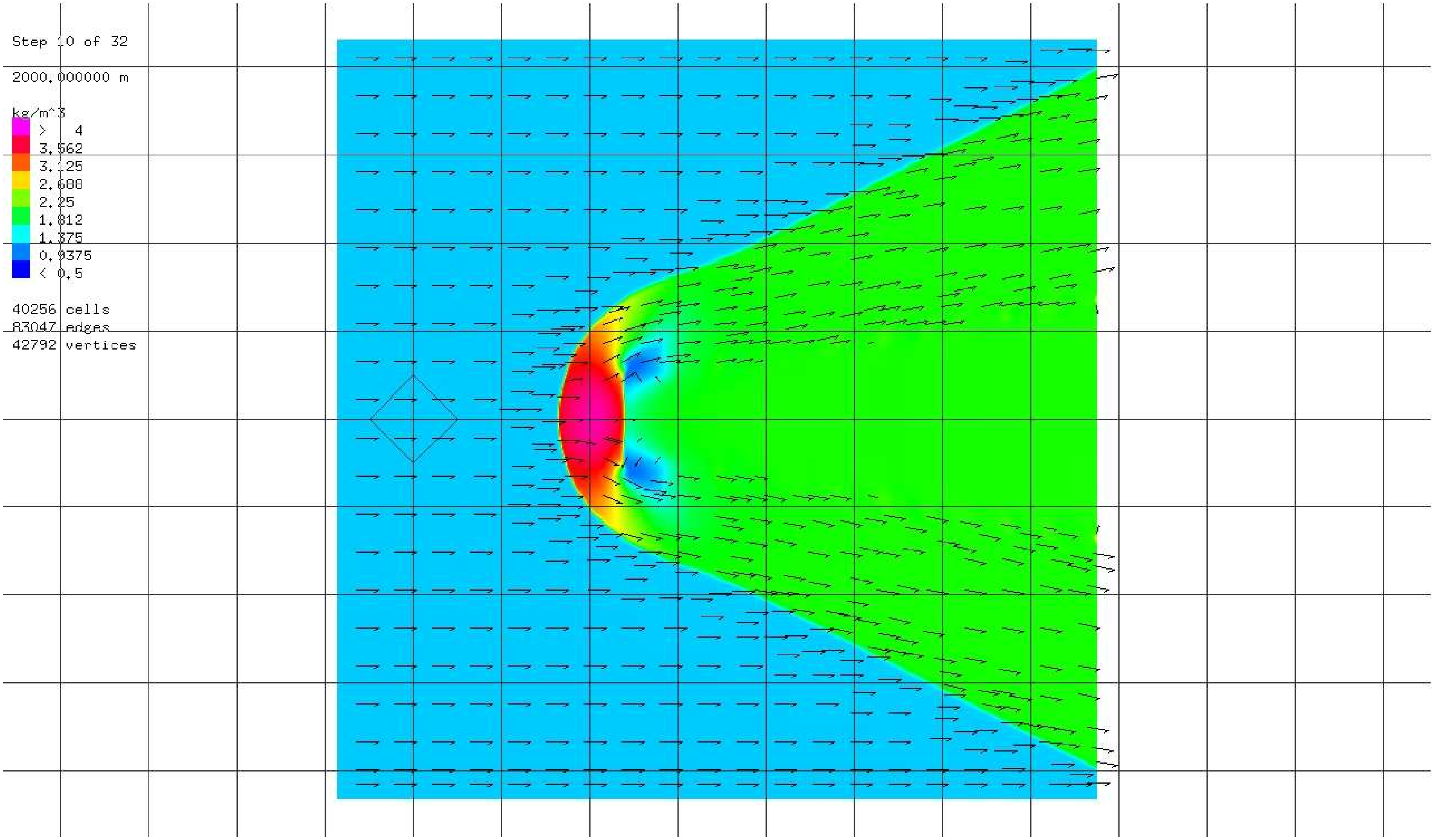
Step 6 of 32
2000.000000 m
kg/m^3
> 4
3.562
3.125
2.688
2.25
1.812
1.375
0.9375
< 0.5
30060 cells
62830 edges
32771 vertices

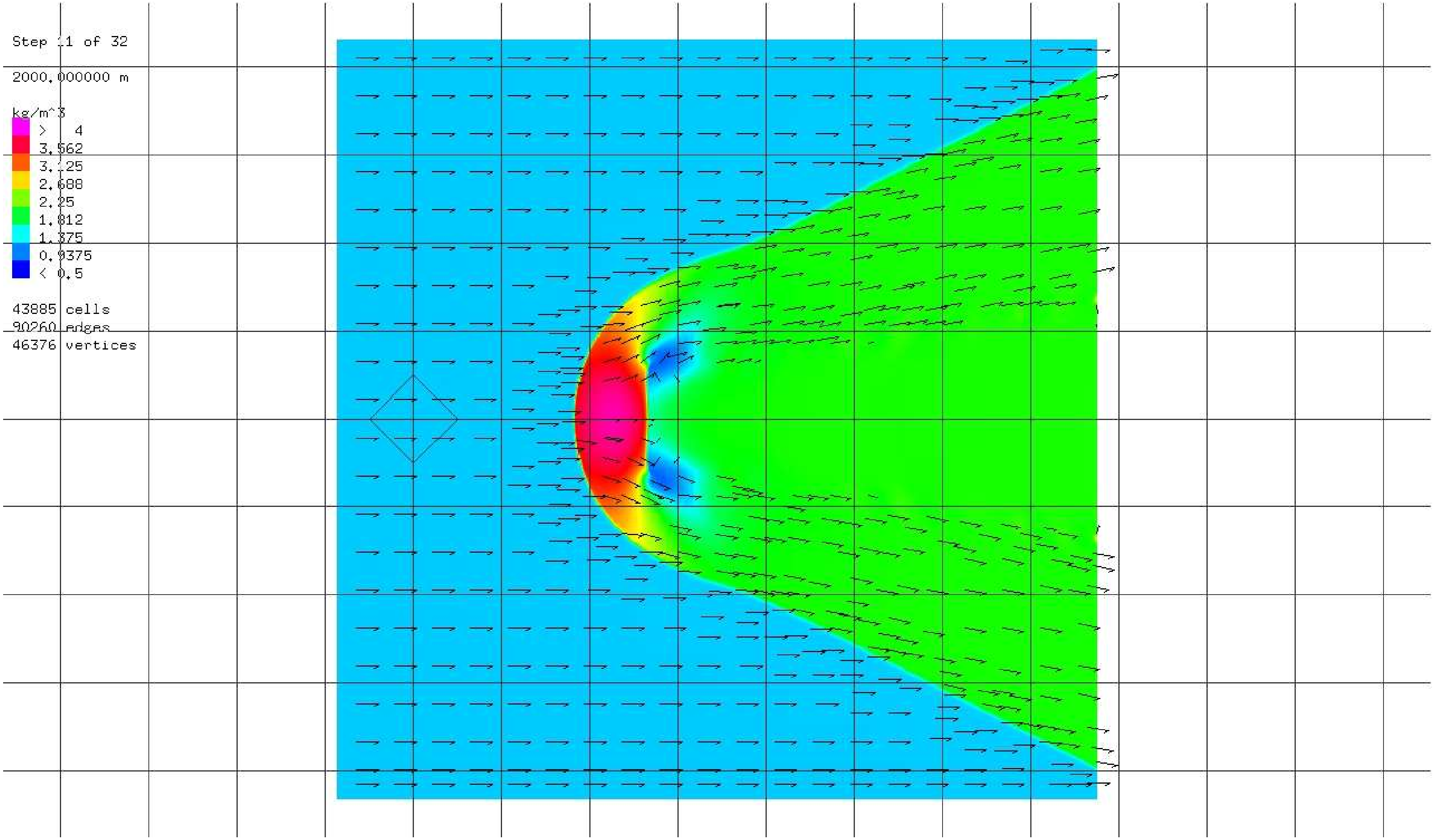


Step 7 of 32
2000,000000 m
kg/m^3
> 4
3,562
3,125
2,688
2,25
1,812
1,375
0,9375
< 0,5
31954 cells
66547 edges
34594 vertices

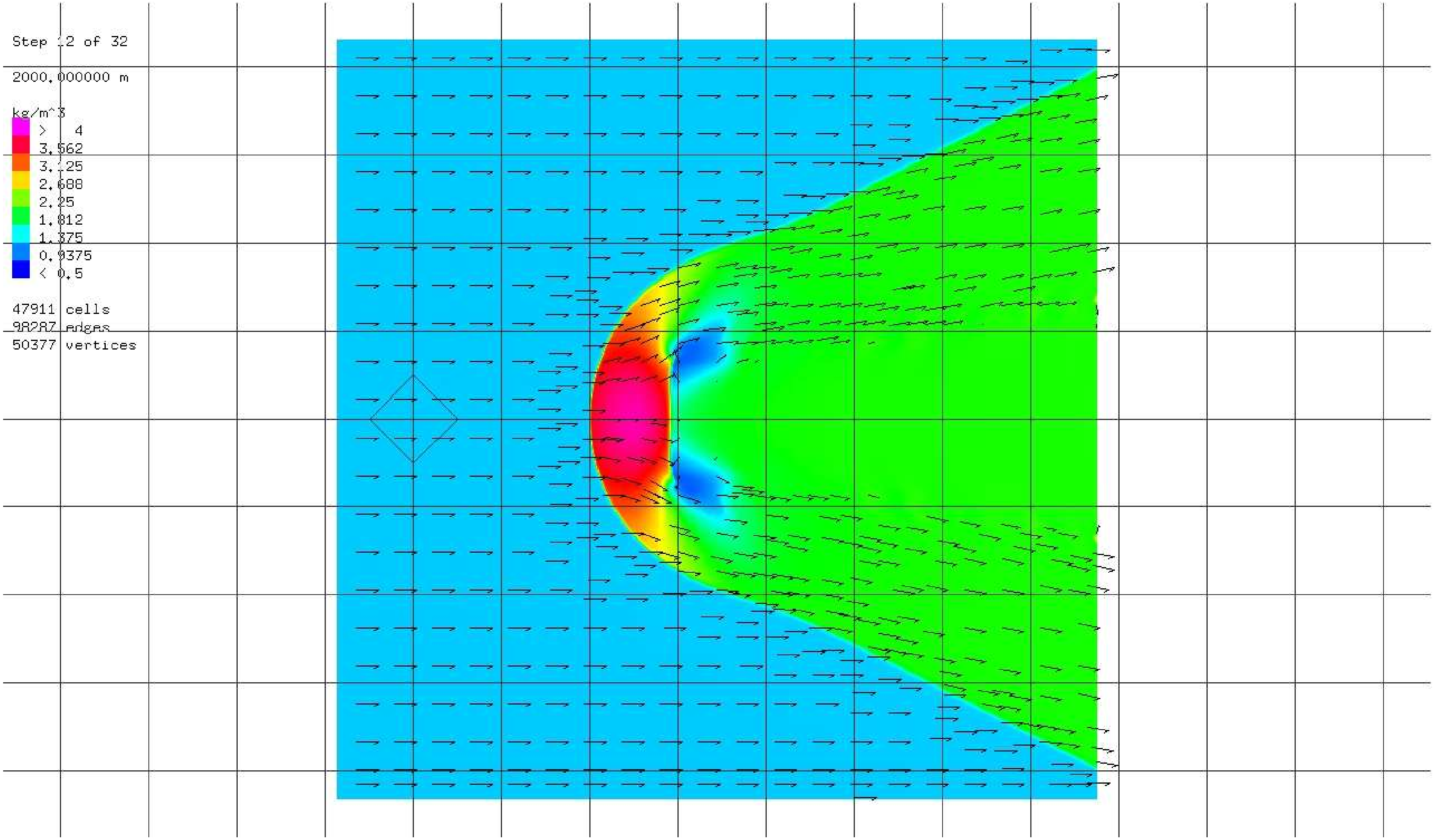

Step 8 of 32
2000.000000 m
kg/m^3
> 4
3.562
3.125
2.688
2.25
1.812
1.375
0.9375
< 0.5
34007 cells
70610 edges
36604 vertices


Step 9 of 32
2000.000000 m
kg/m^3
> 4
3.562
3.125
2.688
2.25
1.812
1.375
0.9375
< 0.5
37044 cells
76656 edges
39613 vertices

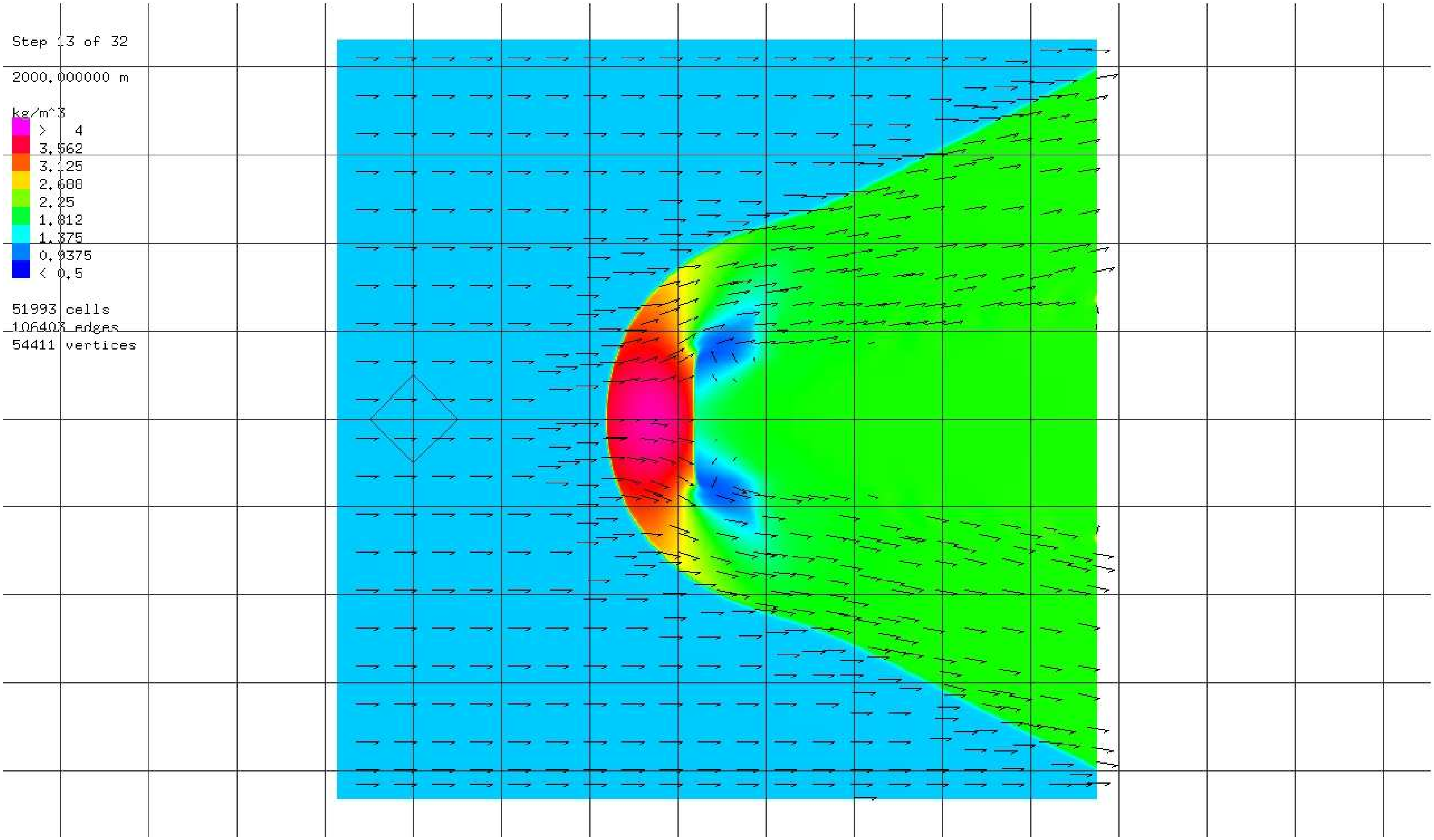

Step 10 of 32
2000.000000 m
kg/m^3
> 4
3.562
3.125
2.688
2.25
1.812
1.375
0.9375
< 0.5
40256 cells
83047 edges
42792 vertices

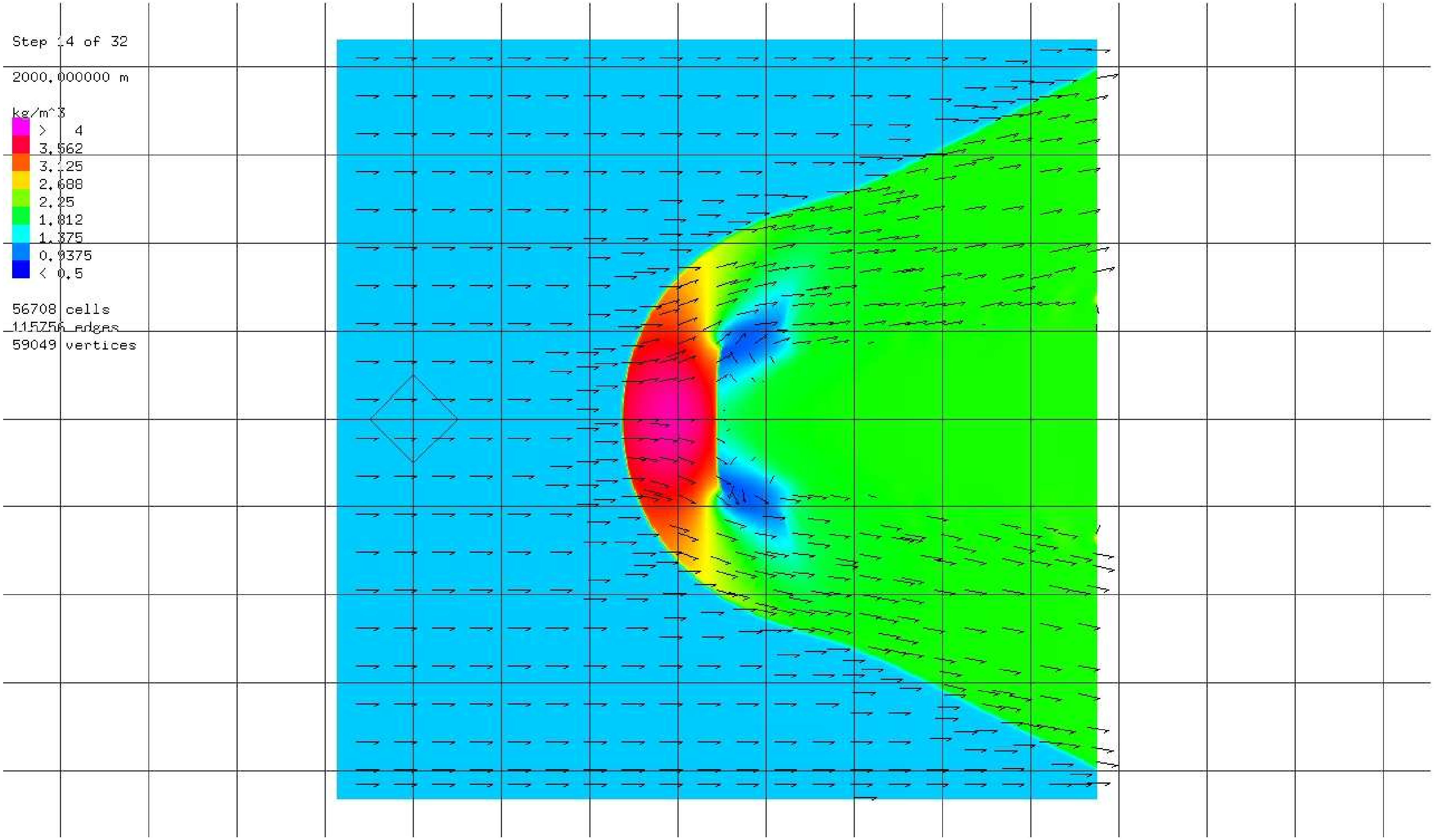


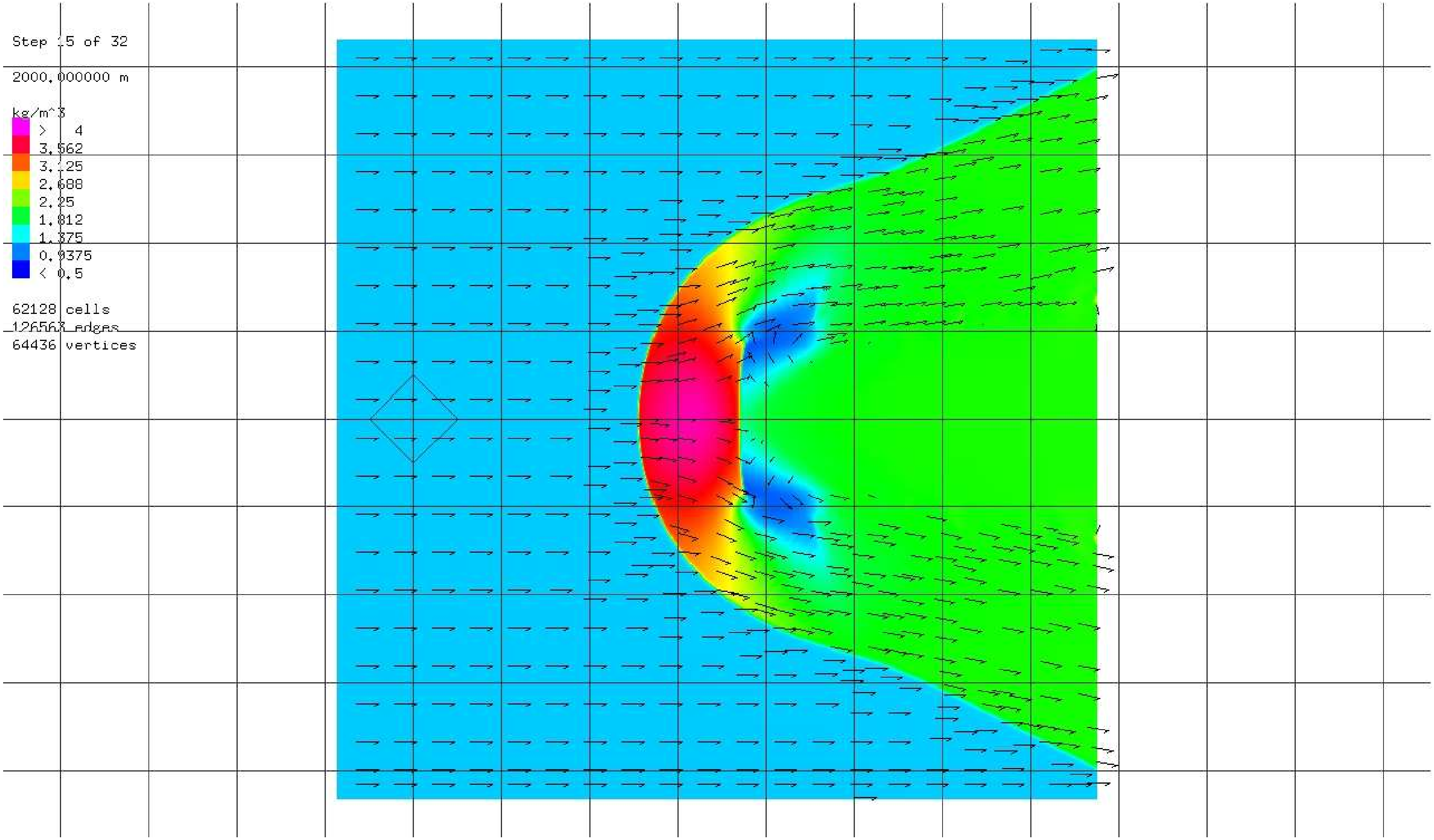
Step 11 of 32
2000.000000 m
kg/m^3
> 4
3.562
3.125
2.688
2.25
1.812
1.375
0.9375
< 0.5
43885 cells
90260 edges
46376 vertices



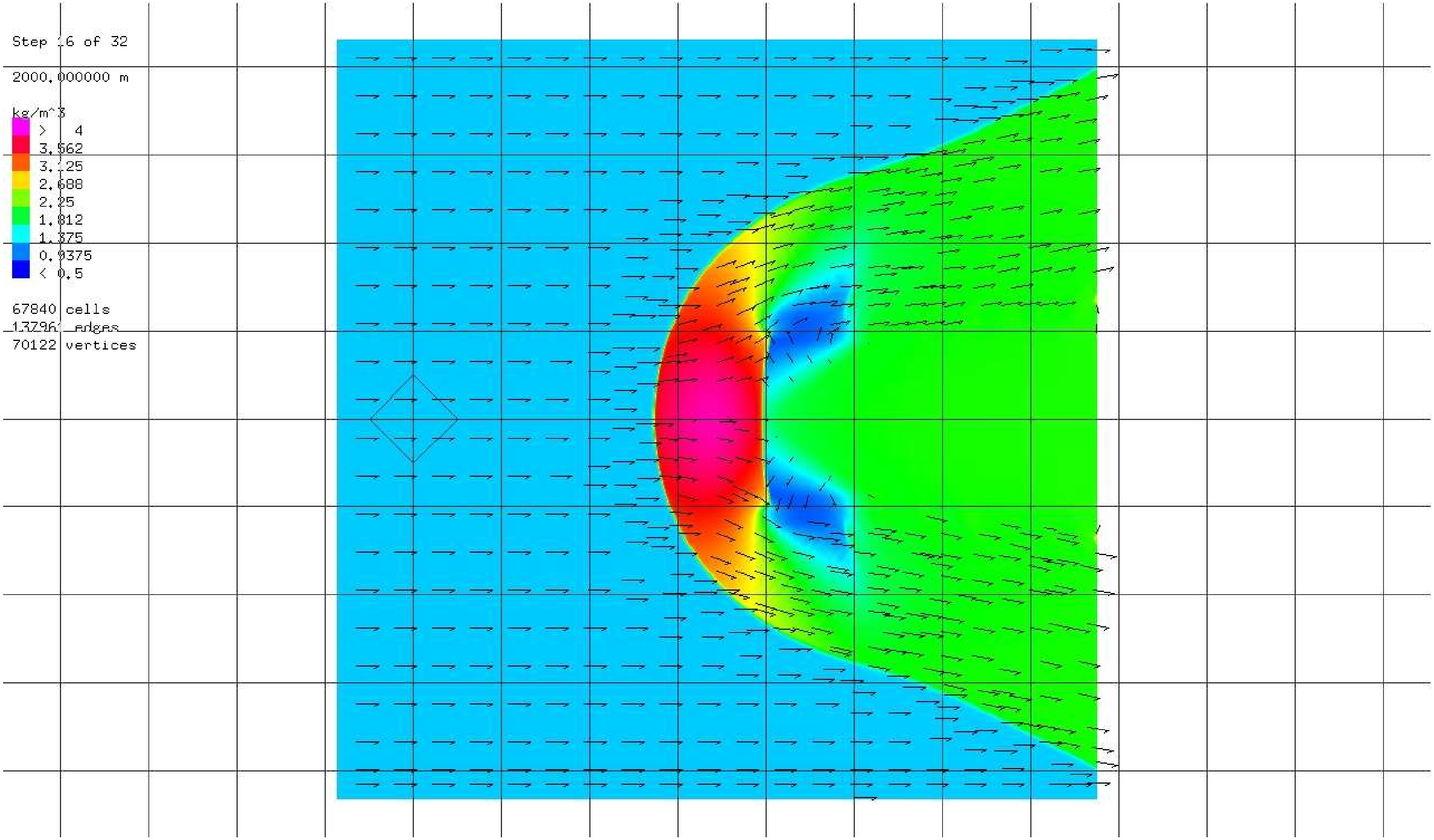
Step 12 of 32
2000.000000 m
kg/m^3
> 4
3.562
3.125
2.688
2.25
1.812
1.375
0.9375
< 0.5
47911 cells
98287 edges
50377 vertices


Step 13 of 32
2000.000000 m
kg/m^3
> 4
3.562
3.125
2.688
2.25
1.812
1.375
0.9375
< 0.5
51993 cells
106403 edges
54411 vertices

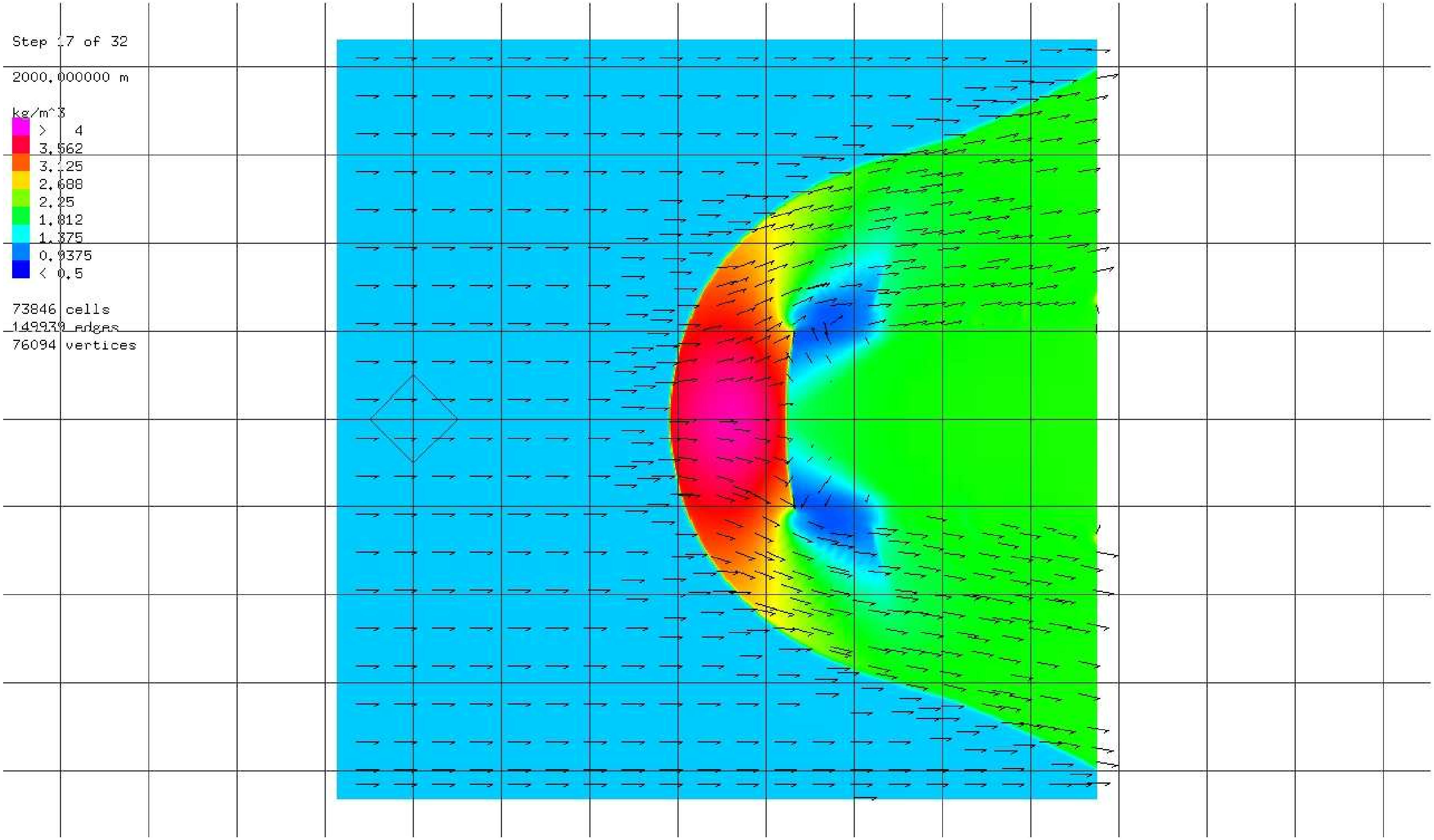

Step 14 of 32
2000.000000 m
kg/m^3
> 4
3.562
3.125
2.688
2.25
1.812
1.375
0.9375
< 0.5
56708 cells
115756 edges
59049 vertices



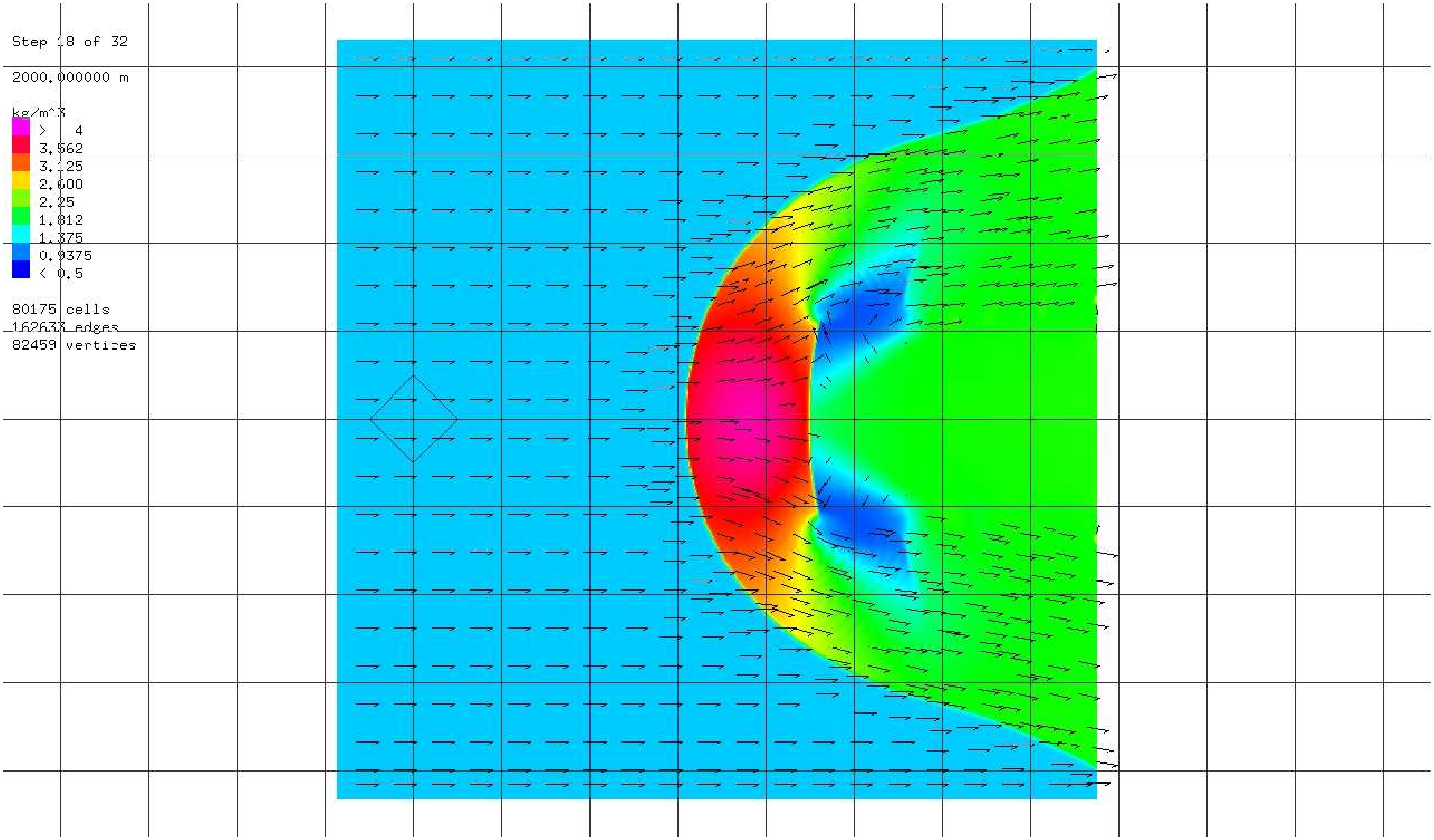
Step 15 of 32
2000.000000 m
kg/m^3
> 4
3.562
3.125
2.688
2.25
1.812
1.375
0.9375
< 0.5
62128 cells
126563 edges
64436 vertices



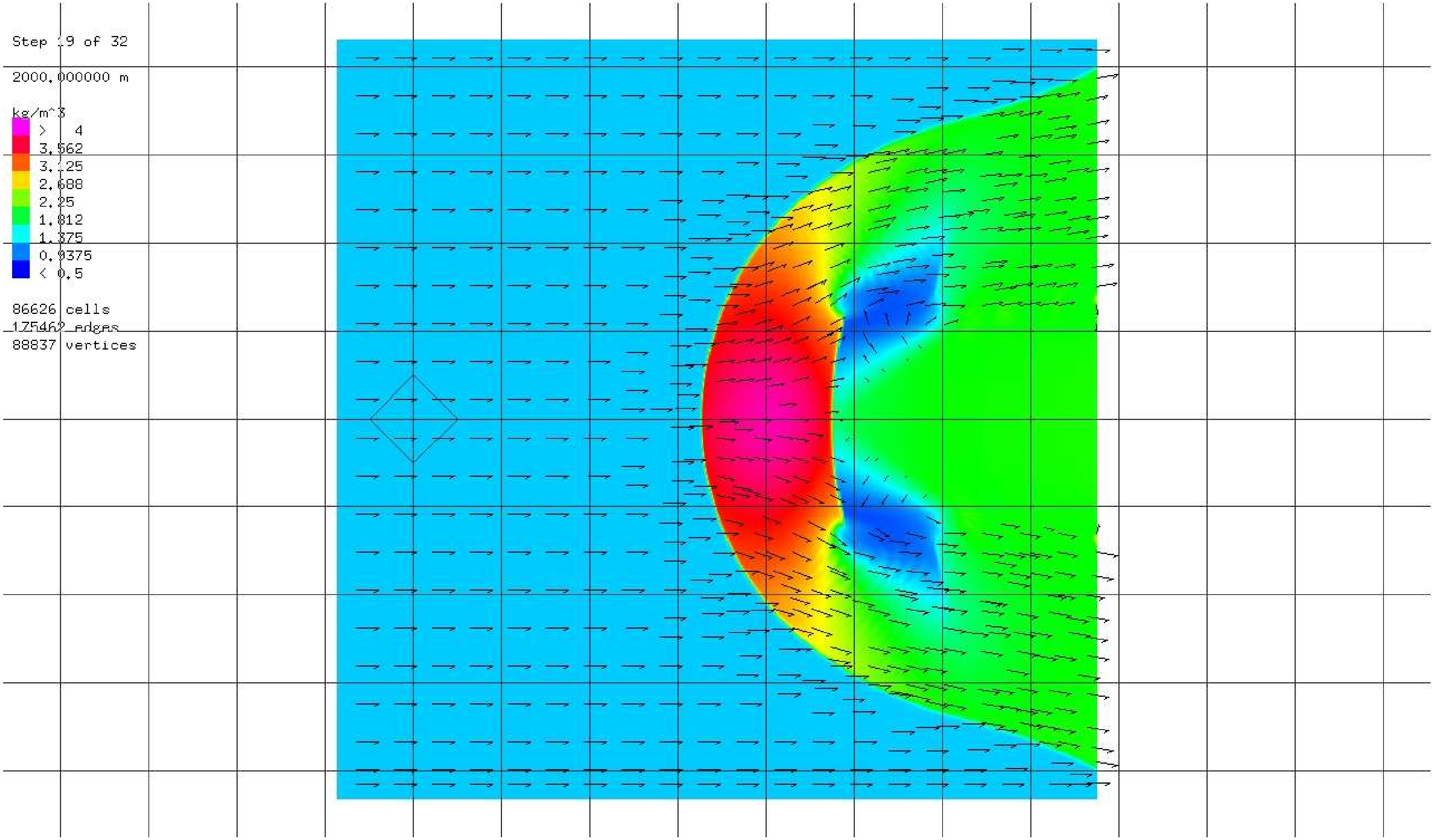
Step 16 of 32
2000.000000 m
kg/m^3
> 4
3.562
3.125
2.688
2.25
1.812
1.375
0.9375
< 0.5
67840 cells
137961 edges
70122 vertices

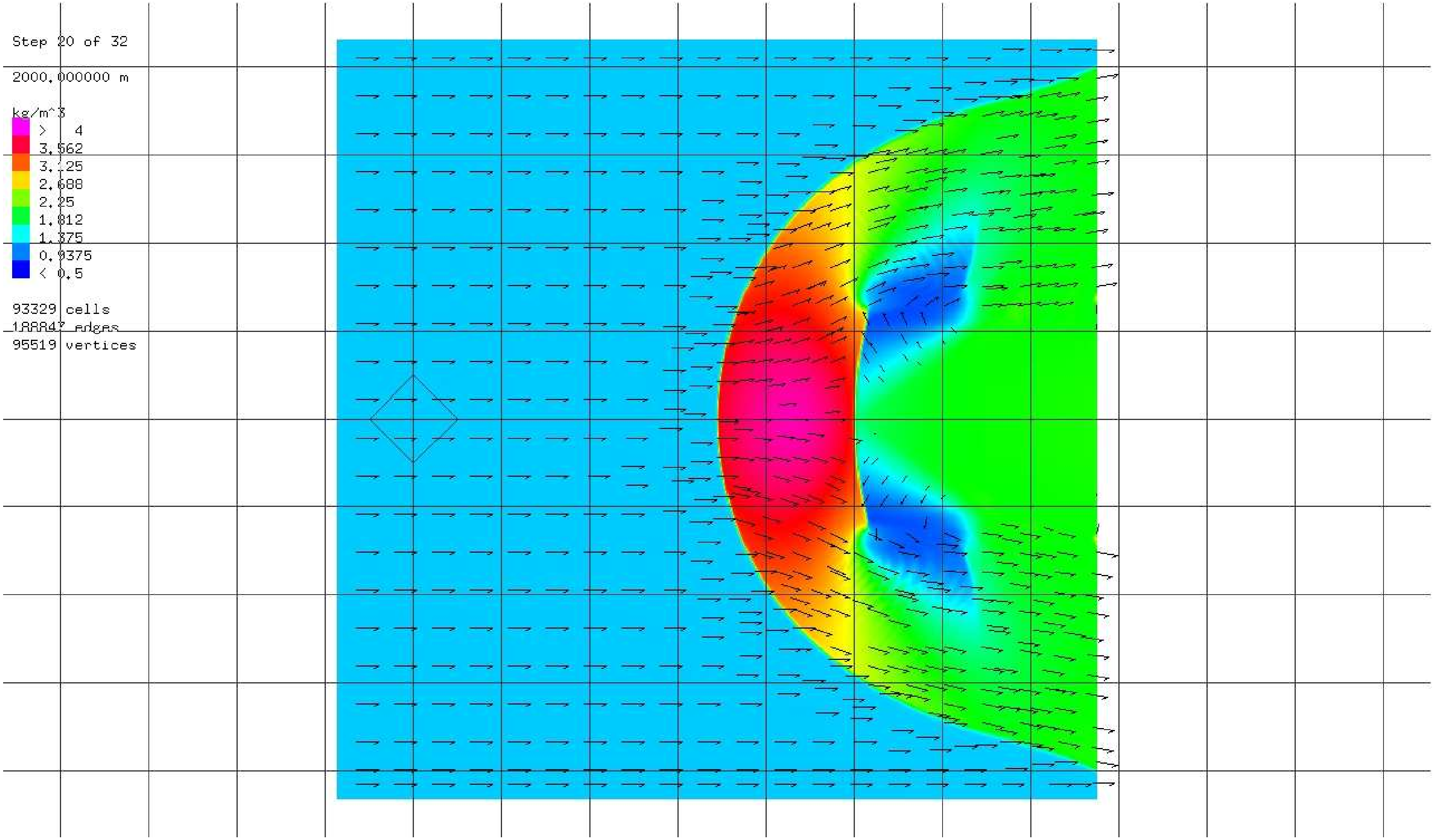


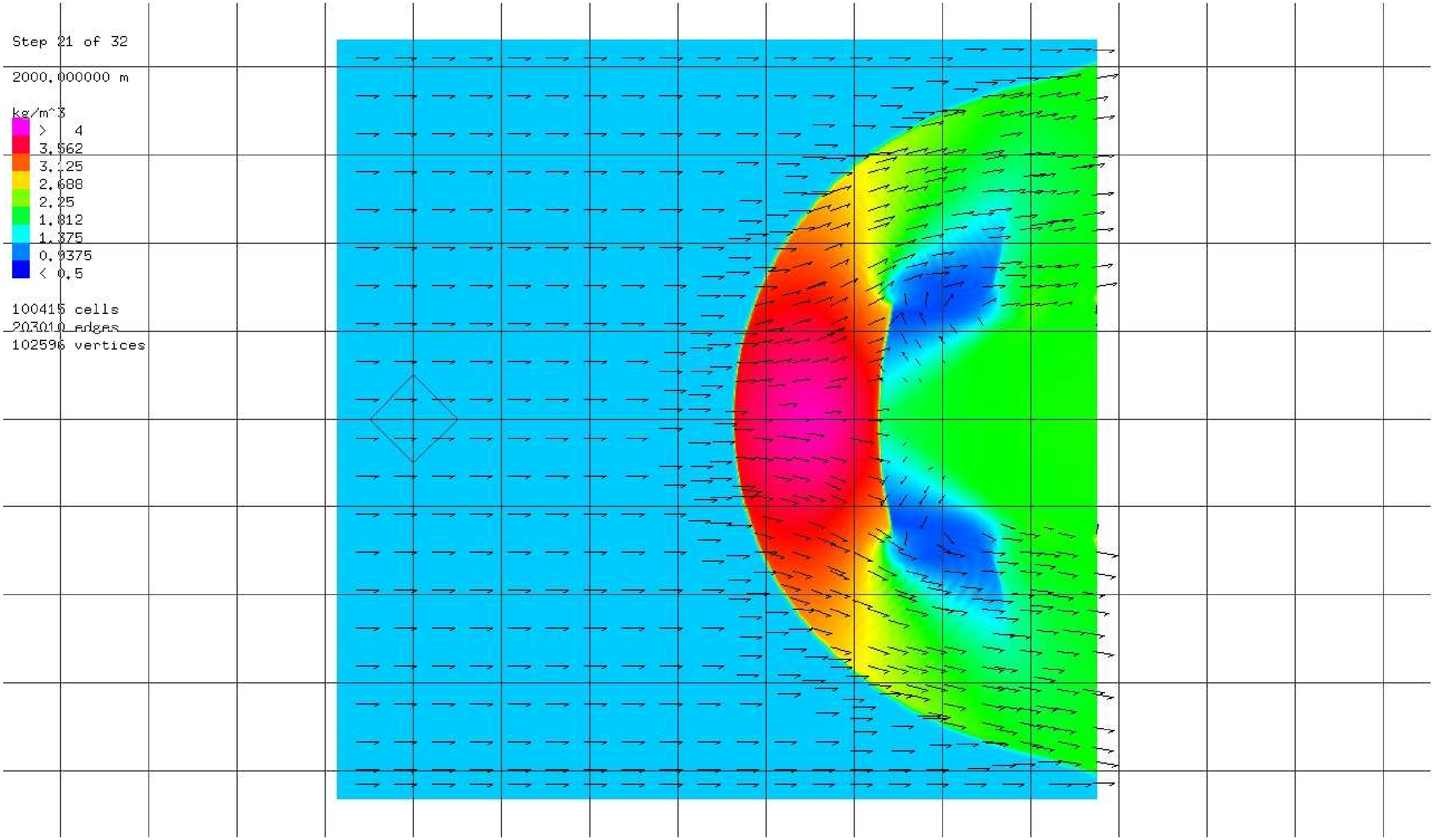
Step 17 of 32
2000.000000 m
kg/m^3
> 4
3.562
3.125
2.688
2.25
1.812
1.375
0.9375
< 0.5
73846 cells
149939 edges
76094 vertices



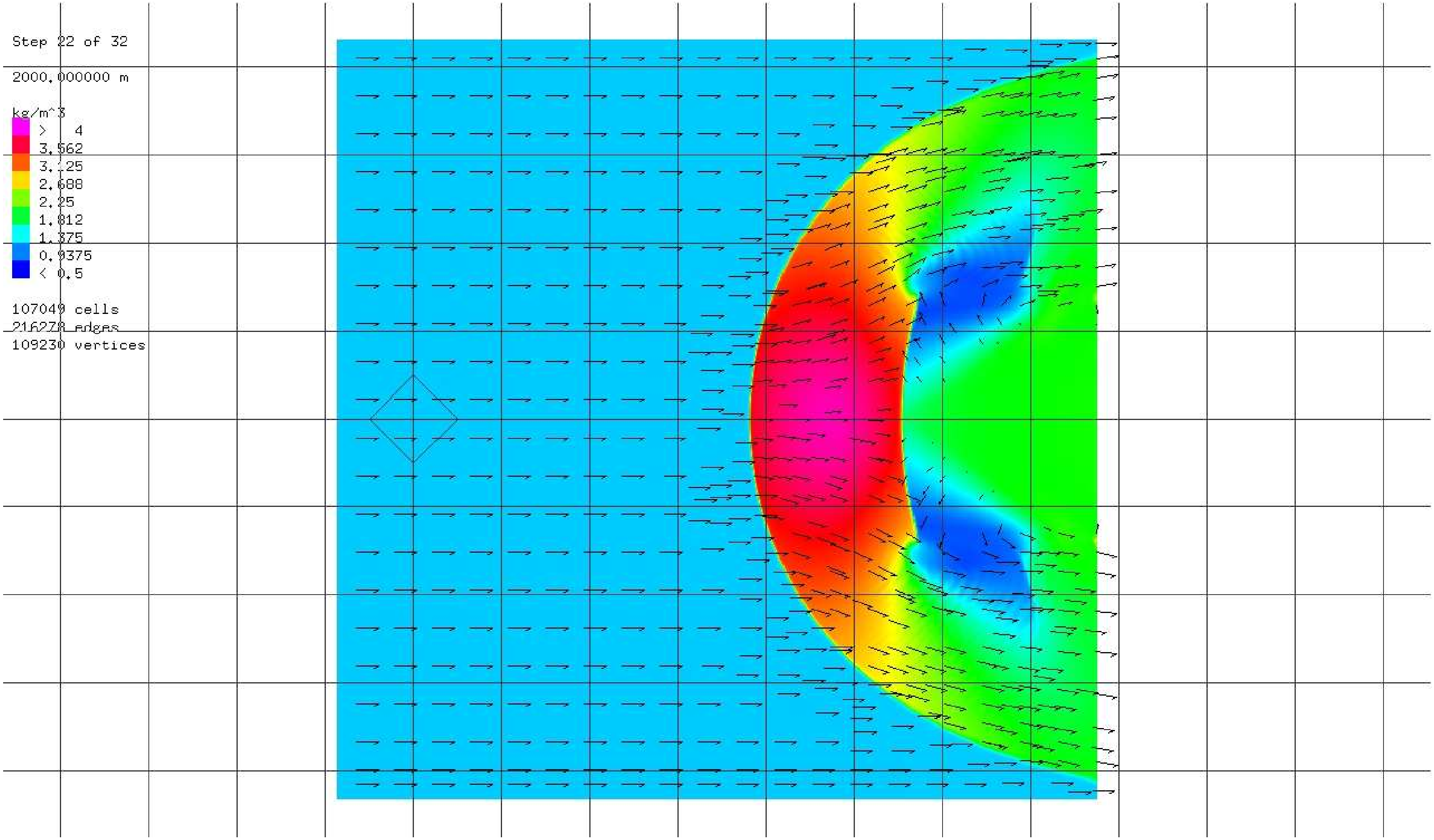
Step 18 of 32
2000.000000 m
kg/m^3
> 4
3.562
3.125
2.688
2.25
1.812
1.375
0.9375
< 0.5
80175 cells
162633 edges
82459 vertices


Step 19 of 32
2000.000000 m
kg/m^3
> 4
3.562
3.125
2.688
2.25
1.812
1.375
0.9375
< 0.5
86626 cells
175462 edges
88837 vertices

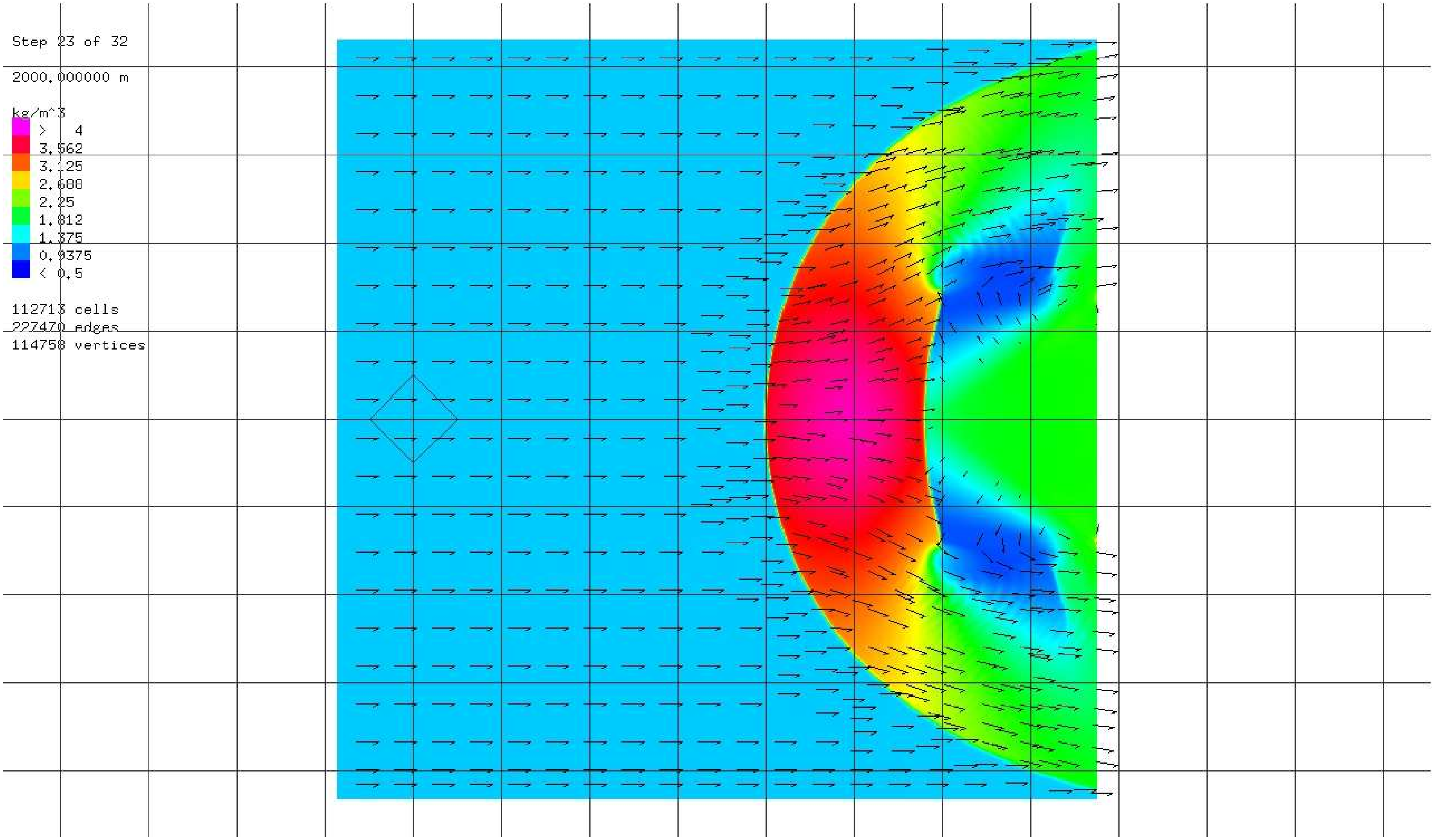

Step 20 of 32
2000.000000 m
kg/m^3
> 4
3.562
3.125
2.688
2.25
1.812
1.375
0.9375
< 0.5
93329 cells
95519 vertices

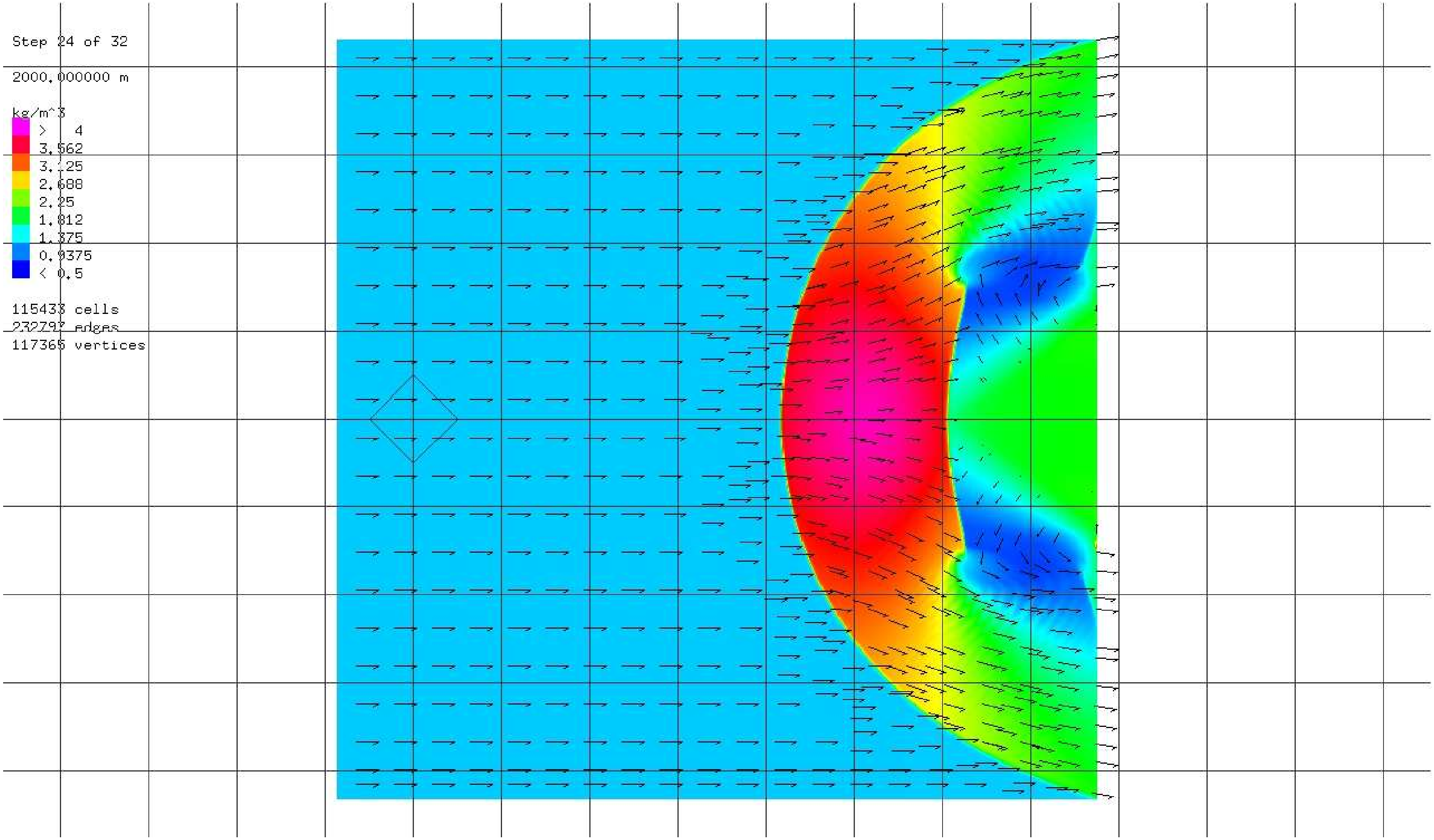


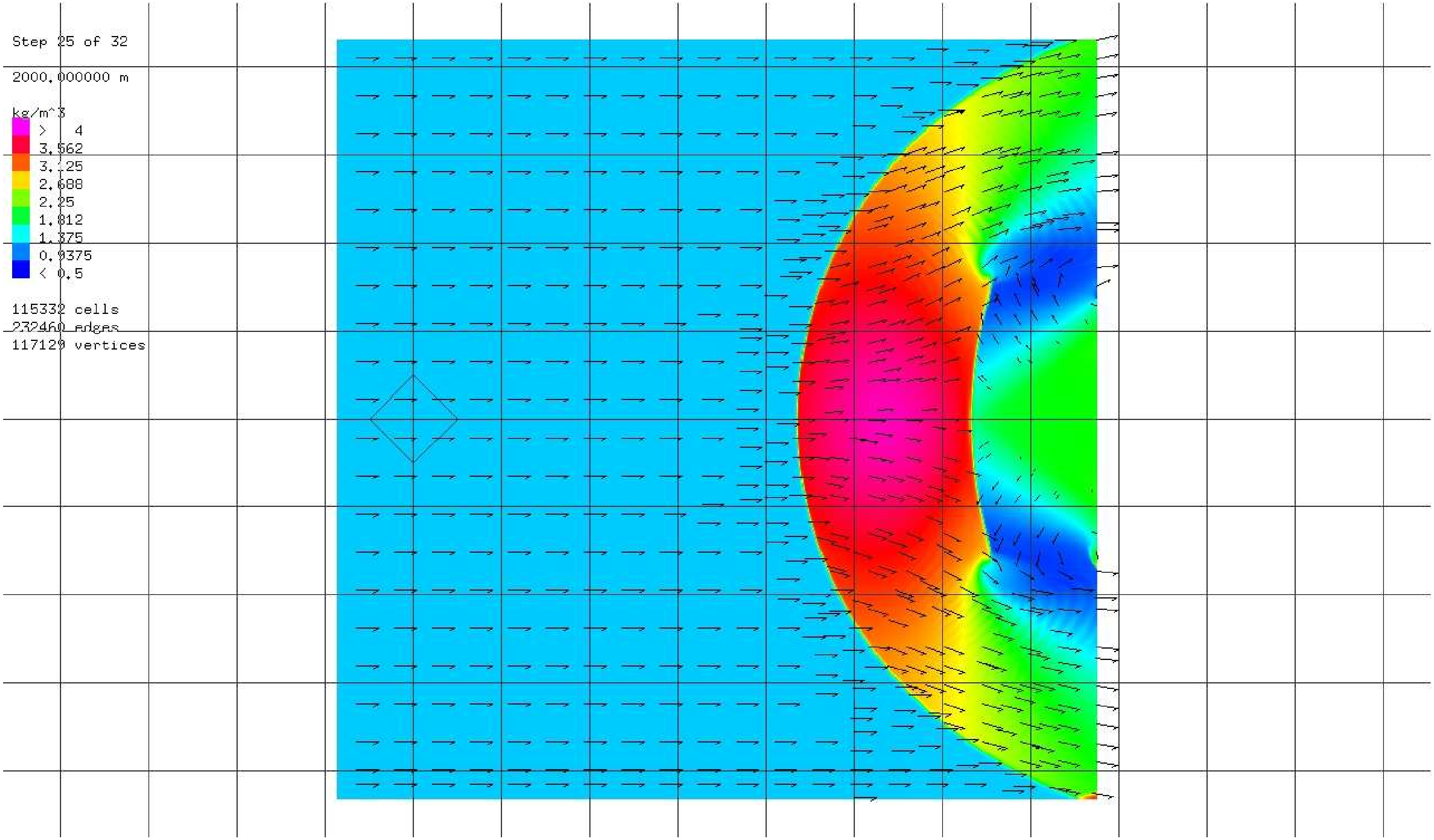
Step 21 of 32
2000.000000 m
kg/m^3
> 4
3.562
3.125
2.688
2.25
1.812
1.375
0.9375
< 0.5
100415 cells
203010 edges
102596 vertices



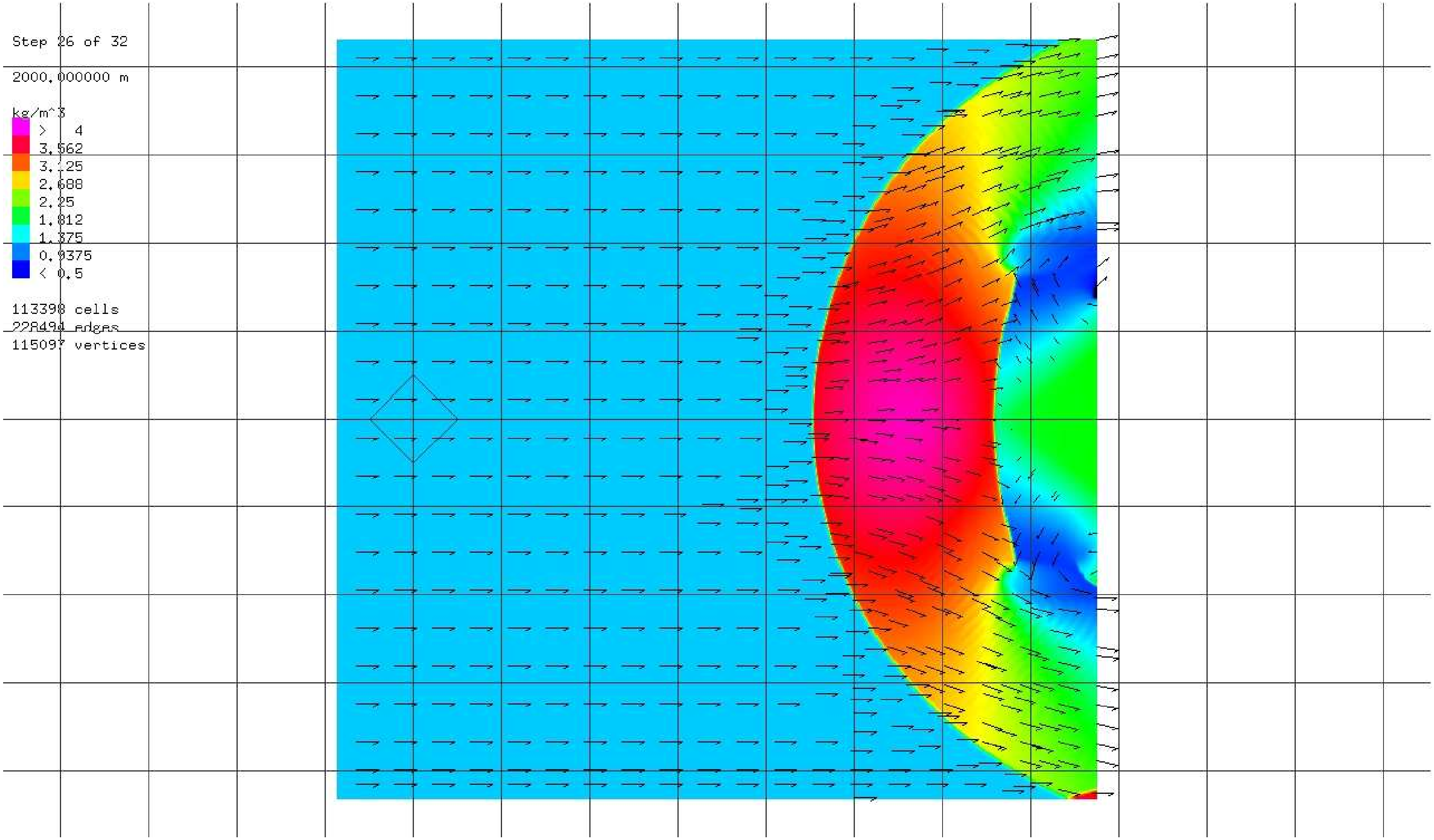
Step 22 of 32
2000.000000 m
kg/m^3
> 4
3.562
3.125
2.688
2.25
1.812
1.375
0.9375
< 0.5
107049 cells
216278 edges
109230 vertices


Step 23 of 32
2000.000000 m
kg/m^3
> 4
3.562
3.125
2.688
2.25
1.812
1.375
0.9375
< 0.5
112713 cells
227470 edges
114758 vertices


Step 24 of 32
2000.000000 m
kg/m^3
> 4
3.562
3.125
2.688
2.25
1.812
1.375
0.9375
< 0.5
115433 cells
232797 edges
117365 vertices



Step 25 of 32
2000.000000 m
kg/m^3
> 4
3.562
3.125
2.688
2.25
1.812
1.375
0.9375
< 0.5
115332 cells
232460 edges
117129 vertices


Step 26 of 32
2000.000000 m
kg/m^3
> 4
3.562
3.125
2.688
2.25
1.812
1.375
0.9375
< 0.5
113398 cells
228494 edges
115097 vertices

Second solution

nuqst-jpg

Essentially same numerical solution for:

✓ Lax-Friedrichs, Godunov, Solomon-Osher, local Lax-Friedrichs

✓ plain first-order, or second-order corrections (slope limiter)

✓ isentropic and non-isentropic Euler, $\gamma = 7/5,\ 5/3, \ldots$

✓ Cartesian or adaptive aligned grids

✓ (t, x) and $(t, x/t)$ coordinates

Same initial data, but
numerical solution $\not\approx$ theoretical solution

$\Rightarrow$ Non-uniqueness not a mere mathematical curiosity, but affects numerics and applications

Note: solution piecewise smooth, unlike de Lellis/Szekelyhidi examples

Lax-Wendroff theorem

Lax-Wendroff theorem: numerical scheme

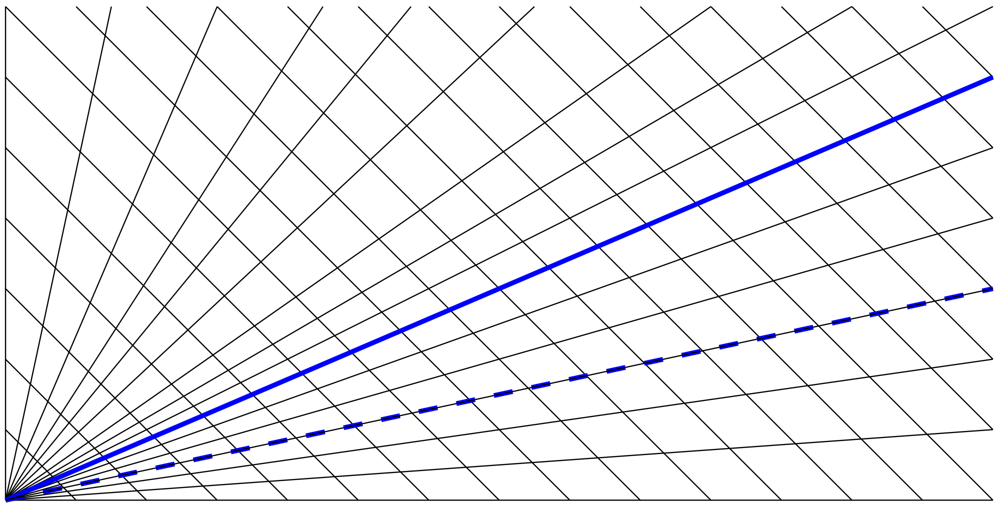
1. conservative,
2. consistent,
3. has discrete entropy inequality,
4. converges as grid becomes infinitely fine,

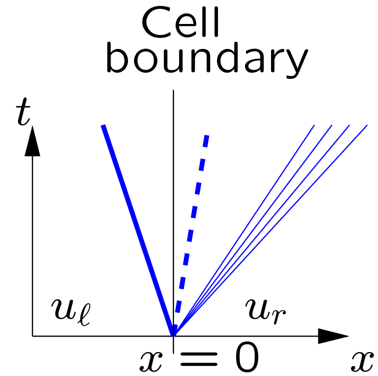
then limit is entropy solution.

Godunov scheme: 1-3 known to be satisfied, 4 seems to apply

$\rightsquigarrow$ If convergence, then second solution is entropy, too.

Trouble for popular numerical schemes





On this grid, Godunov scheme (with exact arithmetic) converges (trivially) to theoretical solution.

On other grids (with realistic arithmetic): convergence to different solution observed.

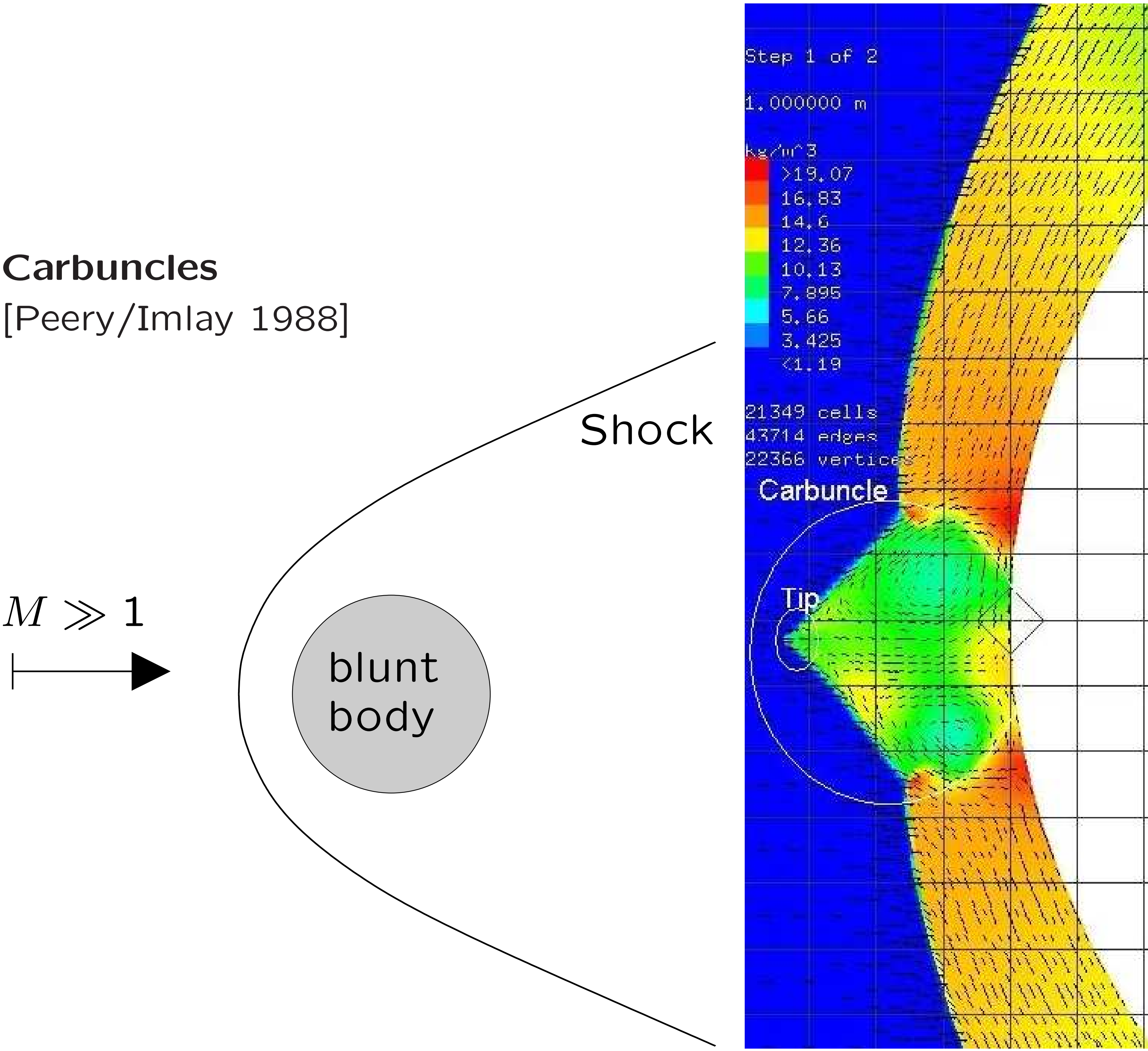
(Proof? Even if wrong, no convergence on reasonably fine grids)

Forget about convergence theory in ≥ 2 dimensions

"The theoretical (steady) solution is 'unstable' and we may expect the second solution to be the unique physically correct one?"

Carbuncles

[Peery/Imlay 1988]



Triggering carbuncles reliably

Carbuncles: present in Godunov scheme, Roe scheme, higher-order schemes, apparently absent in Lax-Friedrichs.

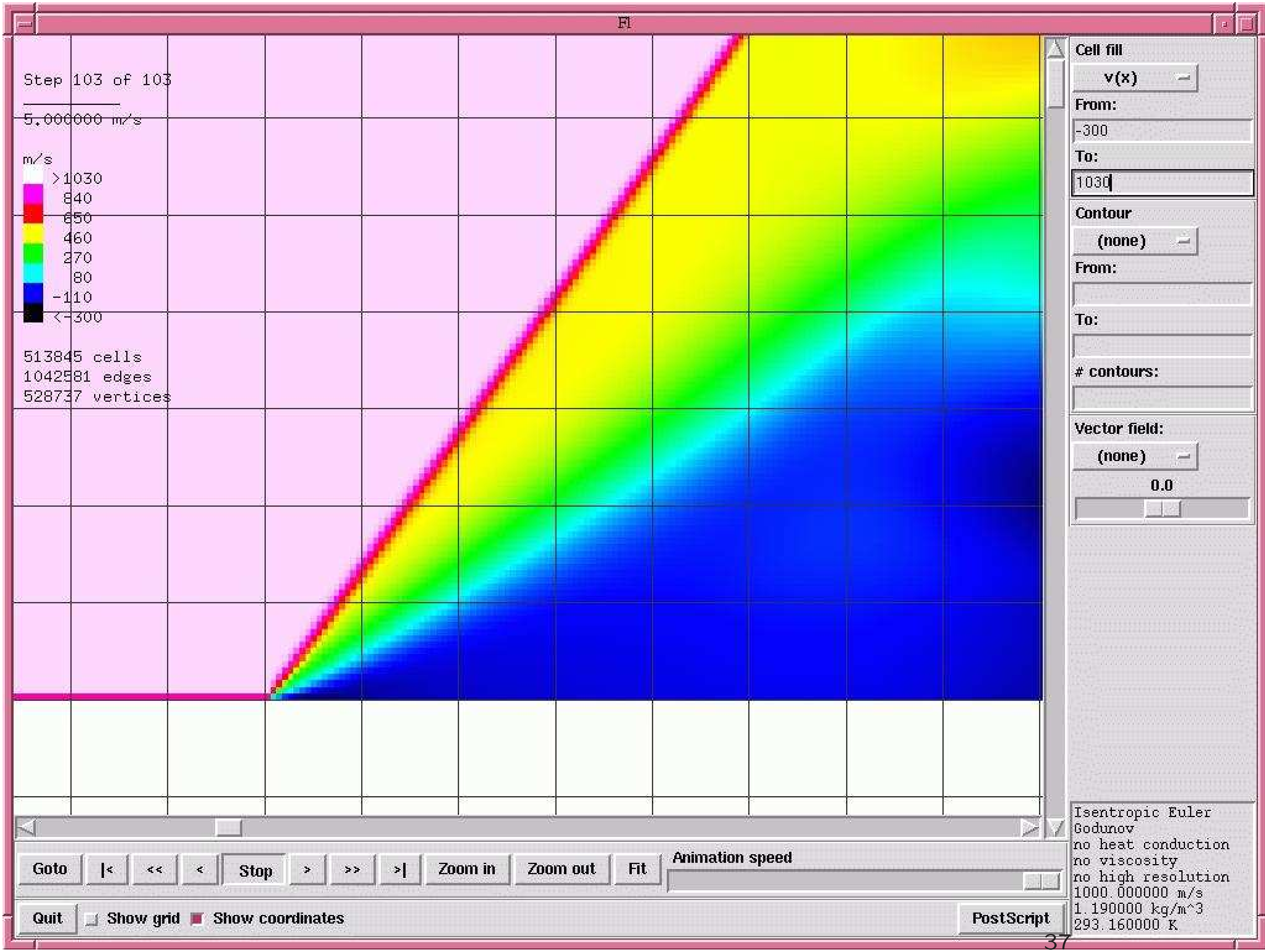
Hard to suppress, or trigger, reliably

Trick: generate a thin filament of reduced horizontal velocity

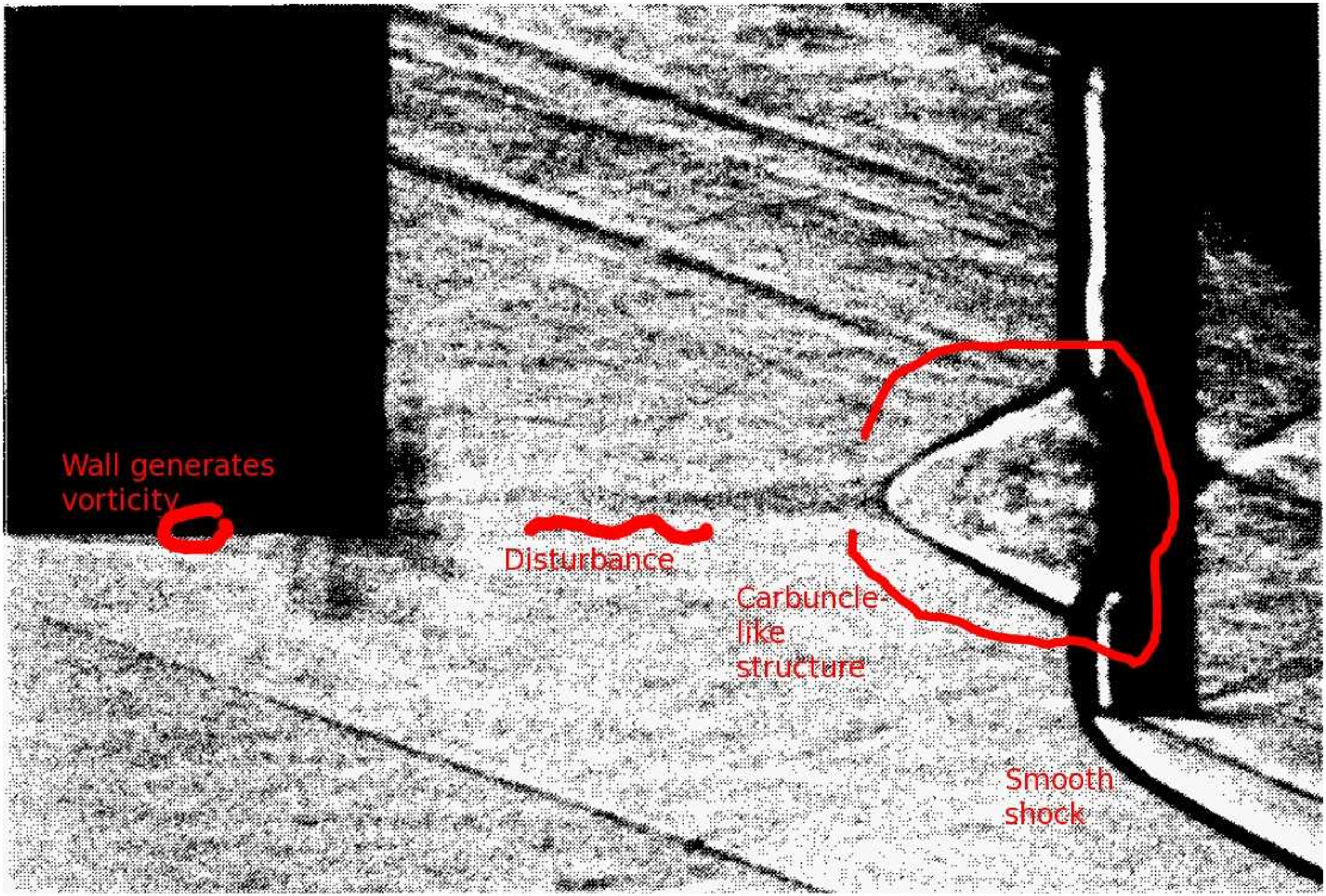
dyncarb-jpg

Result: impinges on shock, produces large-scale perturbation

Similar to initial data in non-uniqueness example


Step 103 of 103
5.000000 m/s
m/s
>1030
840
650
460
270
80
-110
<-300
513845 cells
1042581 edges
528737 vertices
Cell fill
v(x)
From:
-300
To:
1030
Contour
(none)
From:
To:
contours:
Vector field:
(none)
0.0
Isentropic Euler
Godunov
no heat conduction
no viscosity
no high resolution
1000.000000 m/s
1.190000 kg/m^3
293.160000 K
Goto
|<
<<
<
Stop
>
>>
>|
Zoom in
Zoom out
Fit
Animation speed
Quit
Show grid
Show coordinates
PostScript

[Kalkhoran/Sforza/Wang 1991]



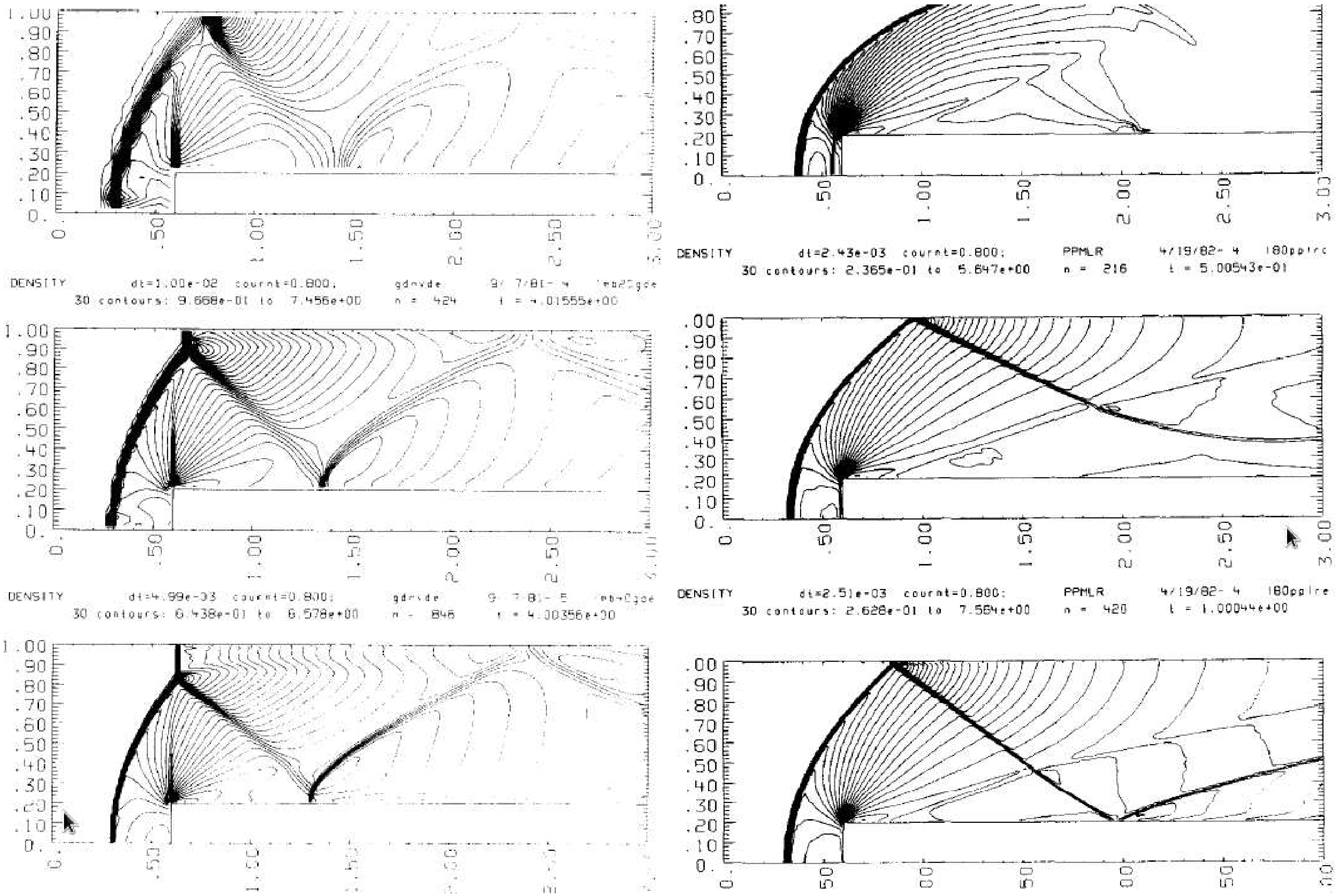
Conclusions

1. "Non-uniqueness will be cured by better analysis and numerics"
2. "Numerical schemes with enough dissipation (Lax-Friedrichs) will not produce carbuncles. Challenge is merely to minimize dissipation while preserving correctness."

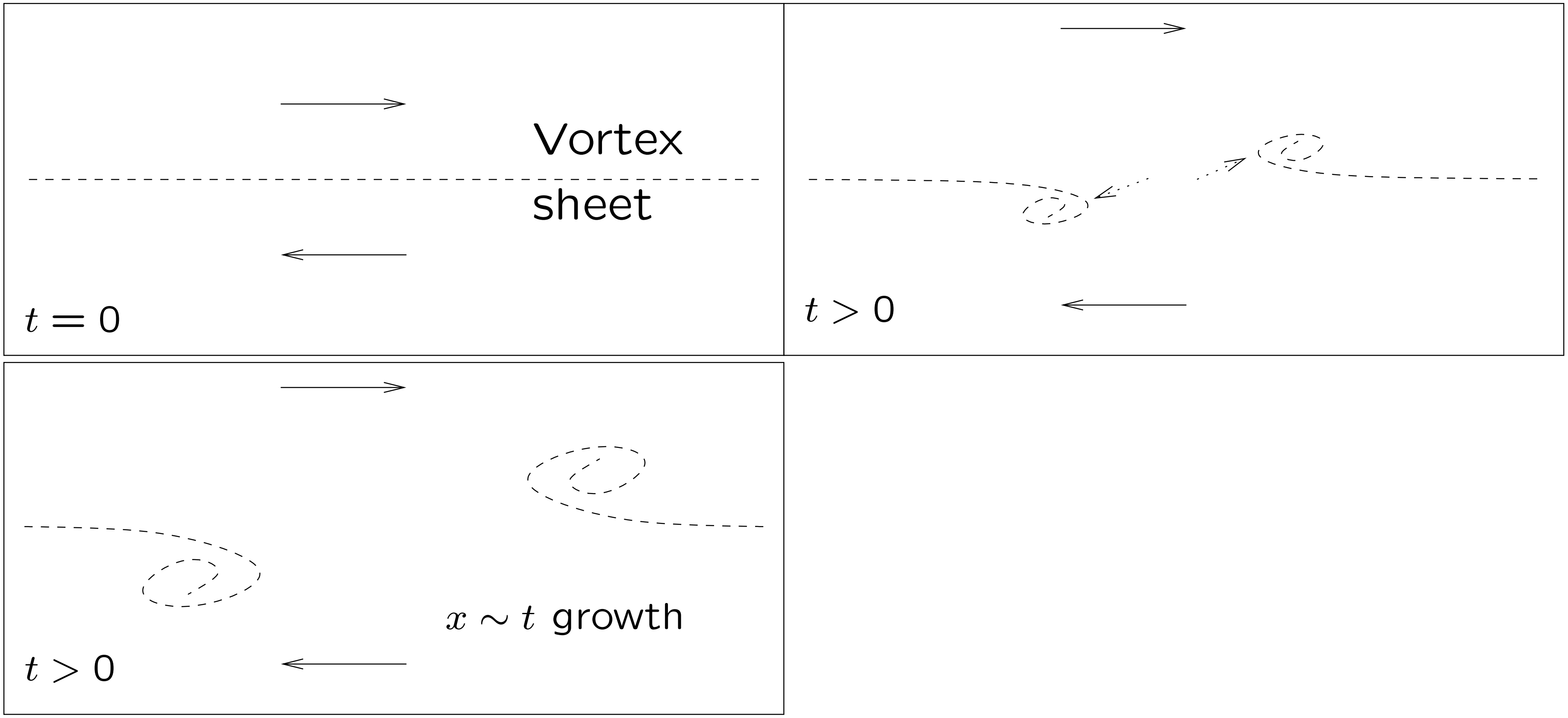
Kalkhoran/Sforza/Wang 1991, Ramalho/Azevedo 2009, Elling 2009:
carbuncle physically meaningful

3. "If we have uniqueness in H^s, but not in $H^{s-\epsilon}$, then H^s is the right space."
Planar shocks more regular than carbuncle, but sometimes carbuncle is correct.

[Colella/Woodward 1983]



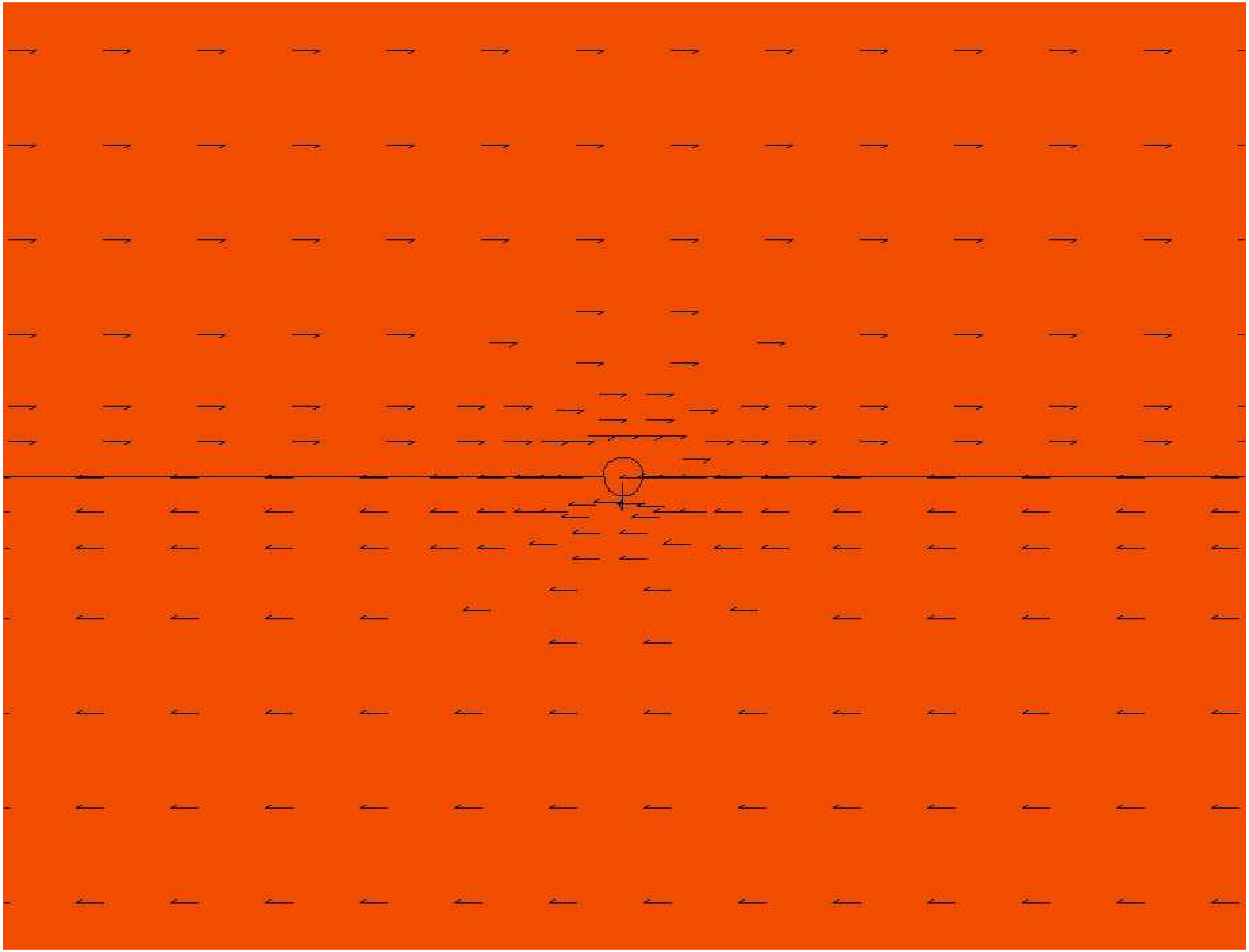
Pullin (1989) separated sheet ssbr/manymany.vs splitsheet

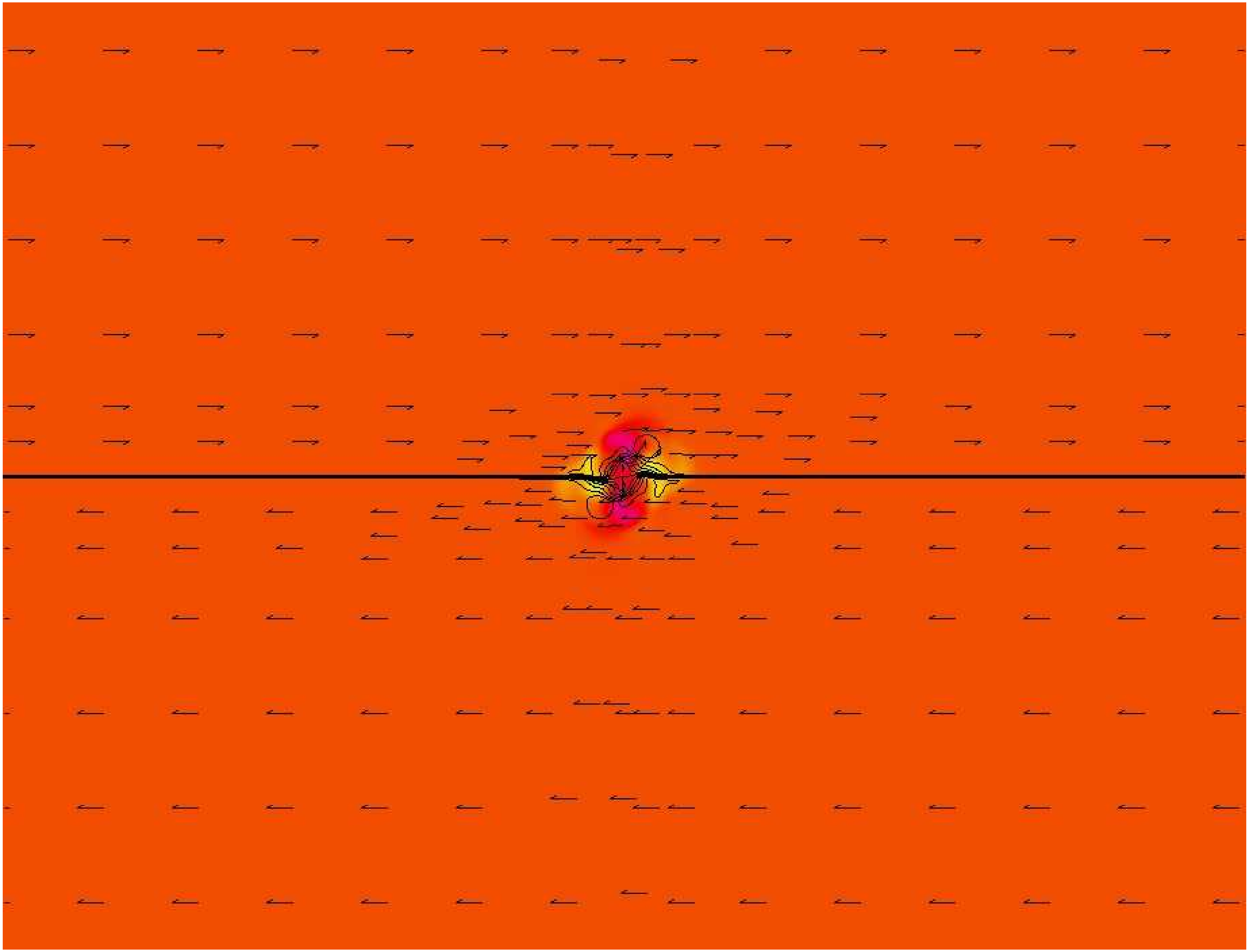


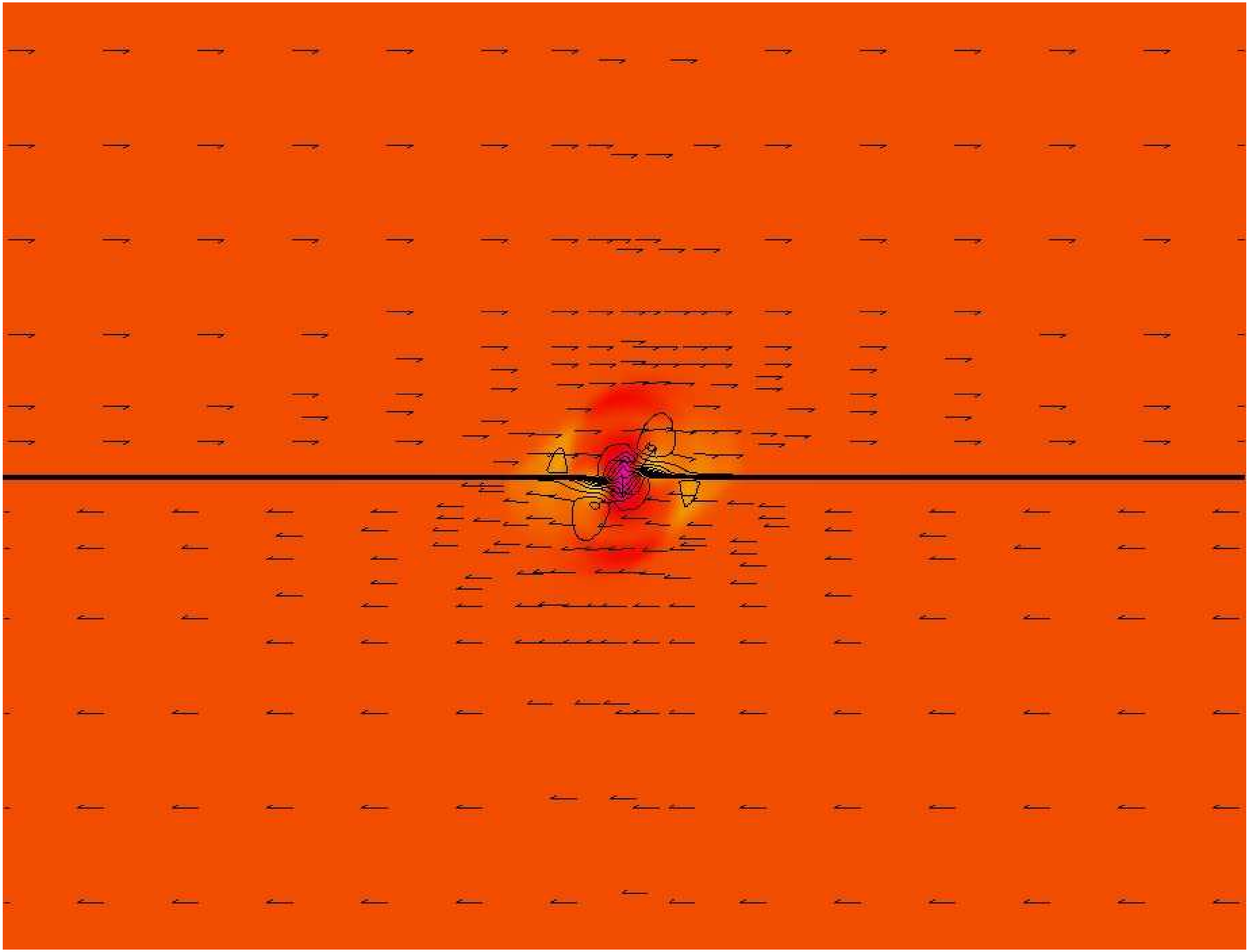
Current state: gap between two groups of counterexamples, rigorous but irregular vs. piecewise smooth but unproven.

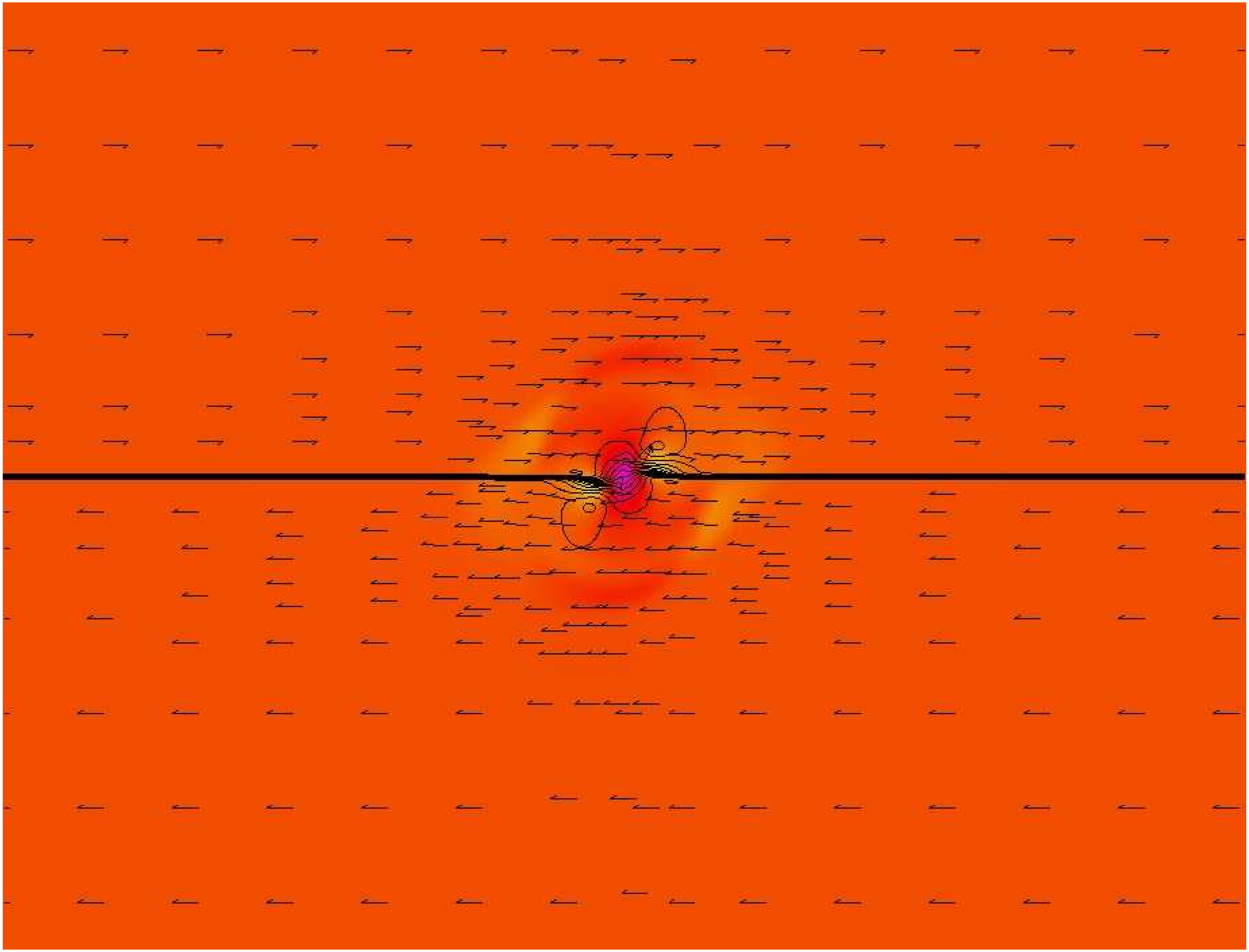
"De Lellis/Szekelyhidi solutions 'crazy'. Non-uniqueness can probably be avoided by narrowing function space or finding stronger admissibility condition."

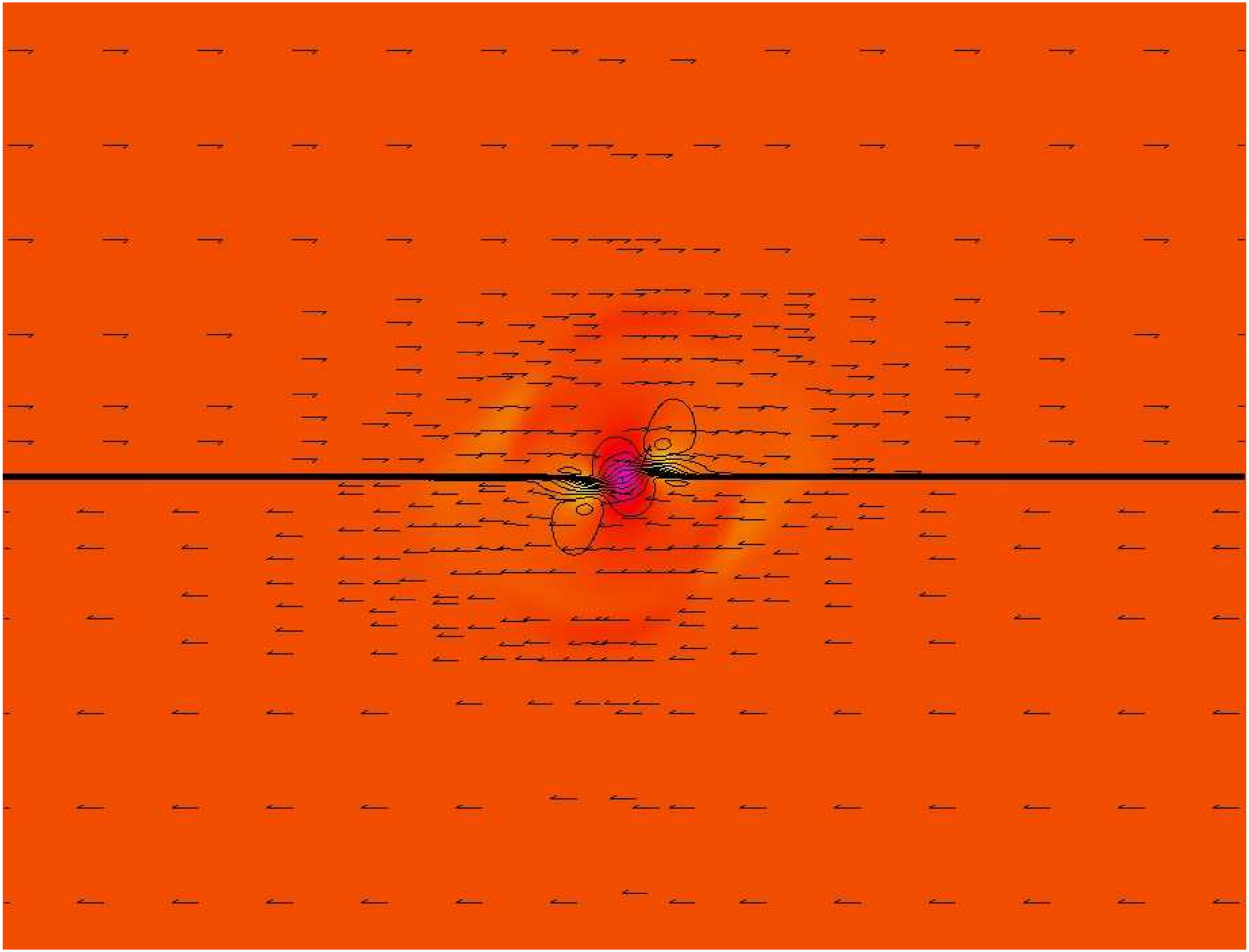
$\rightarrow$ Pullin solution contains only physically reasonable features

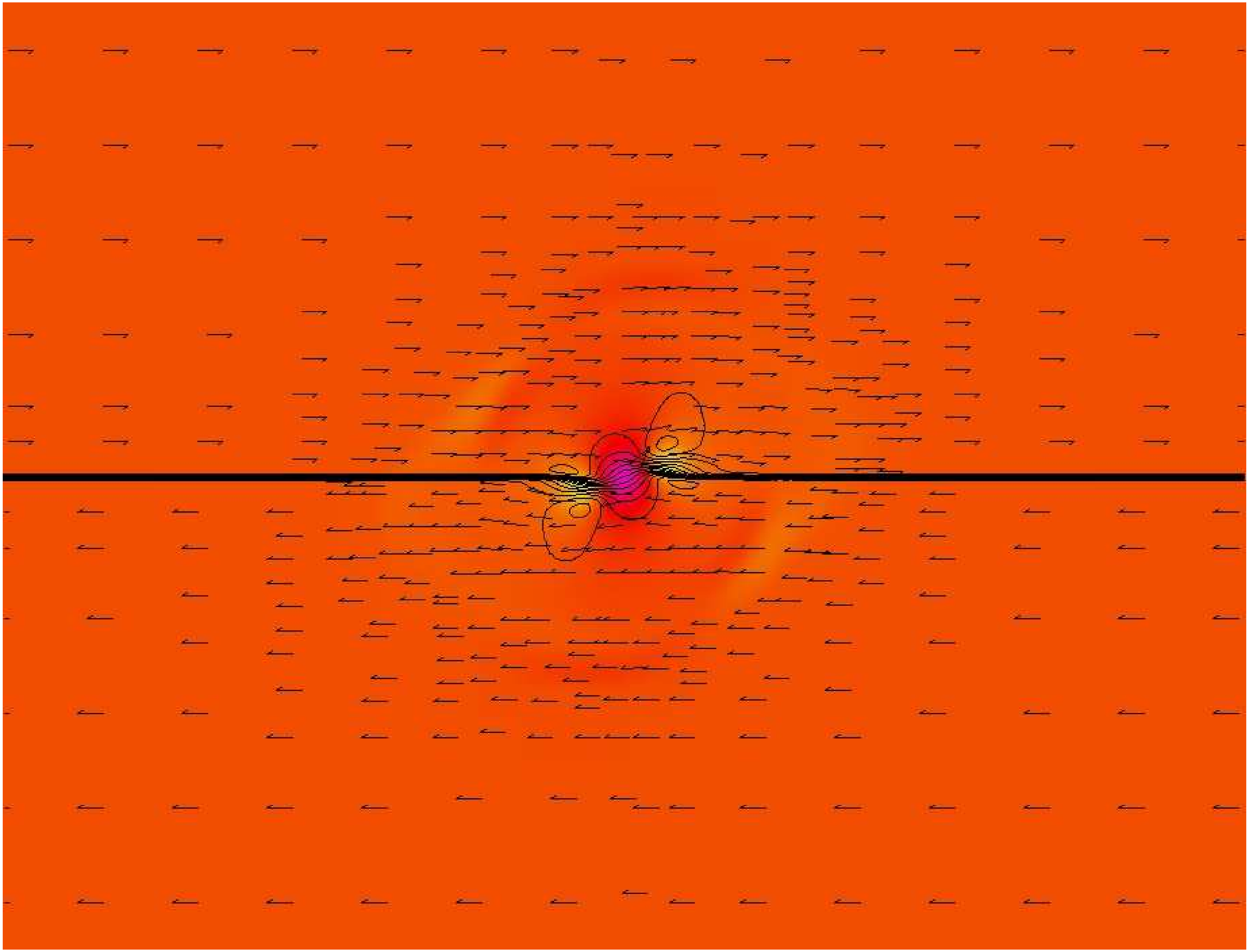


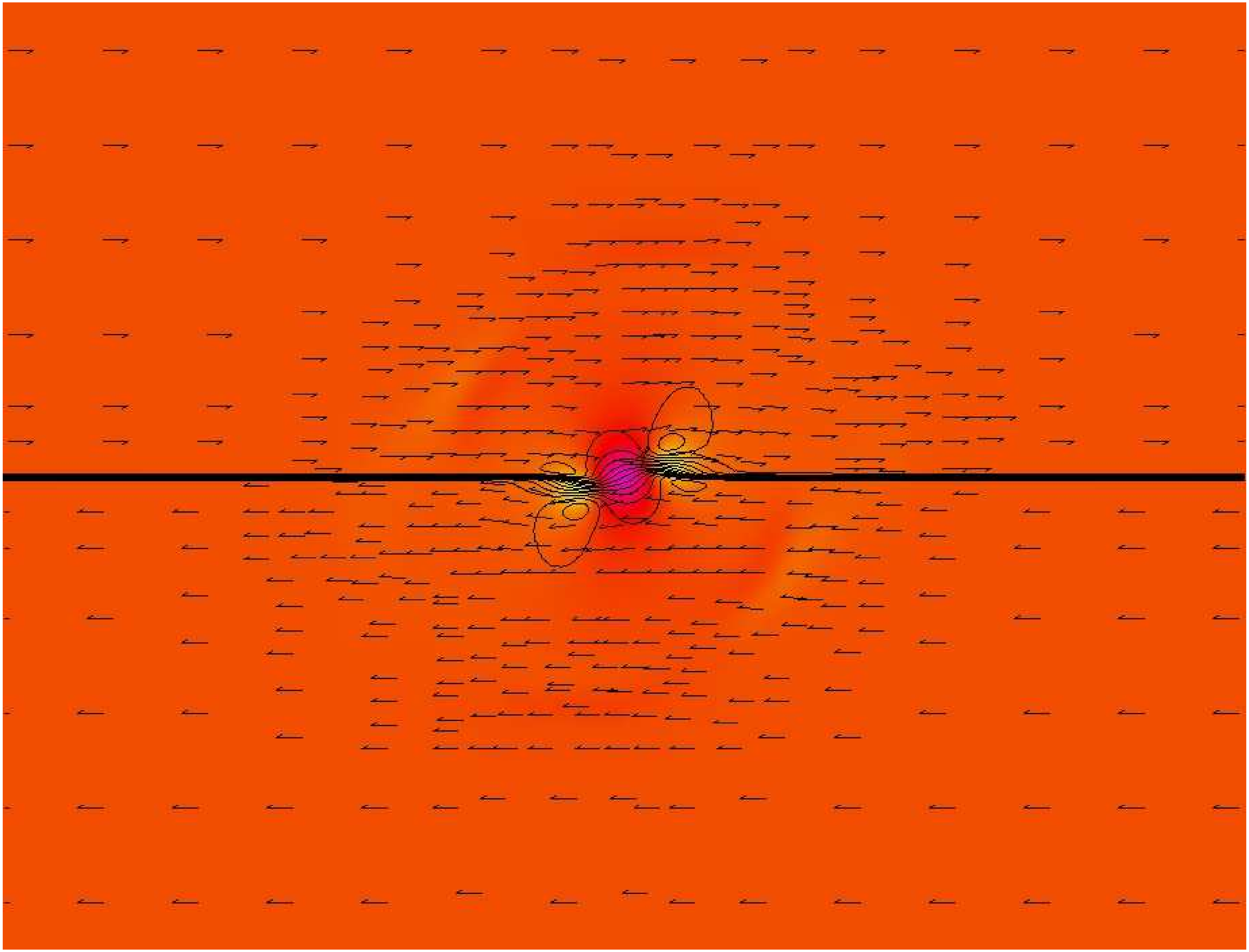


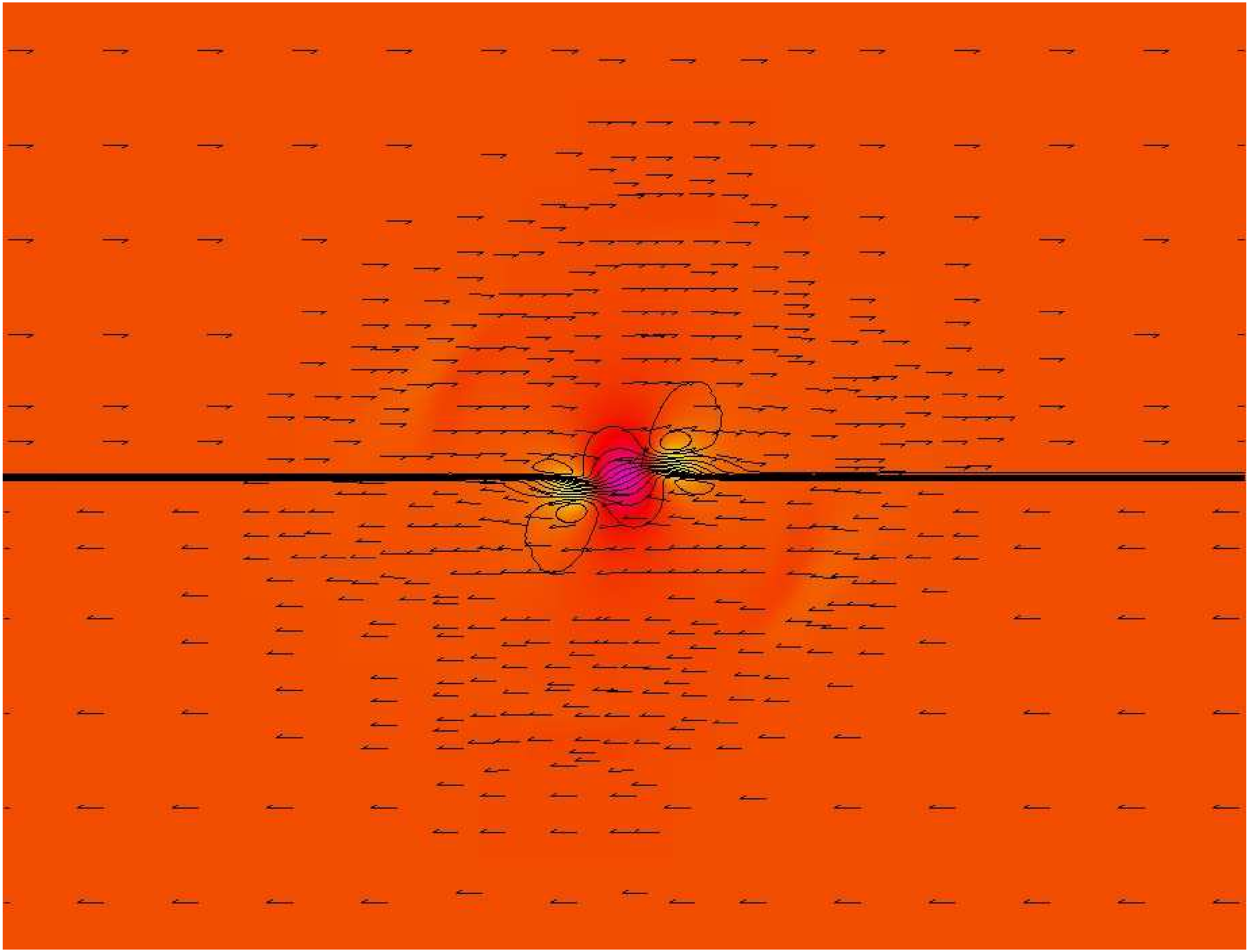


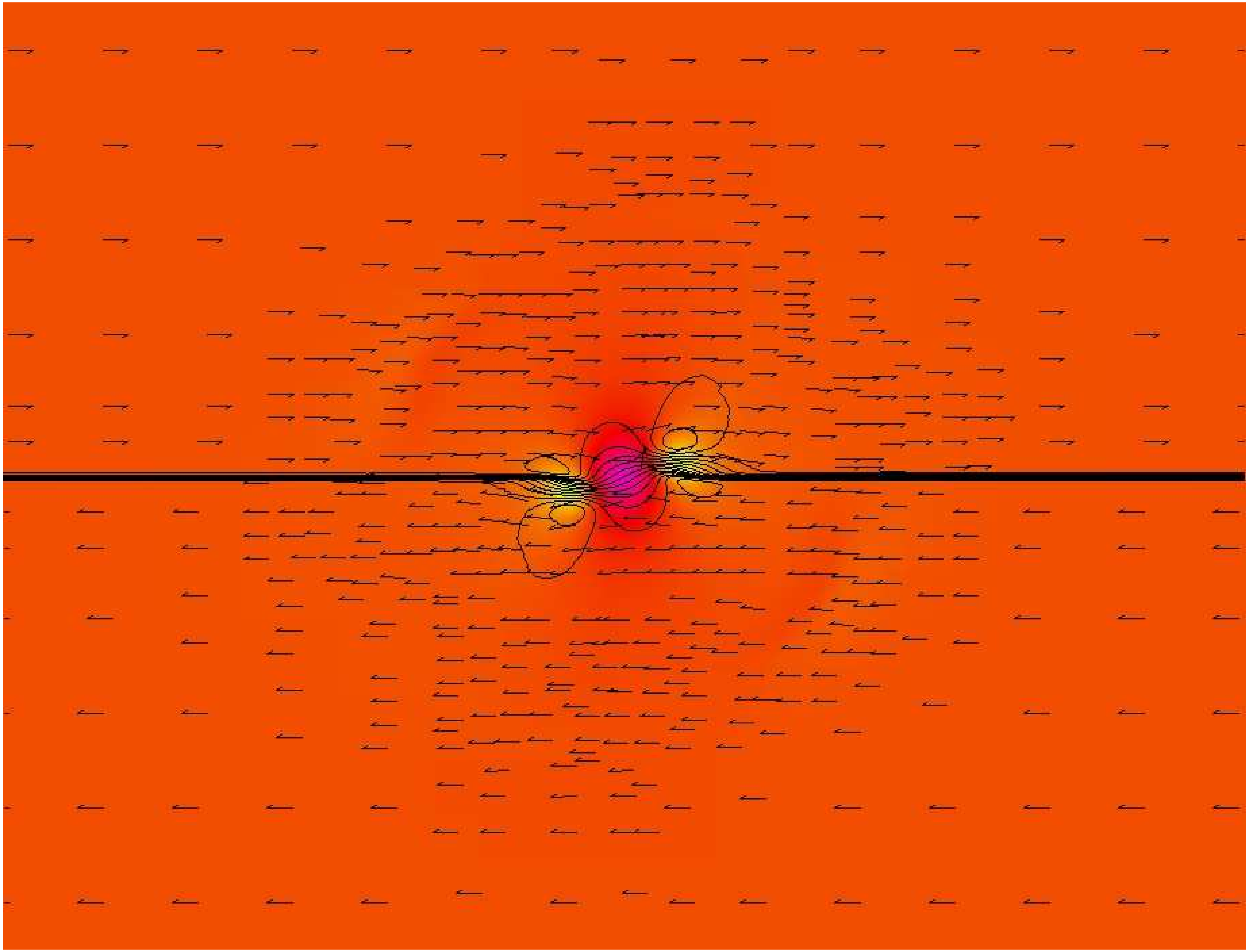


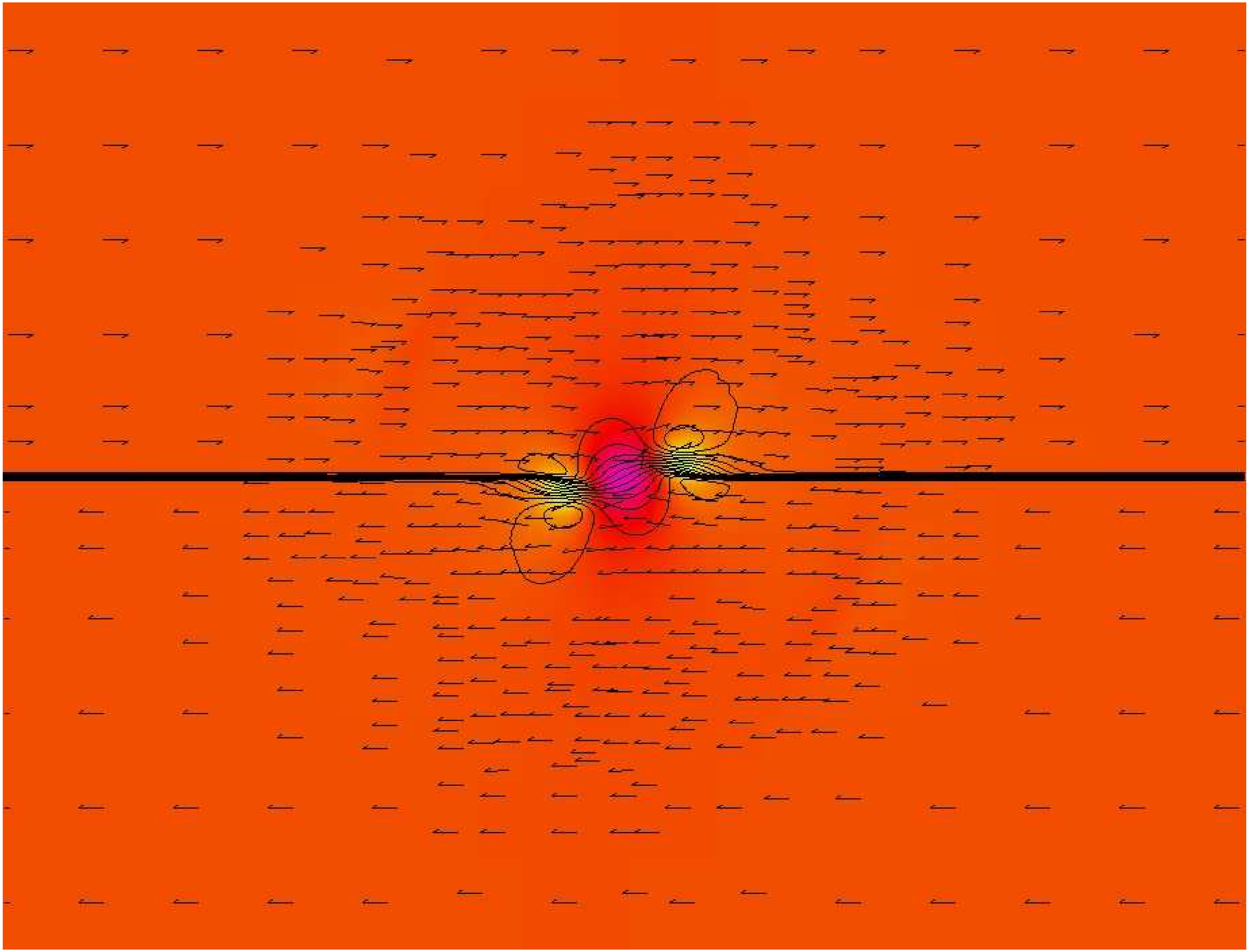


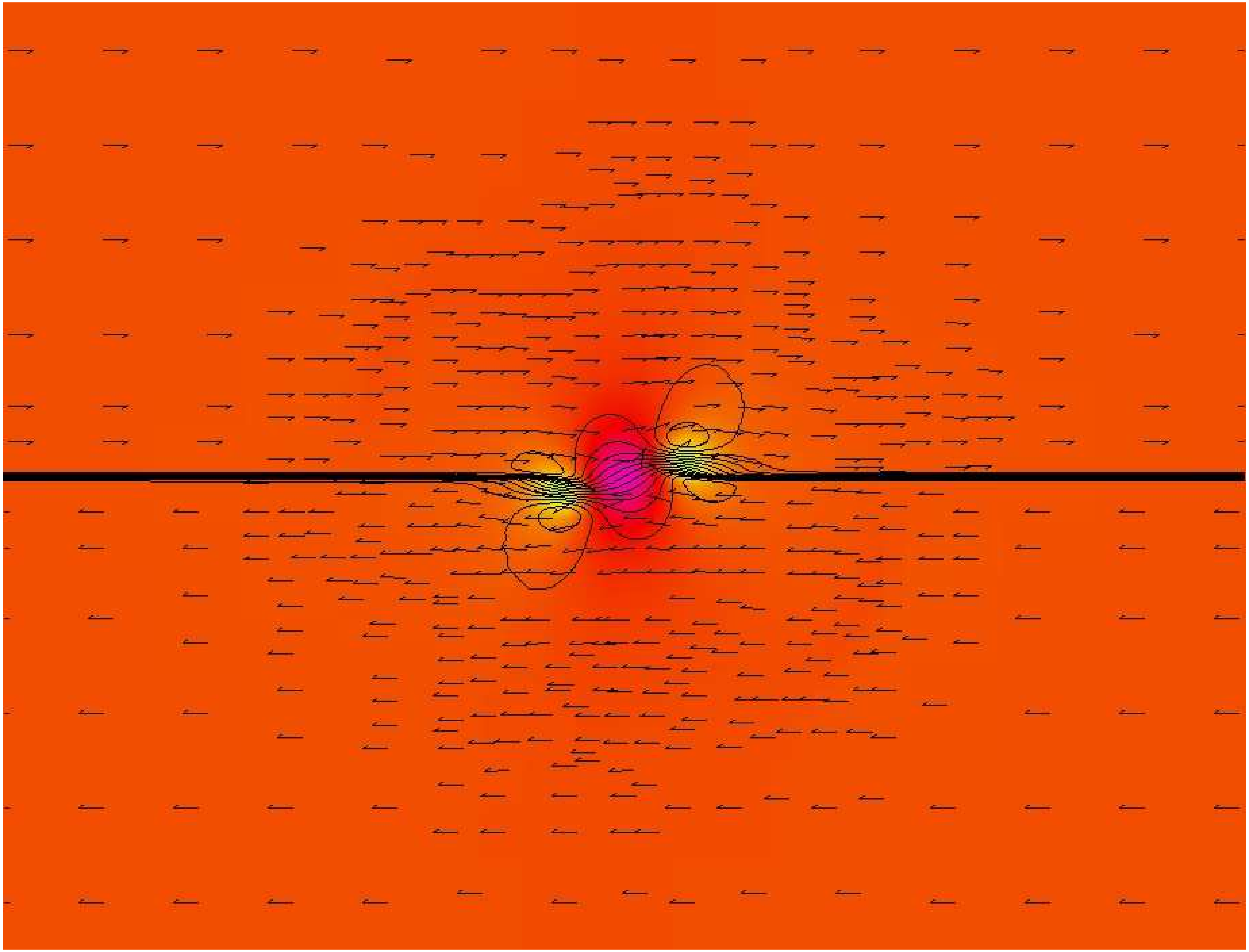


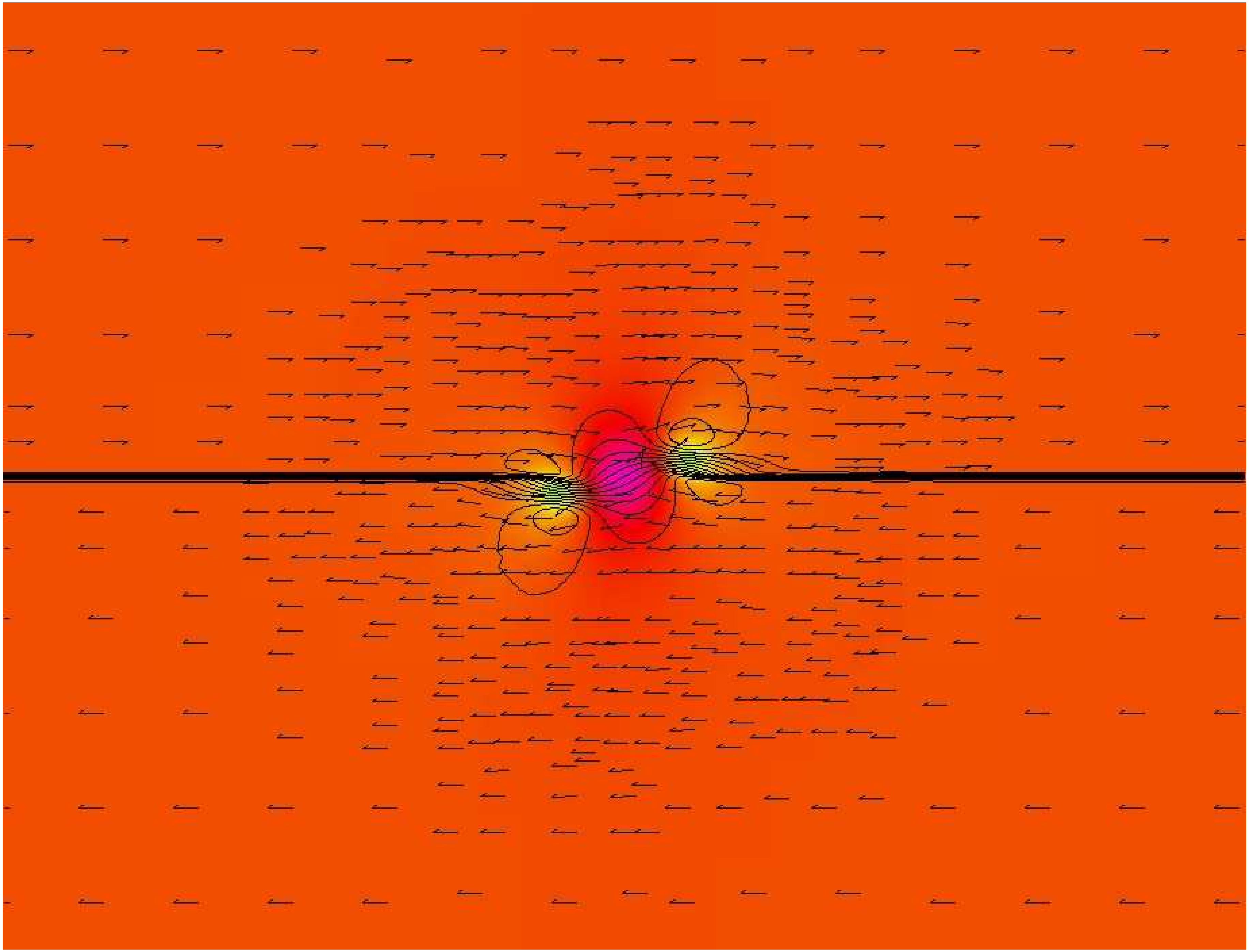


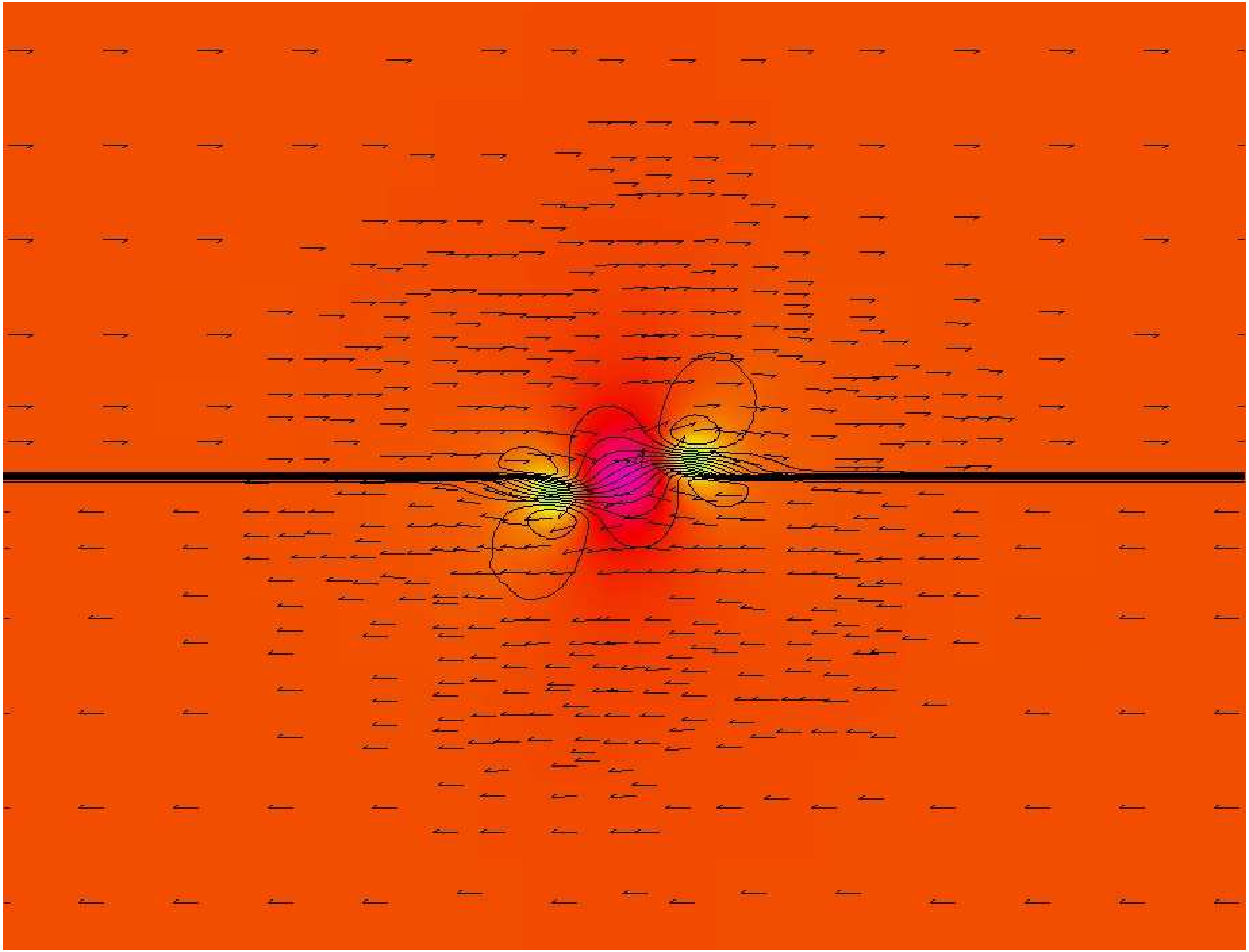


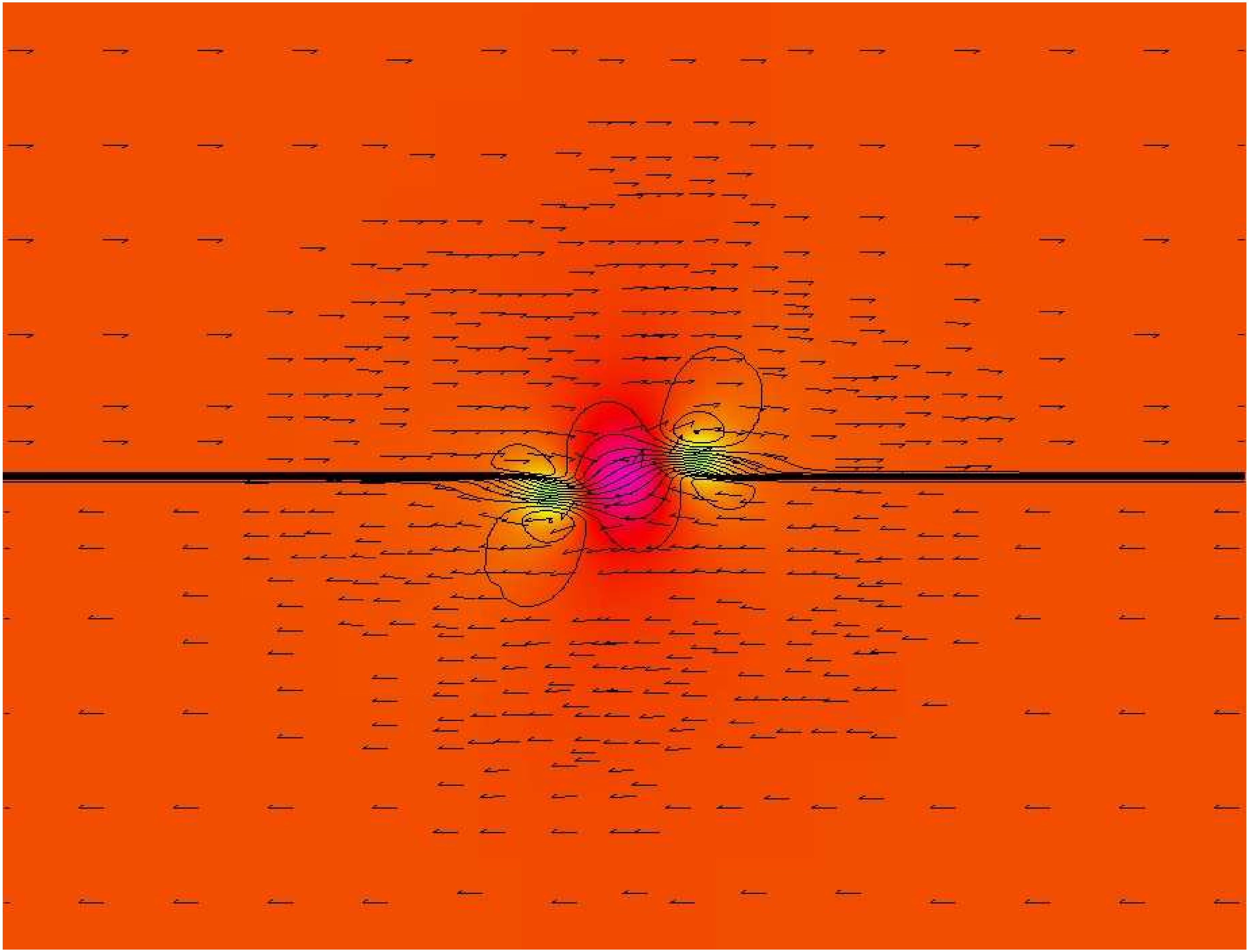


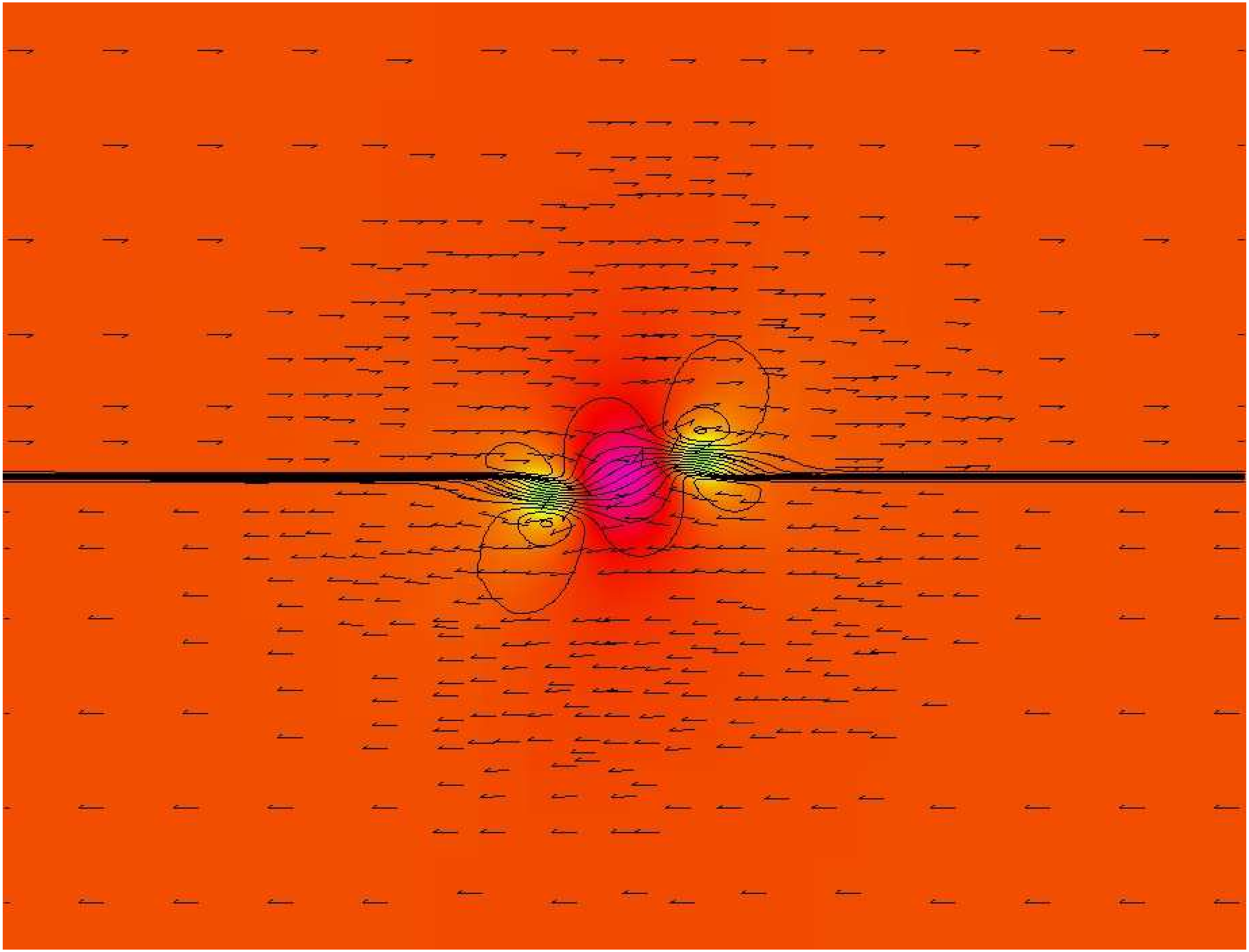


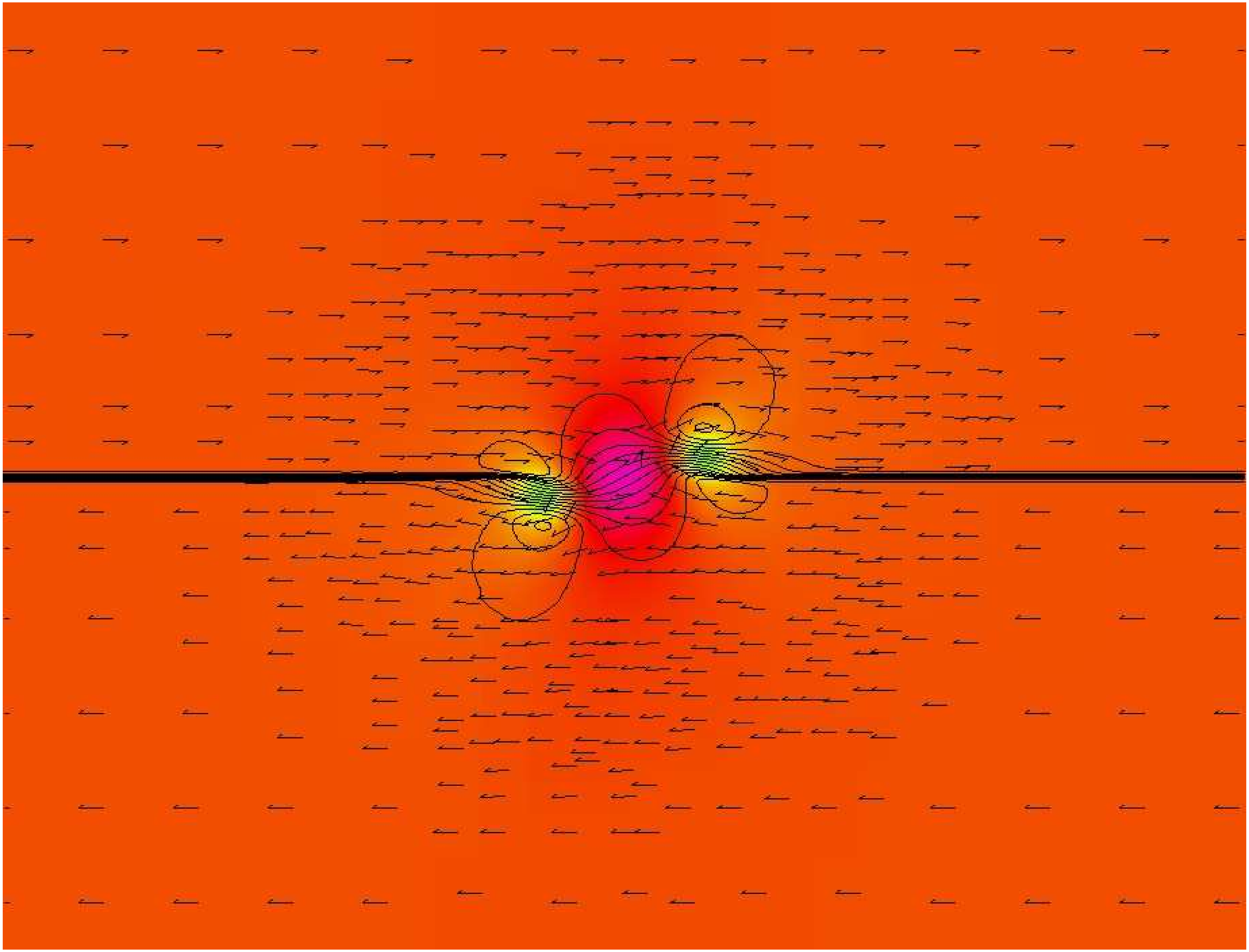


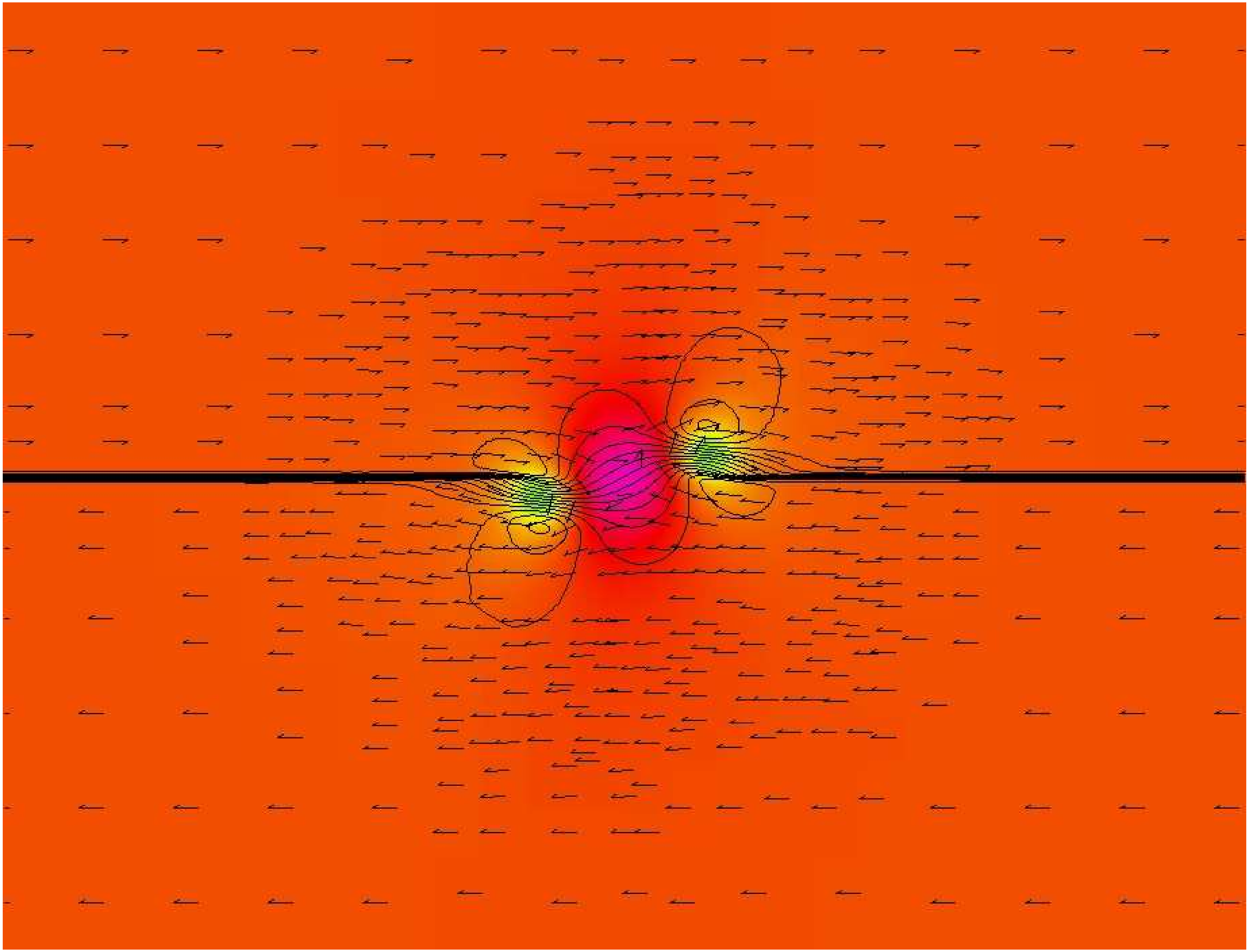


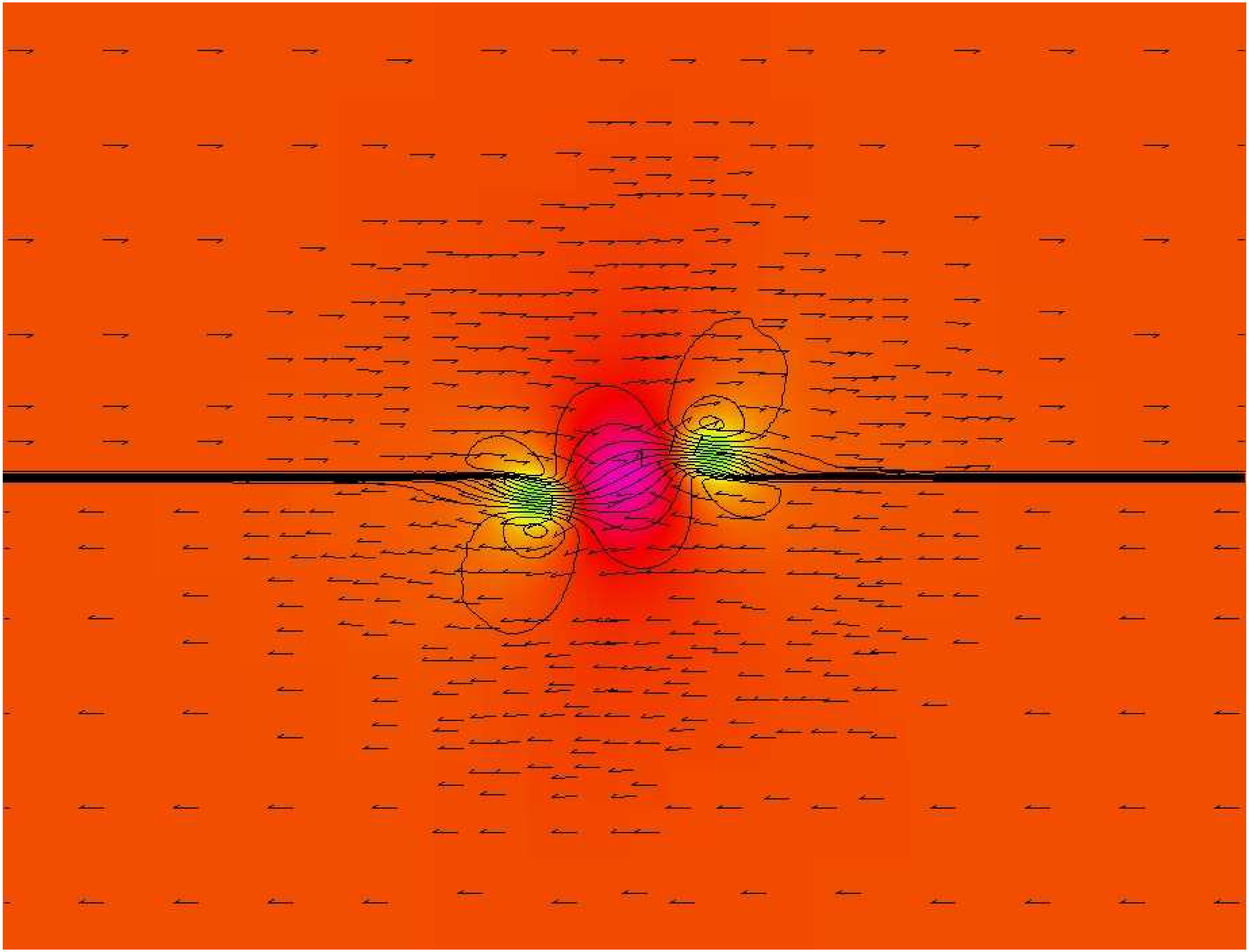


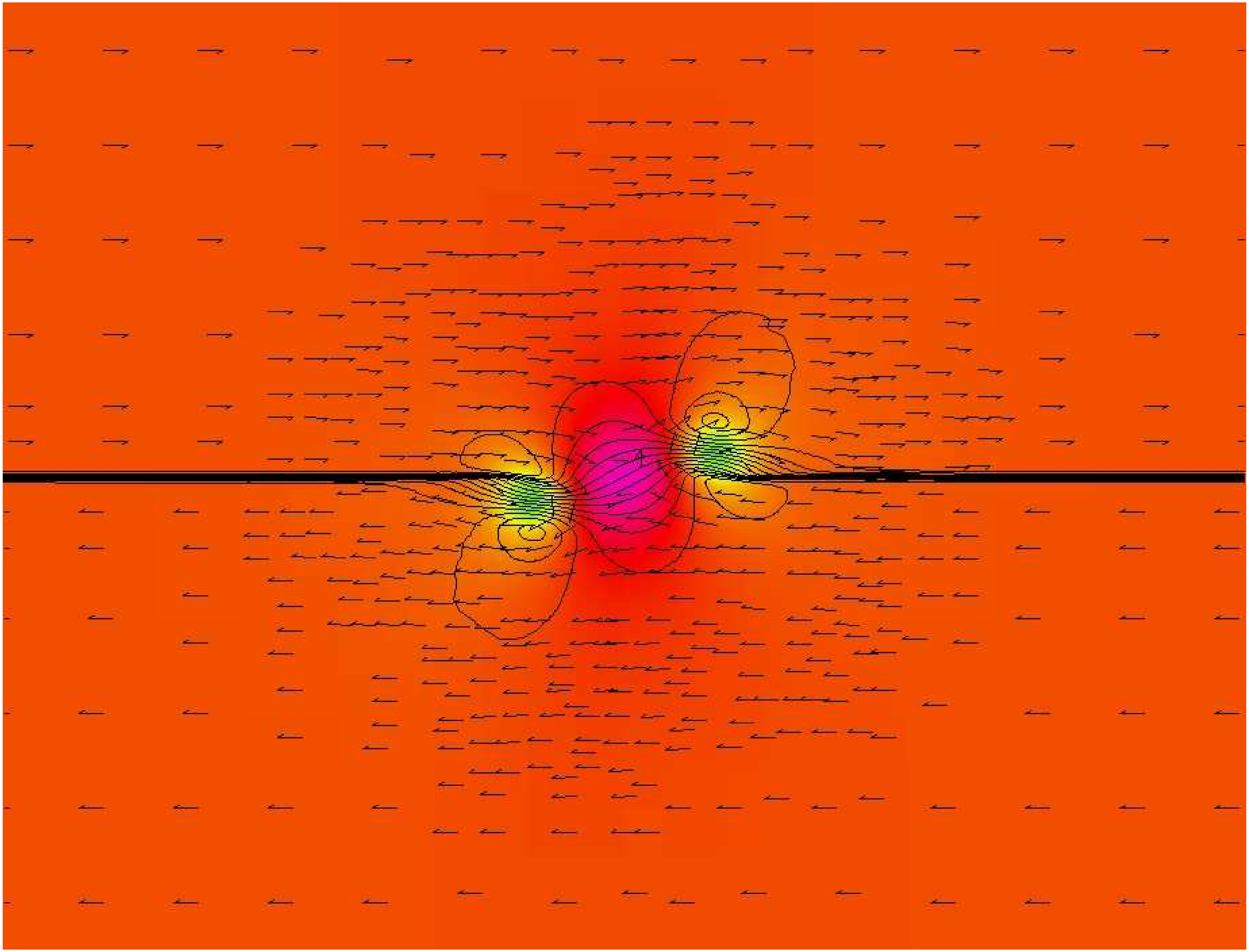


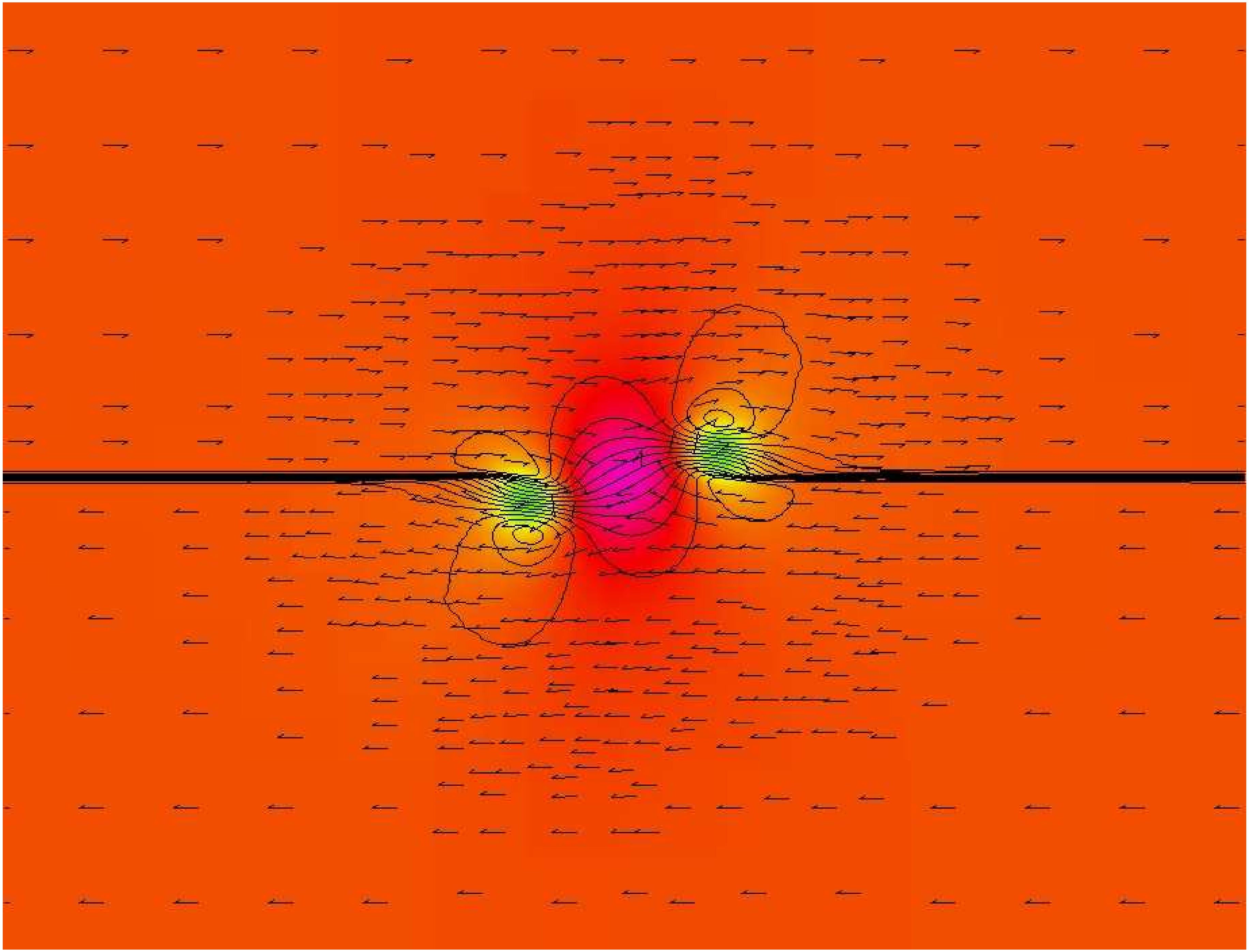


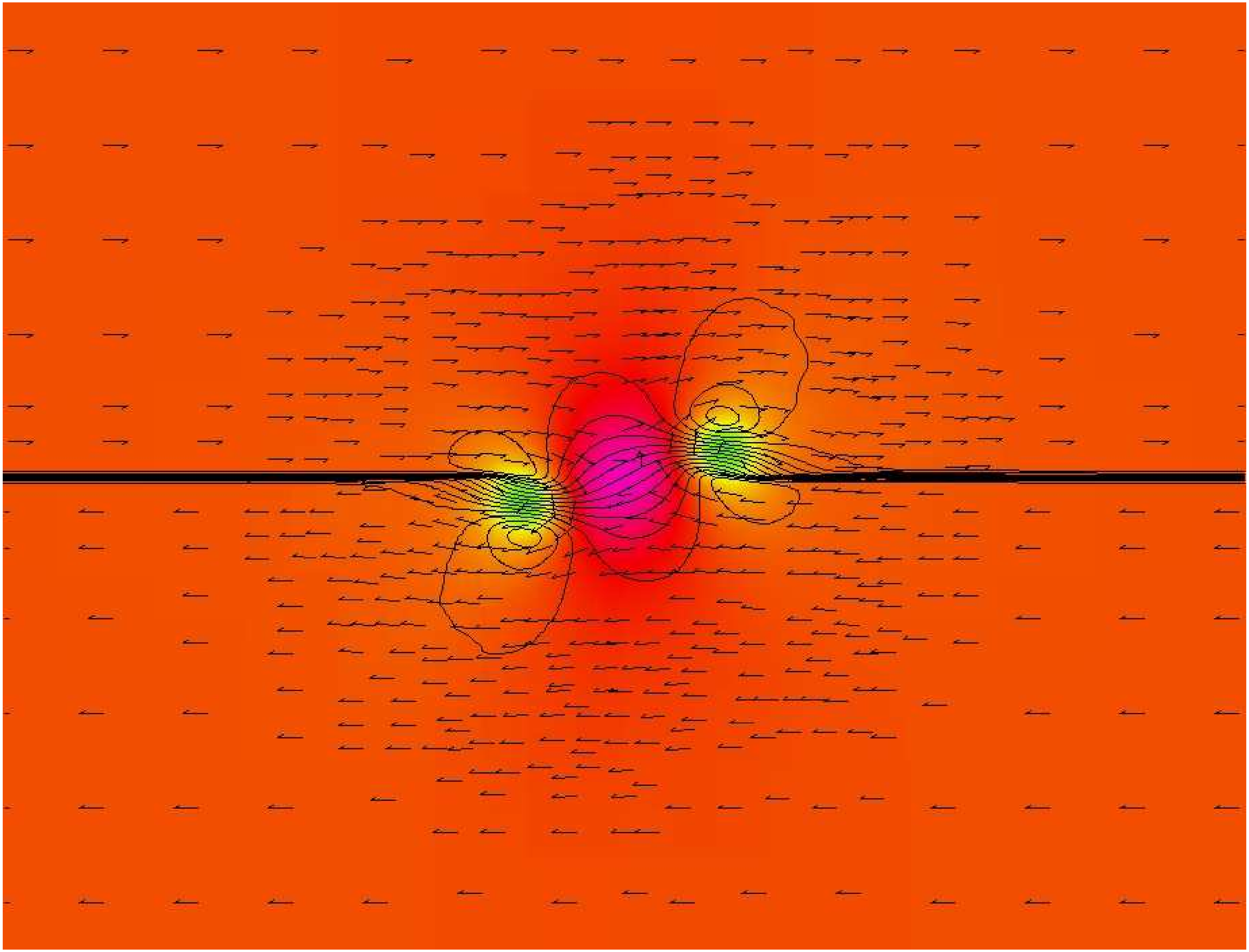


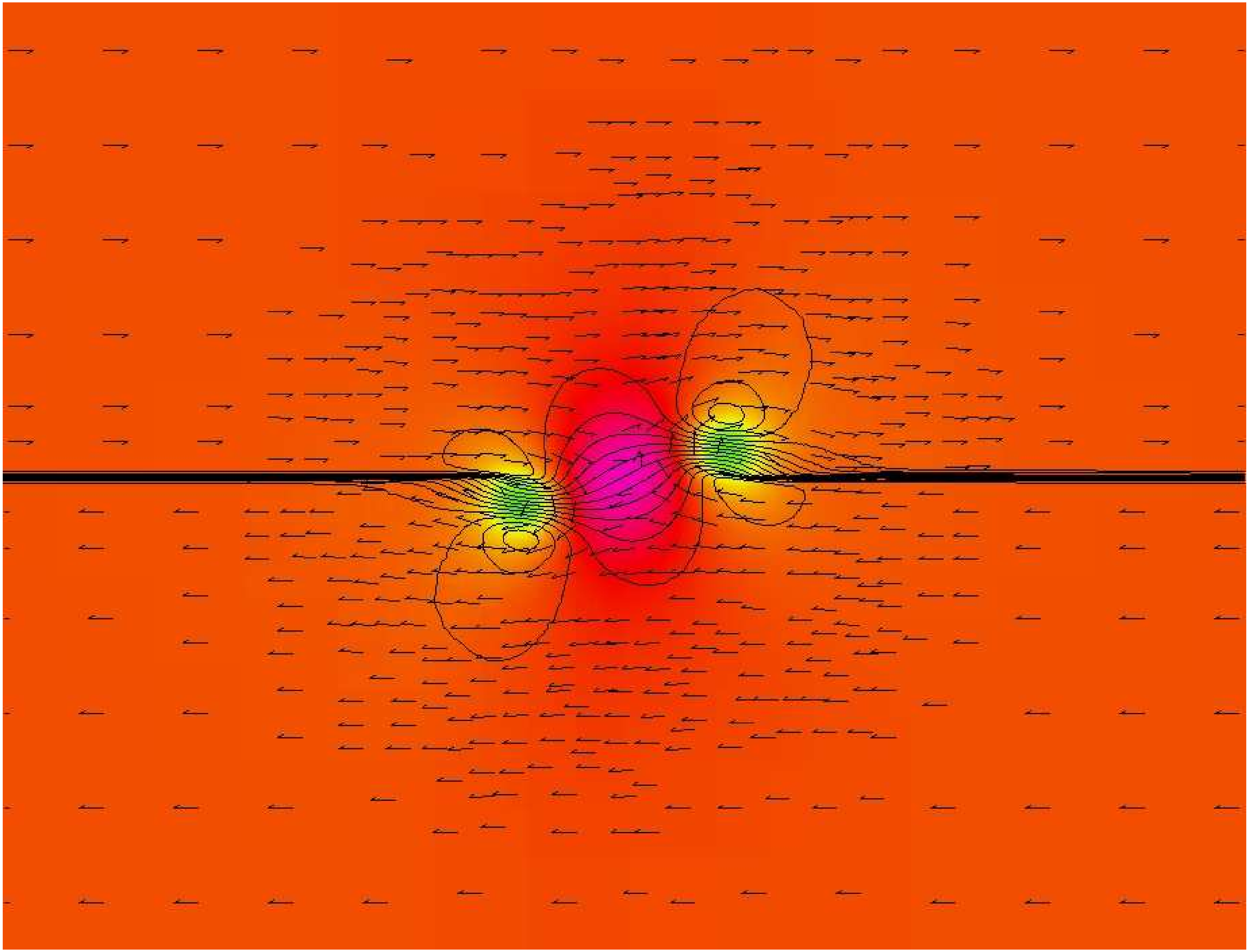


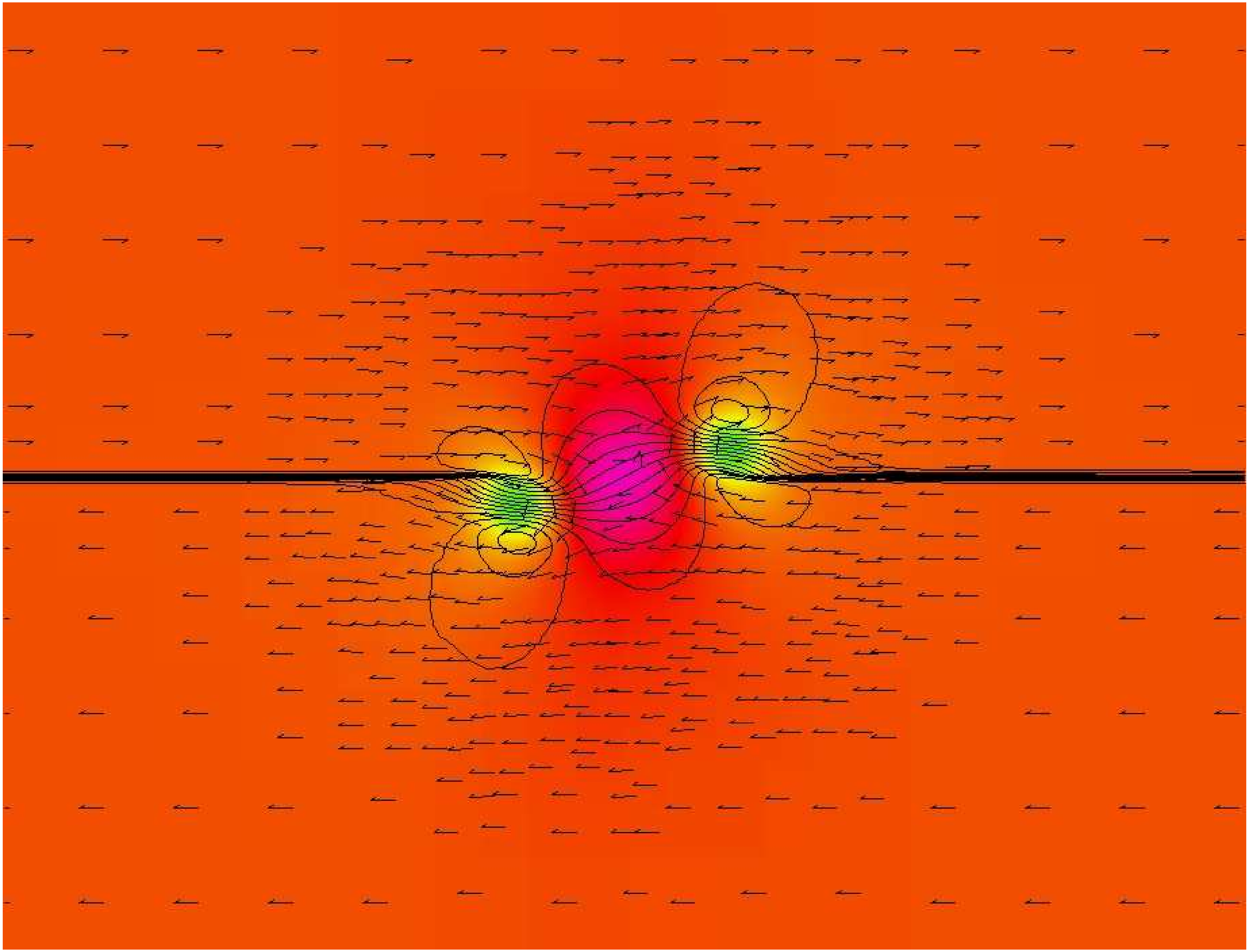


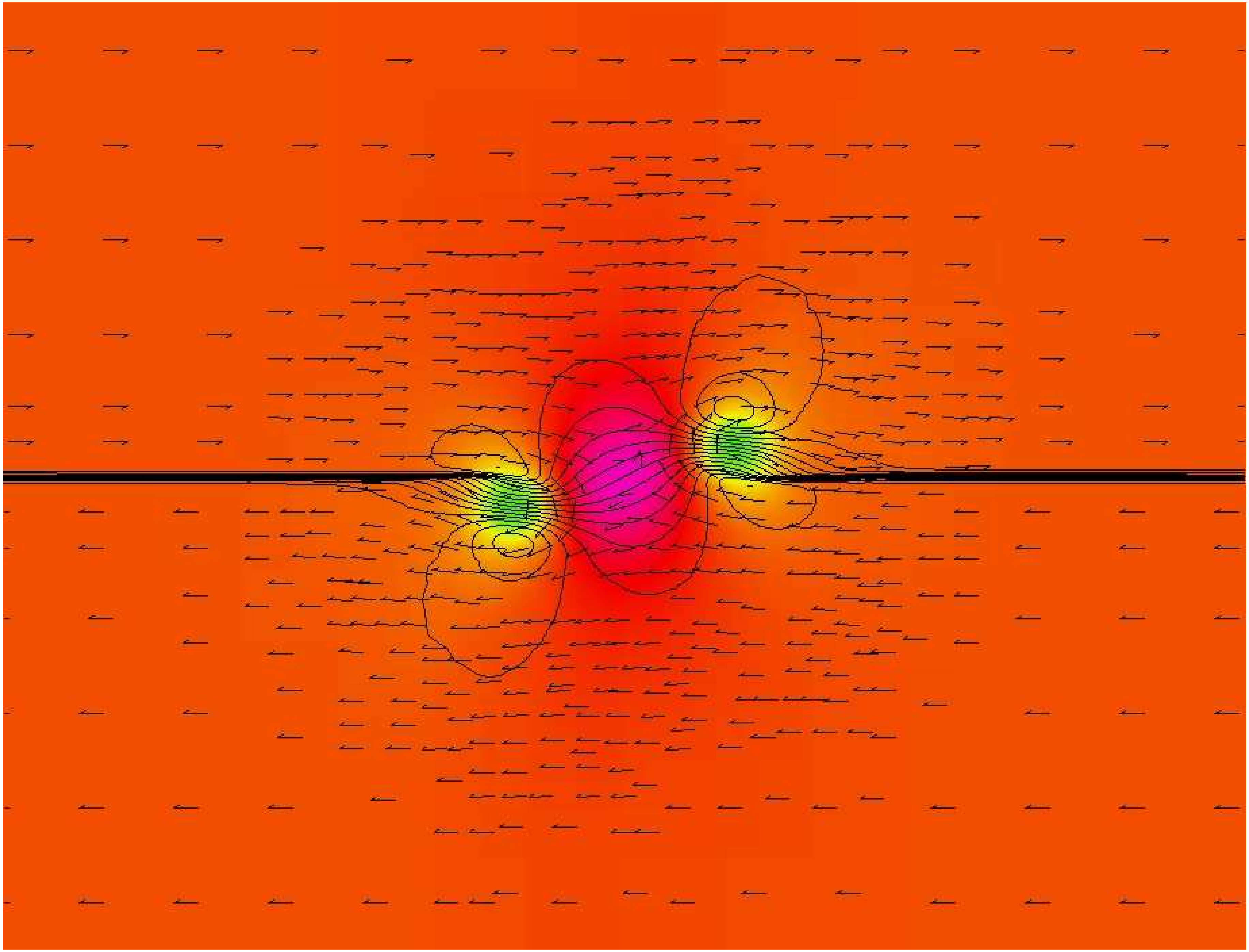


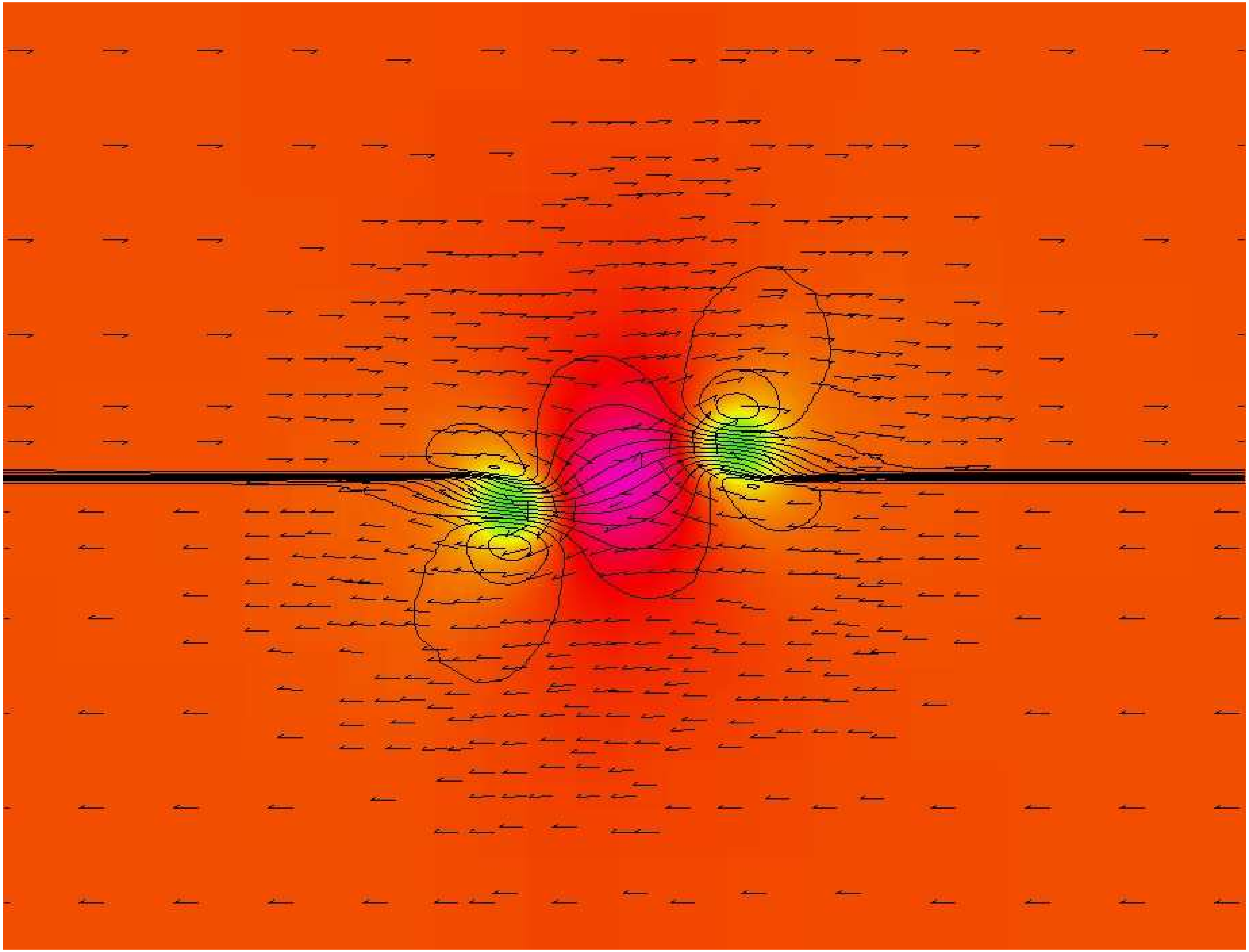


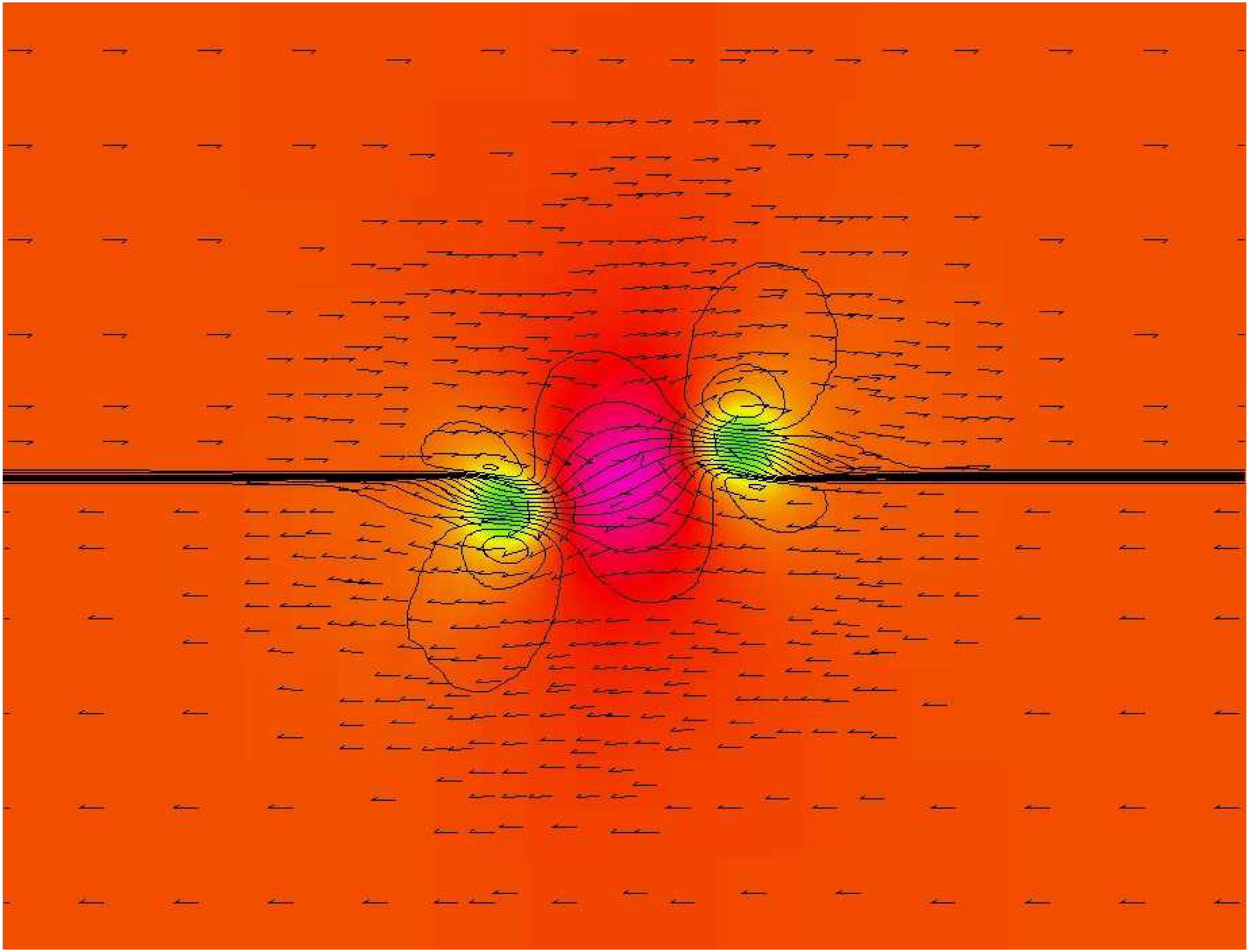


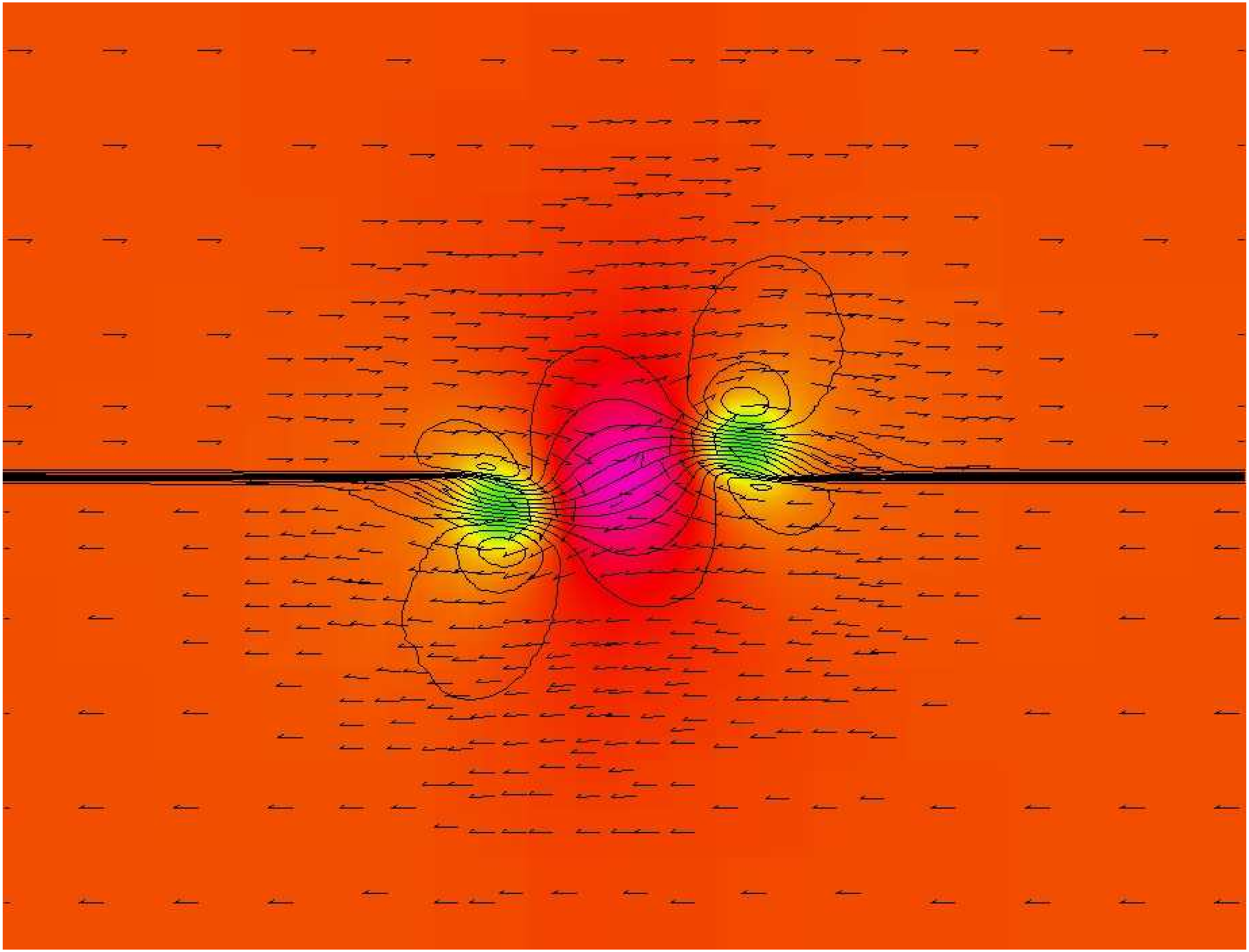


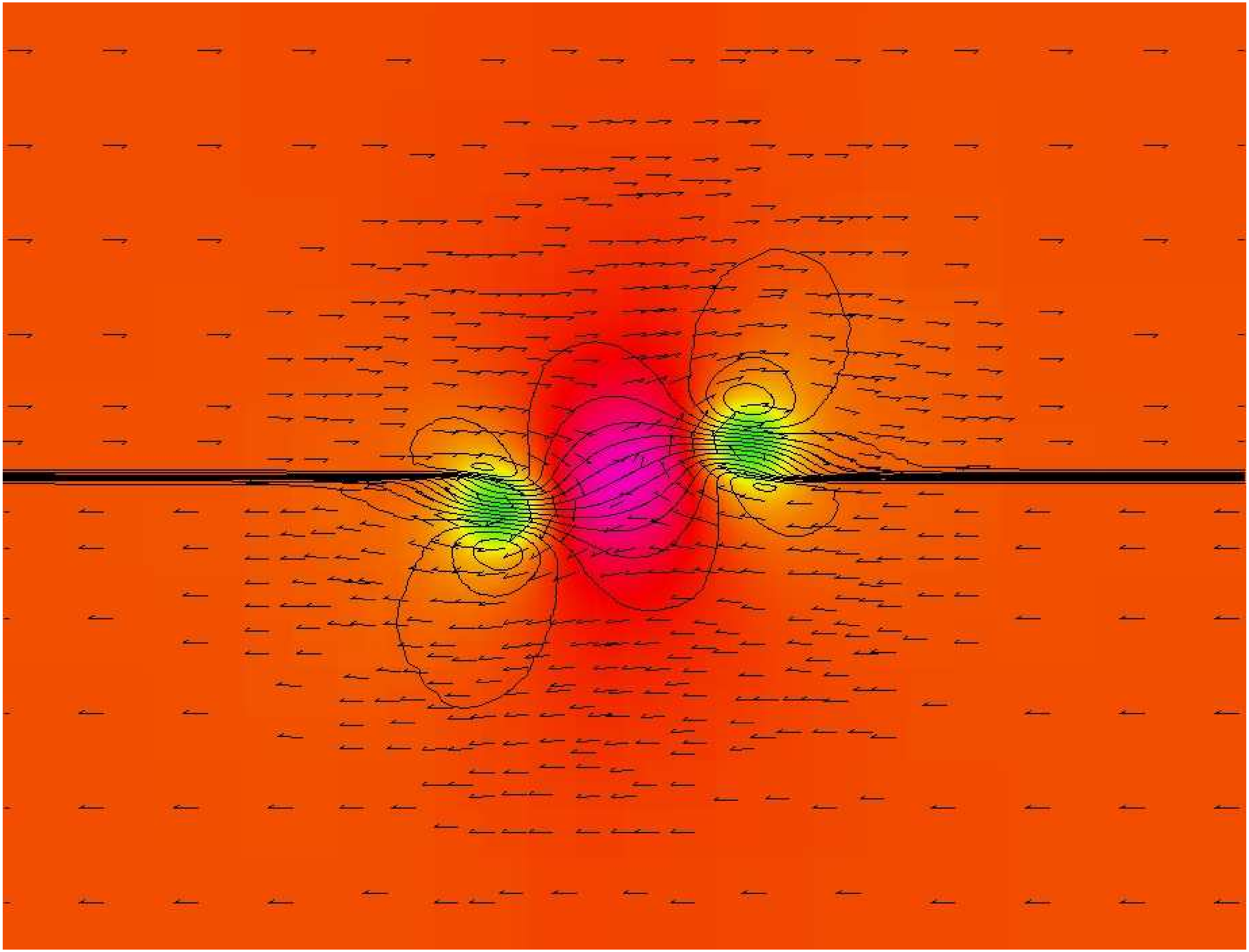


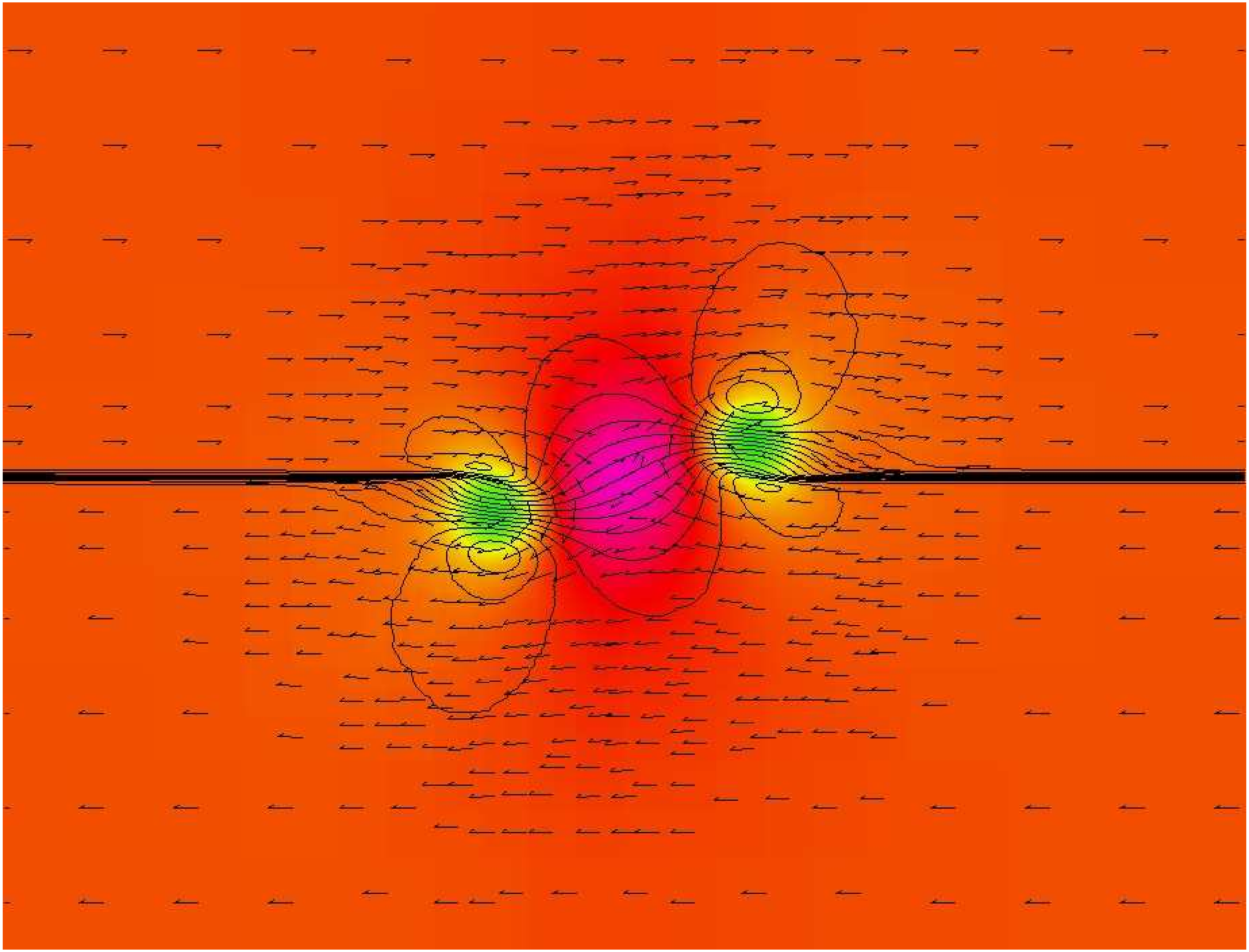


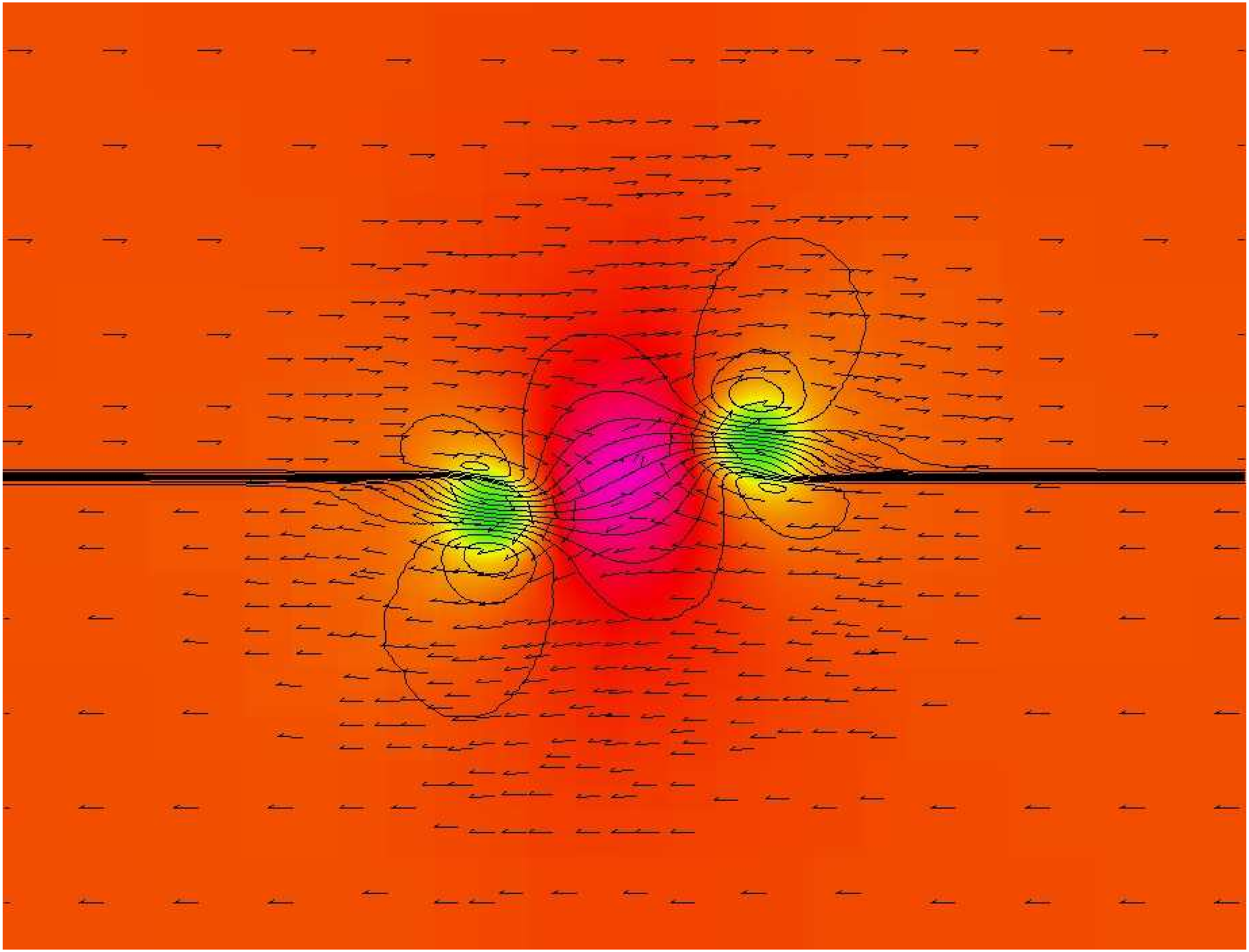


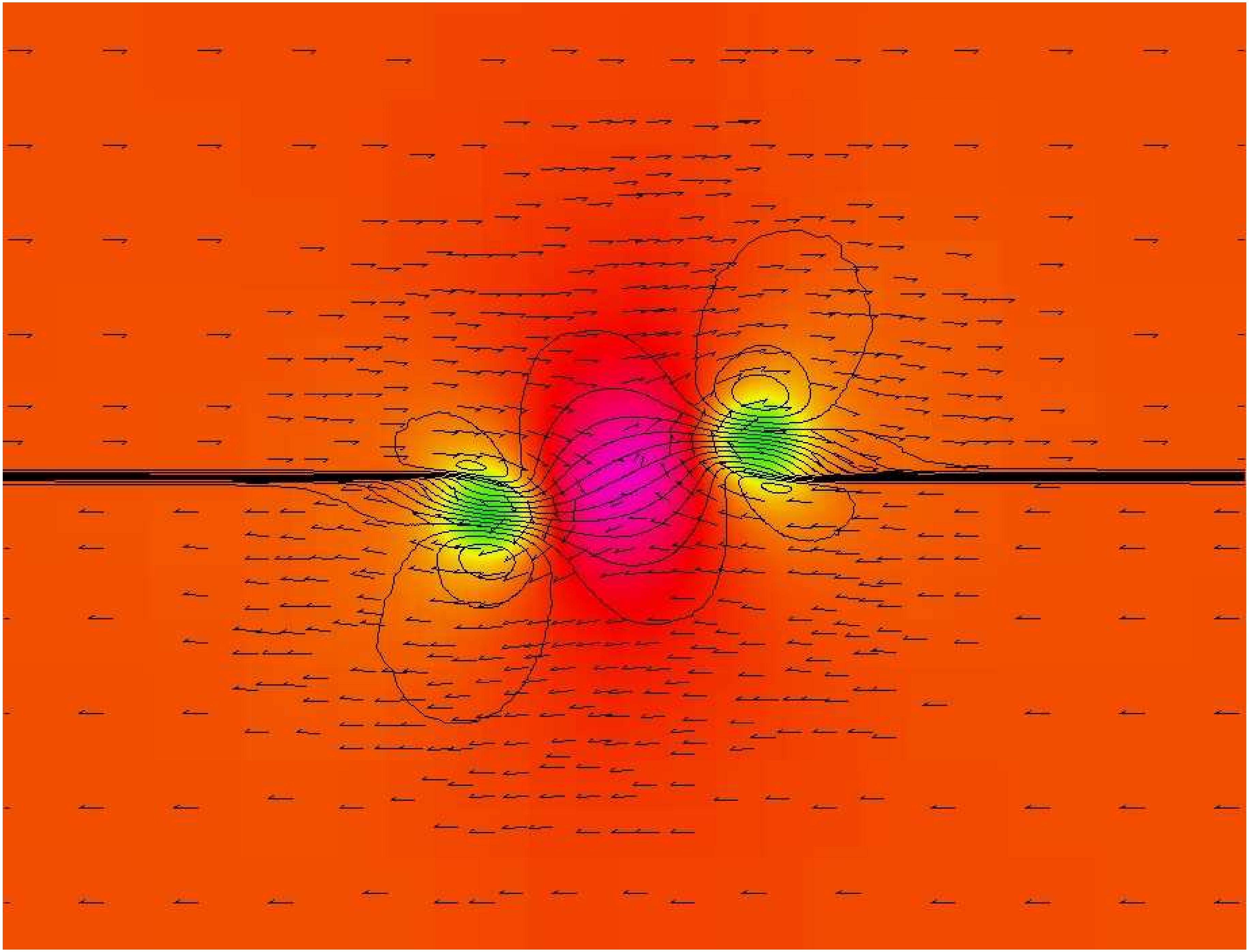


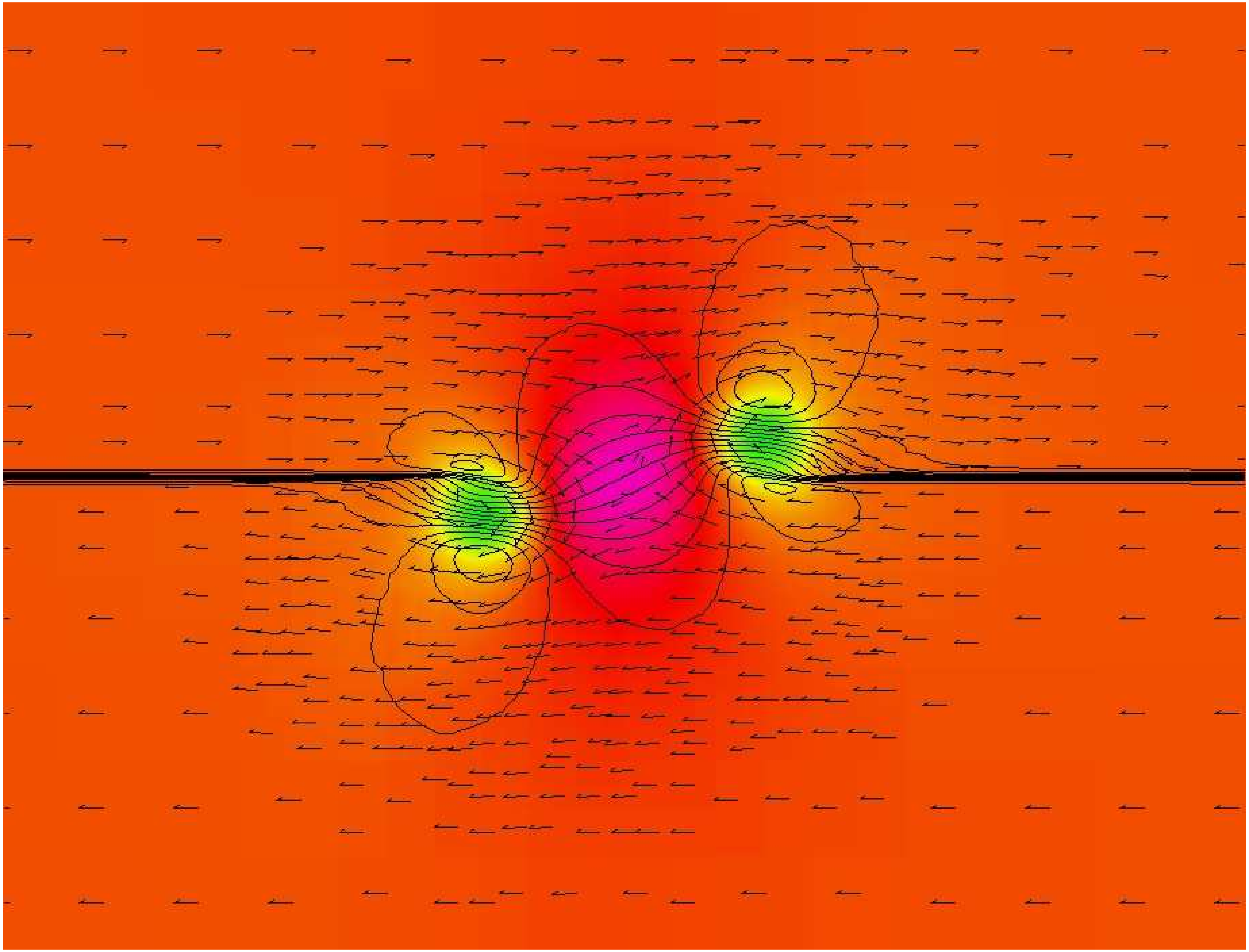


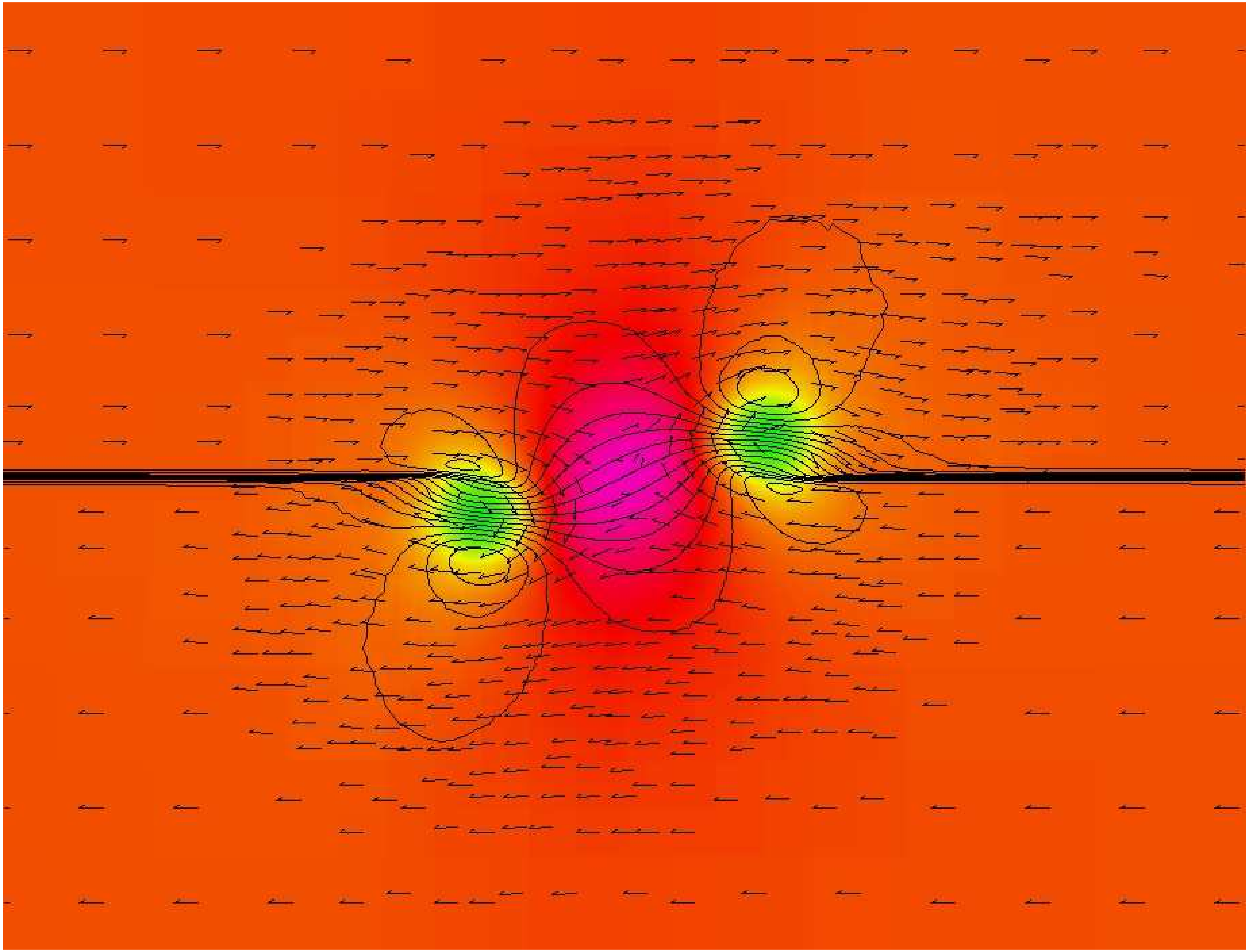


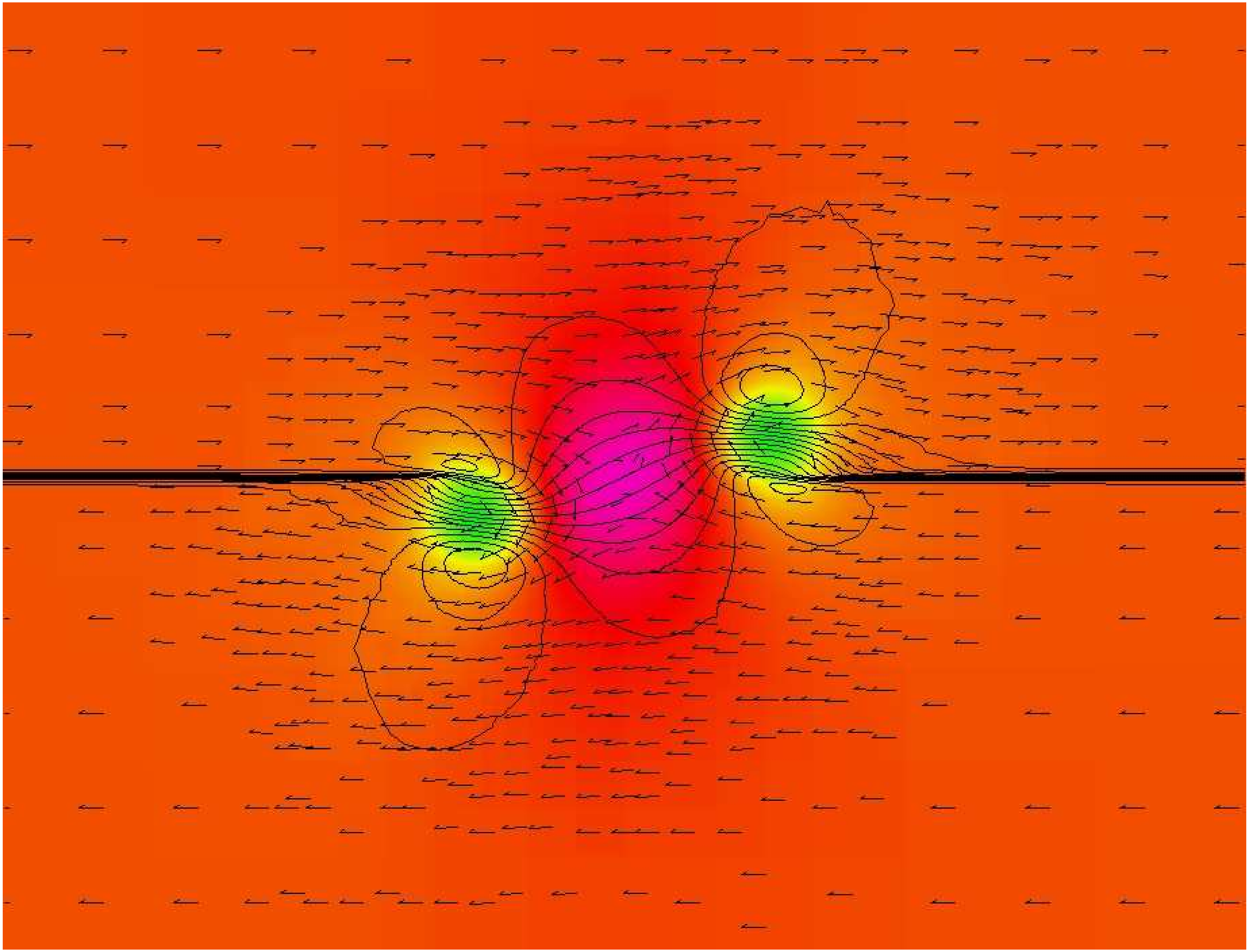


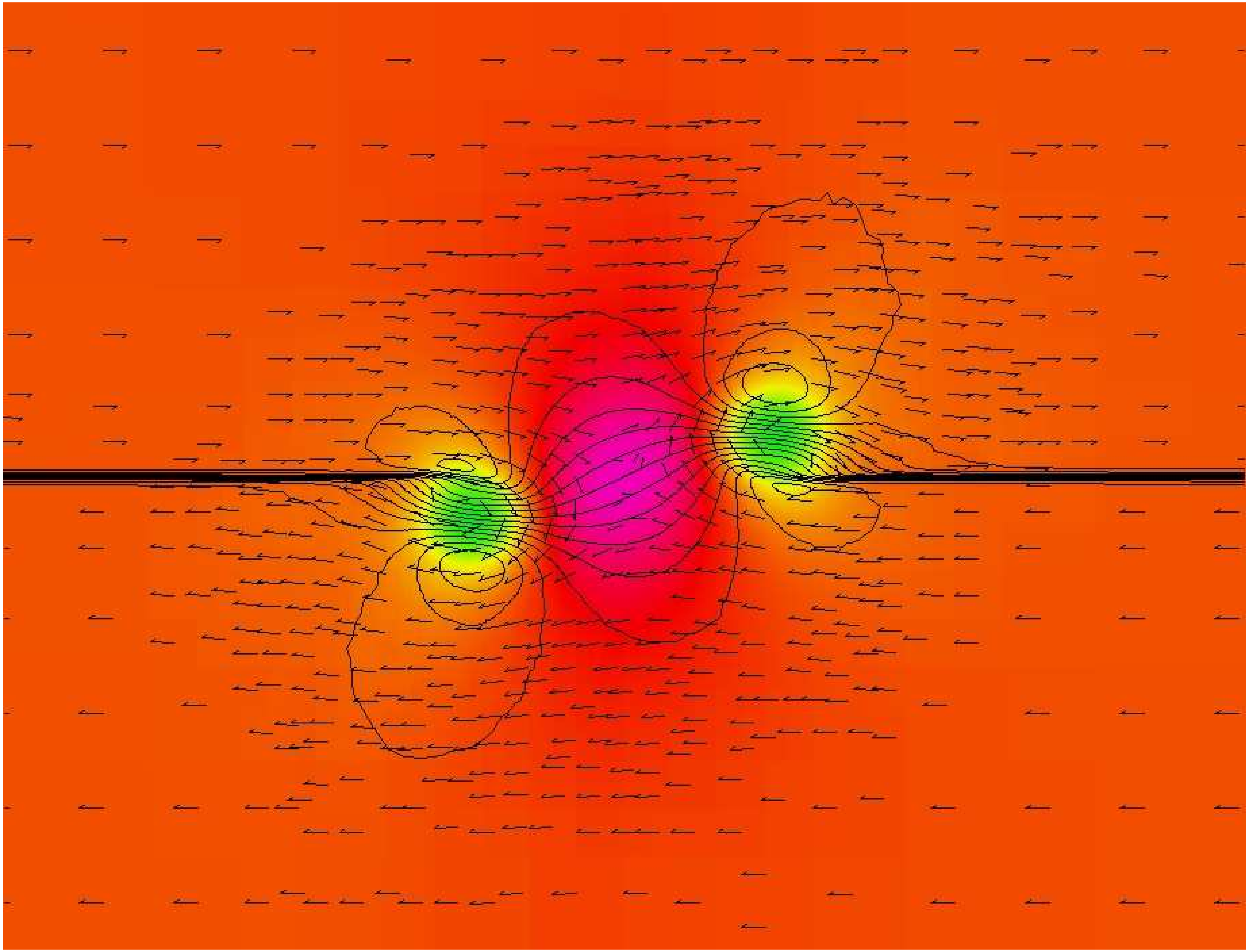


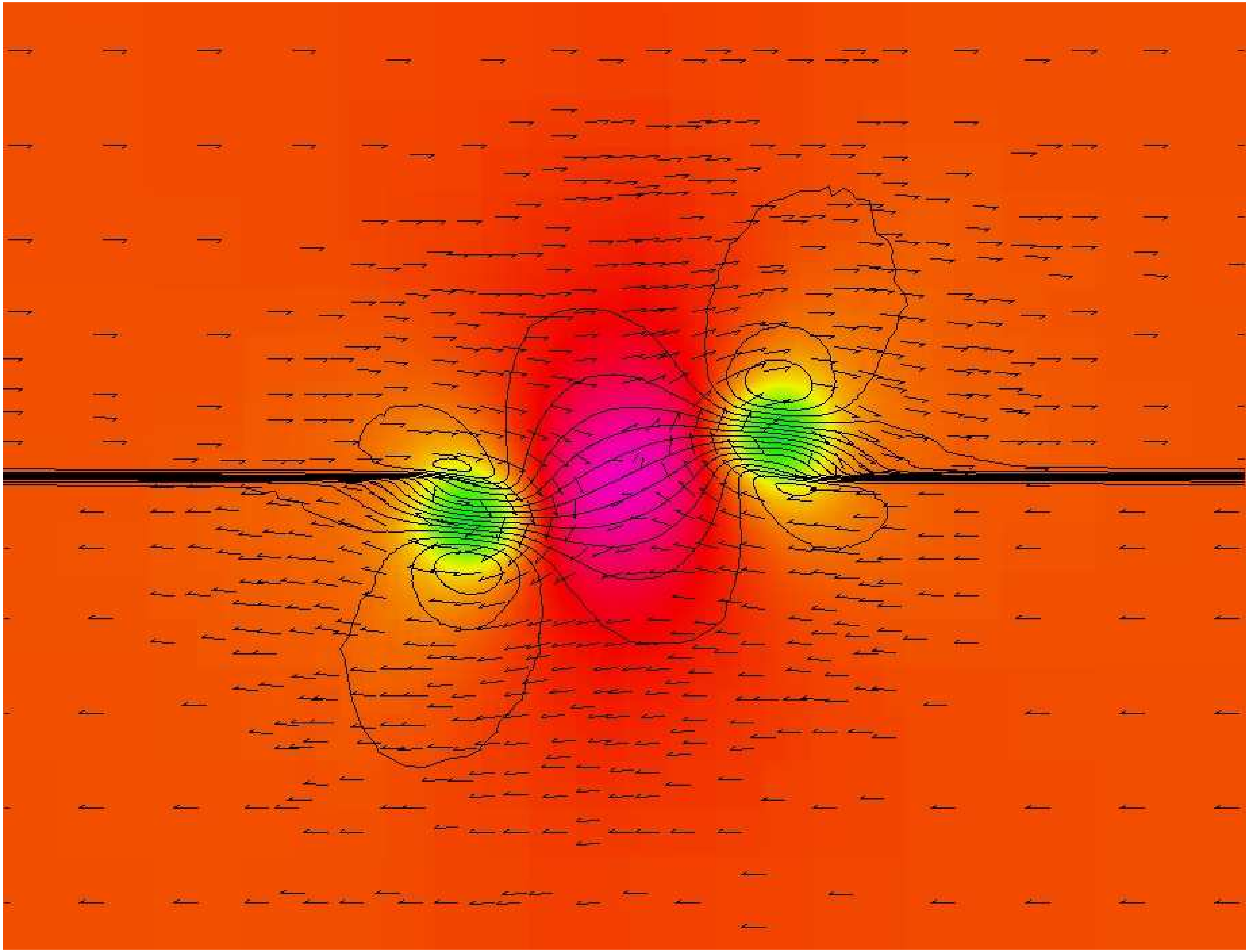


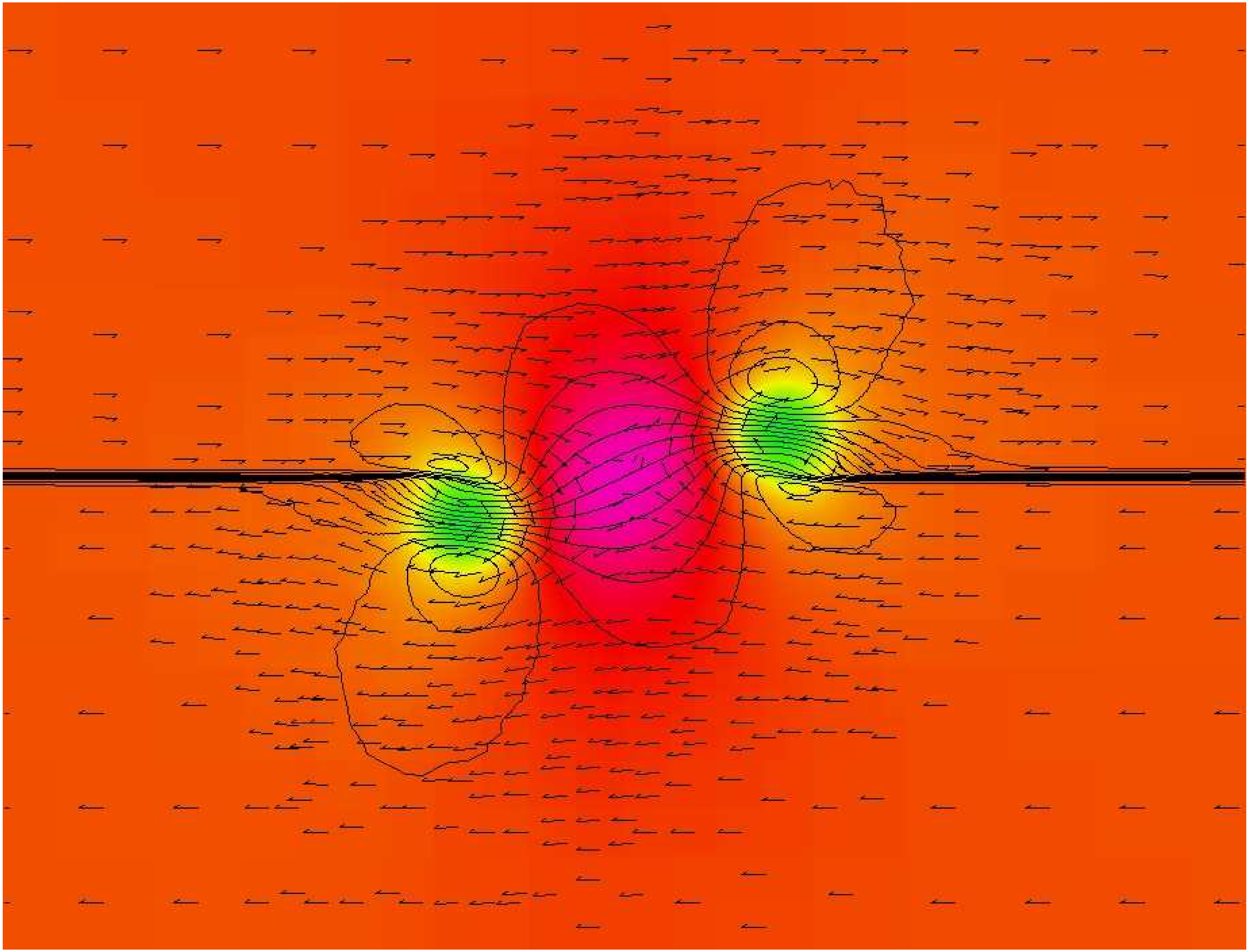


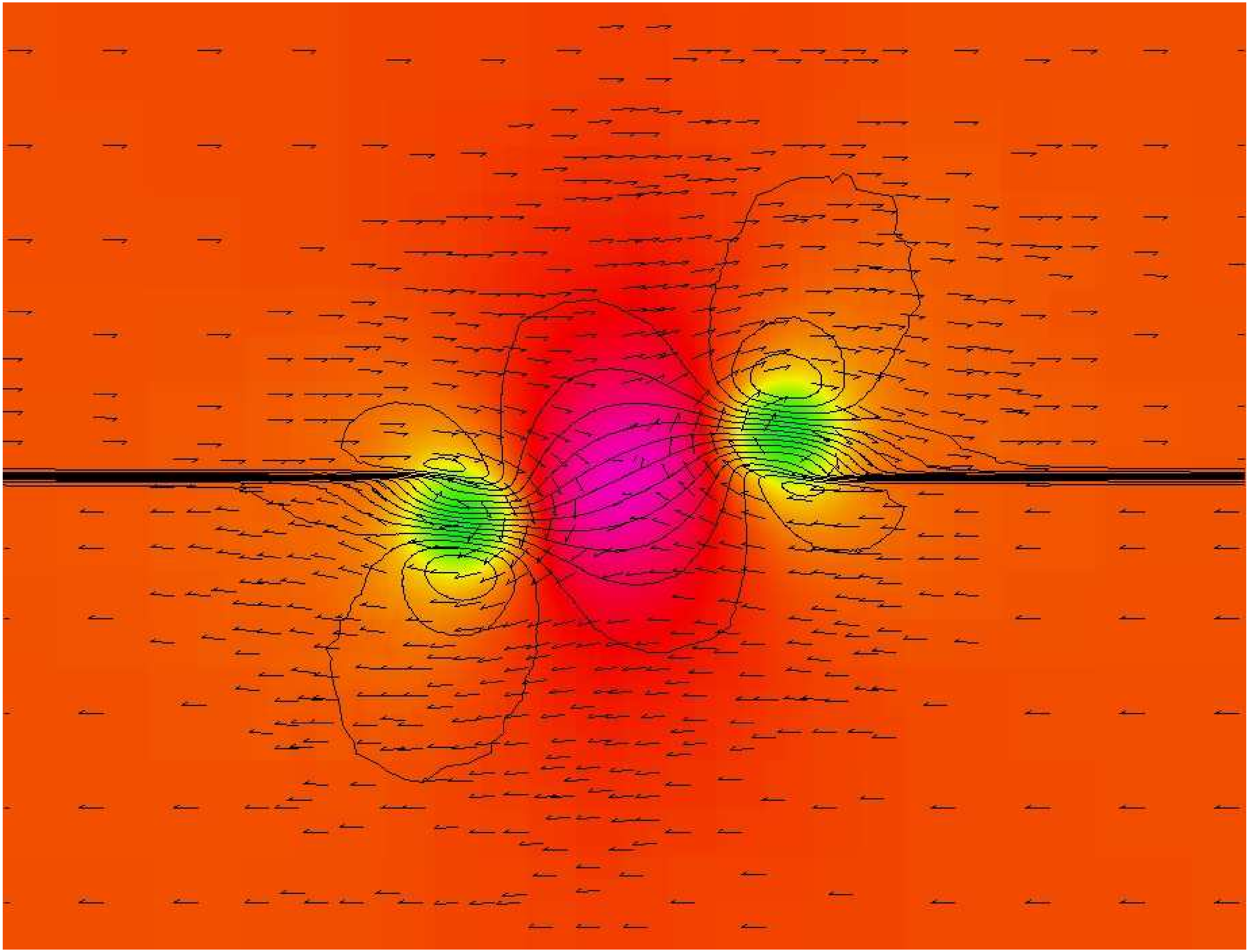


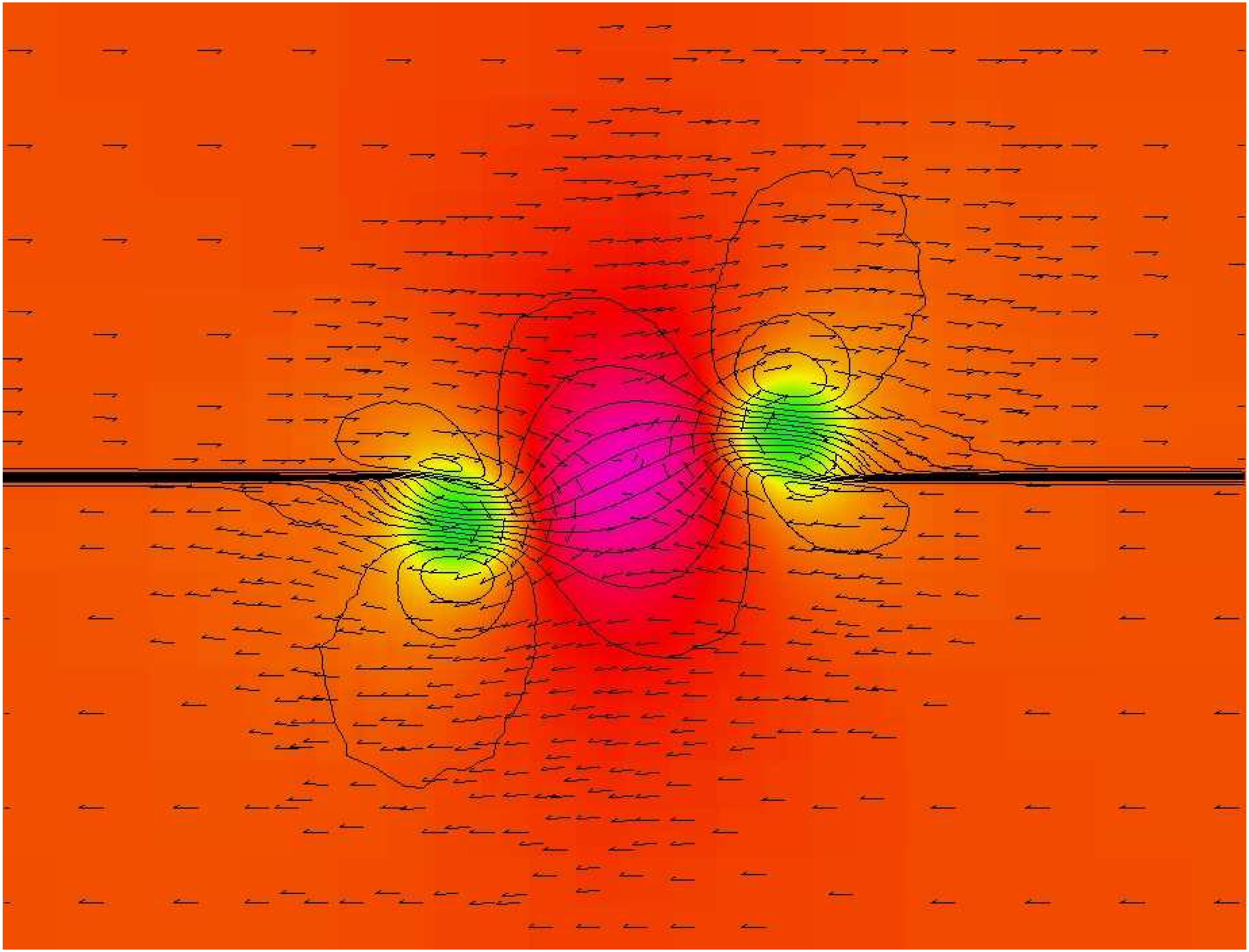


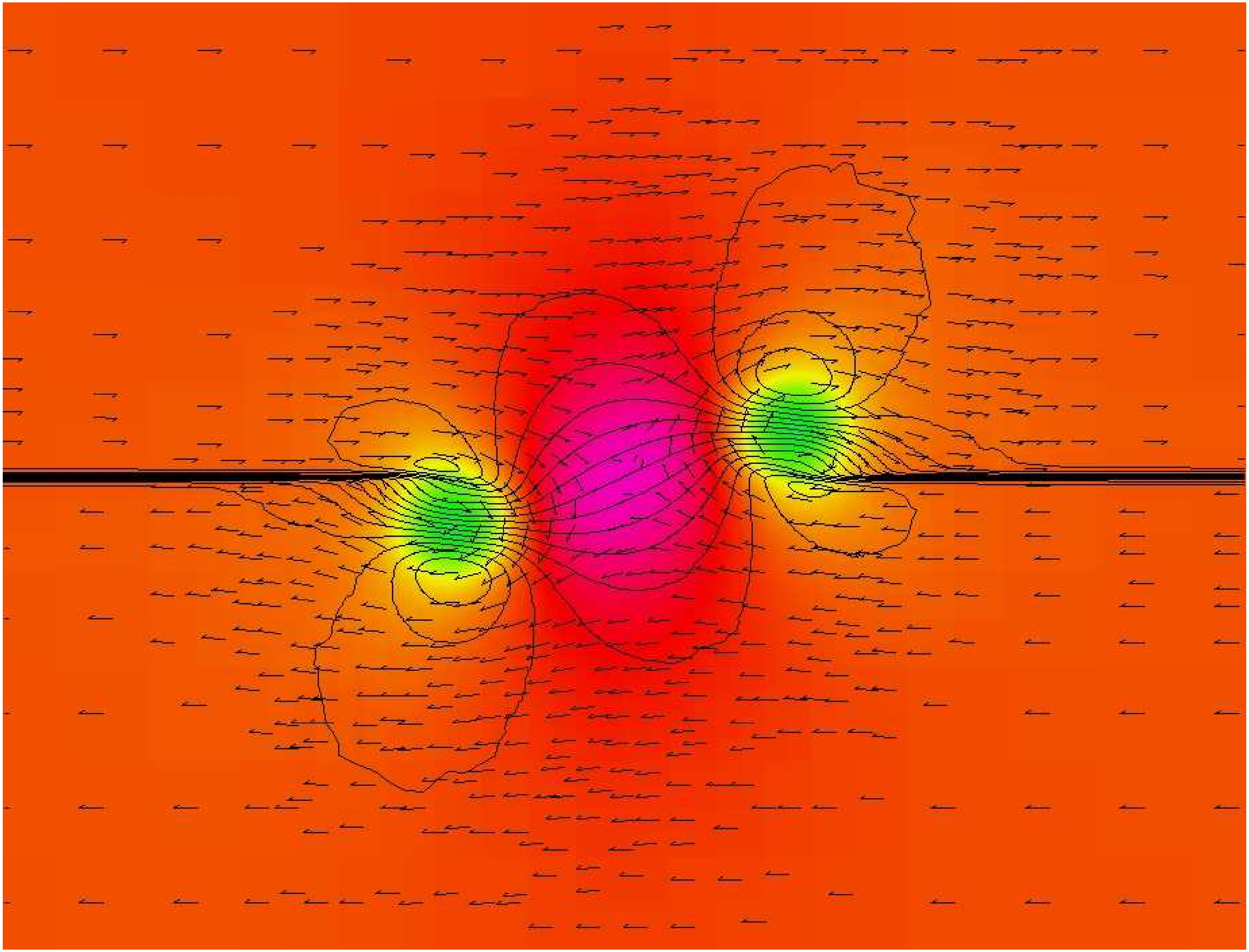


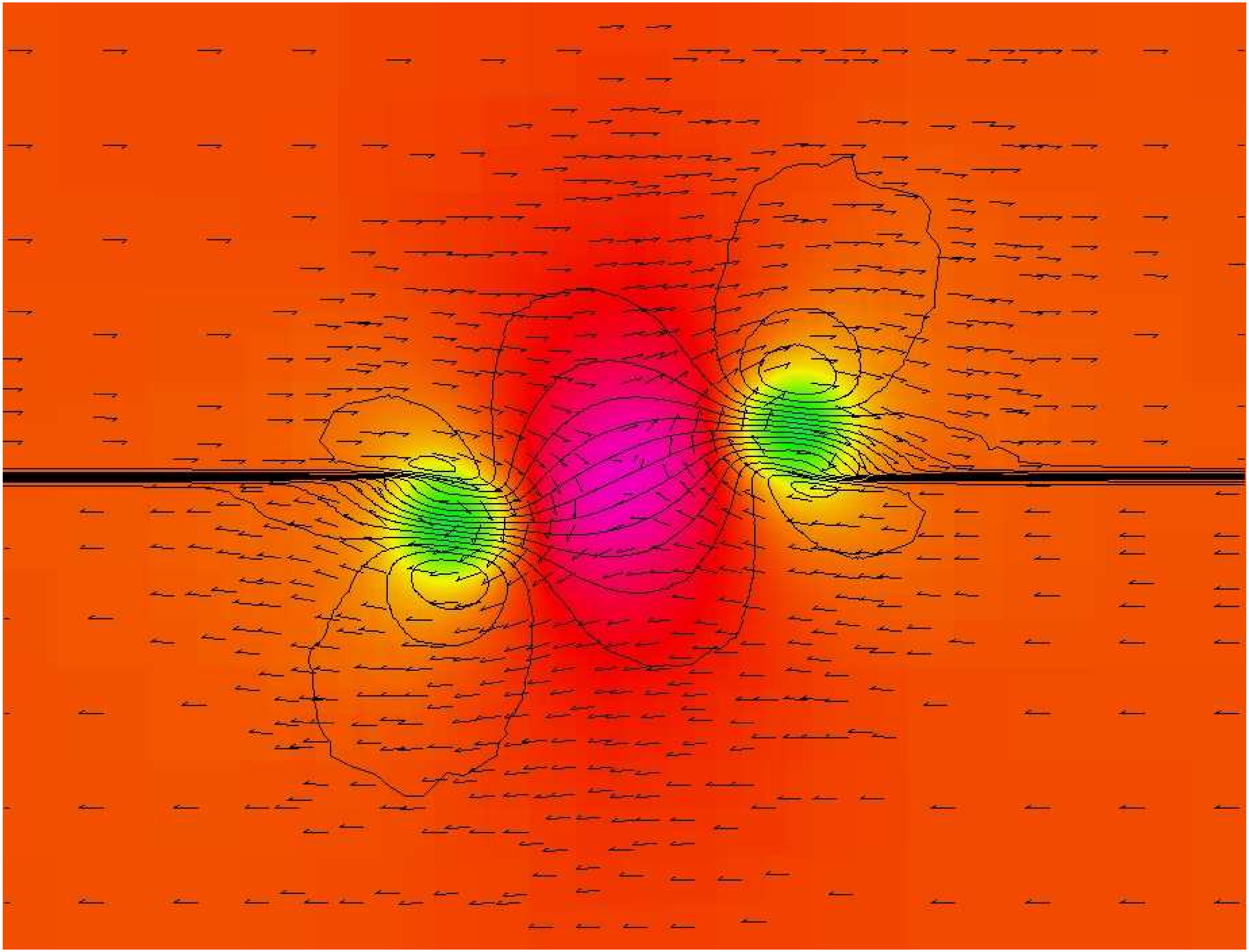




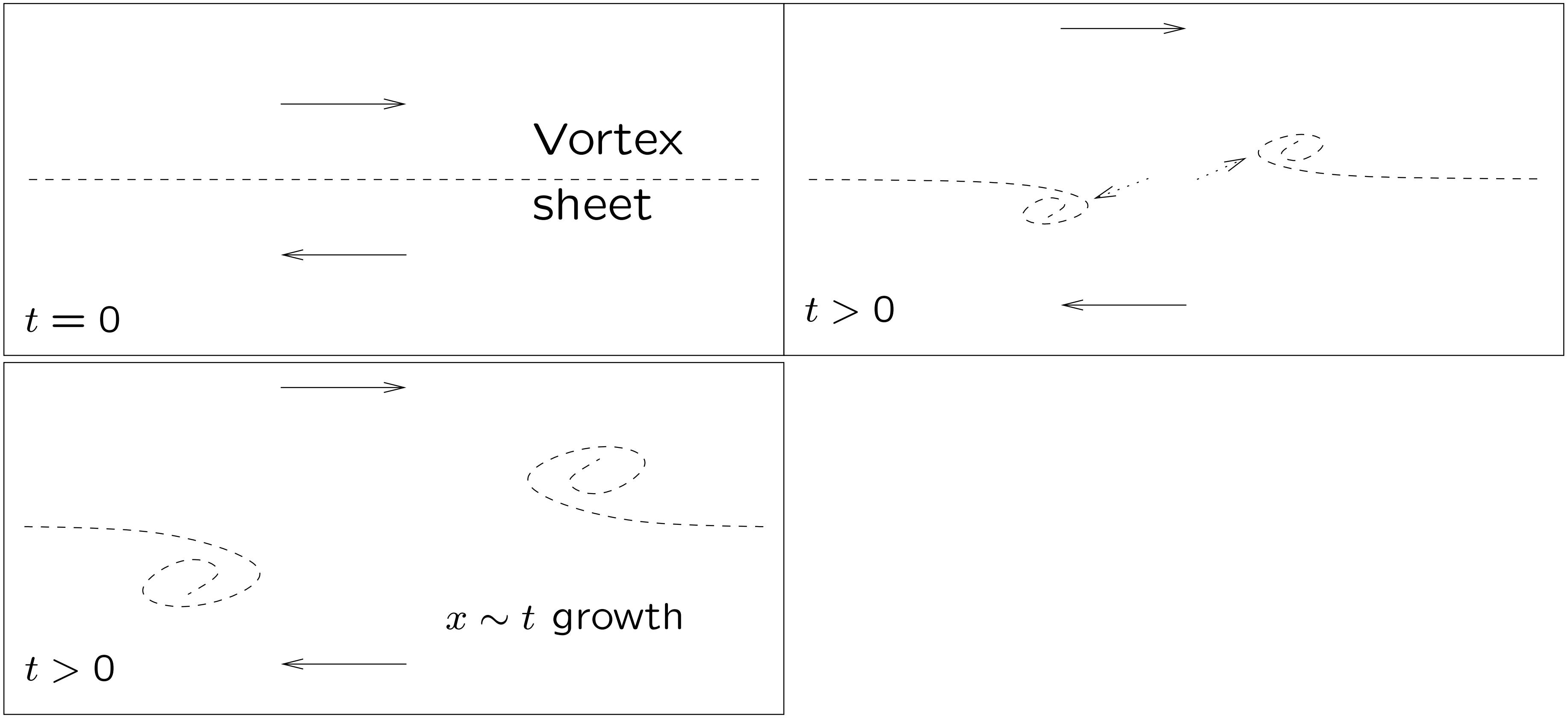








Pullin (1989) separated sheet

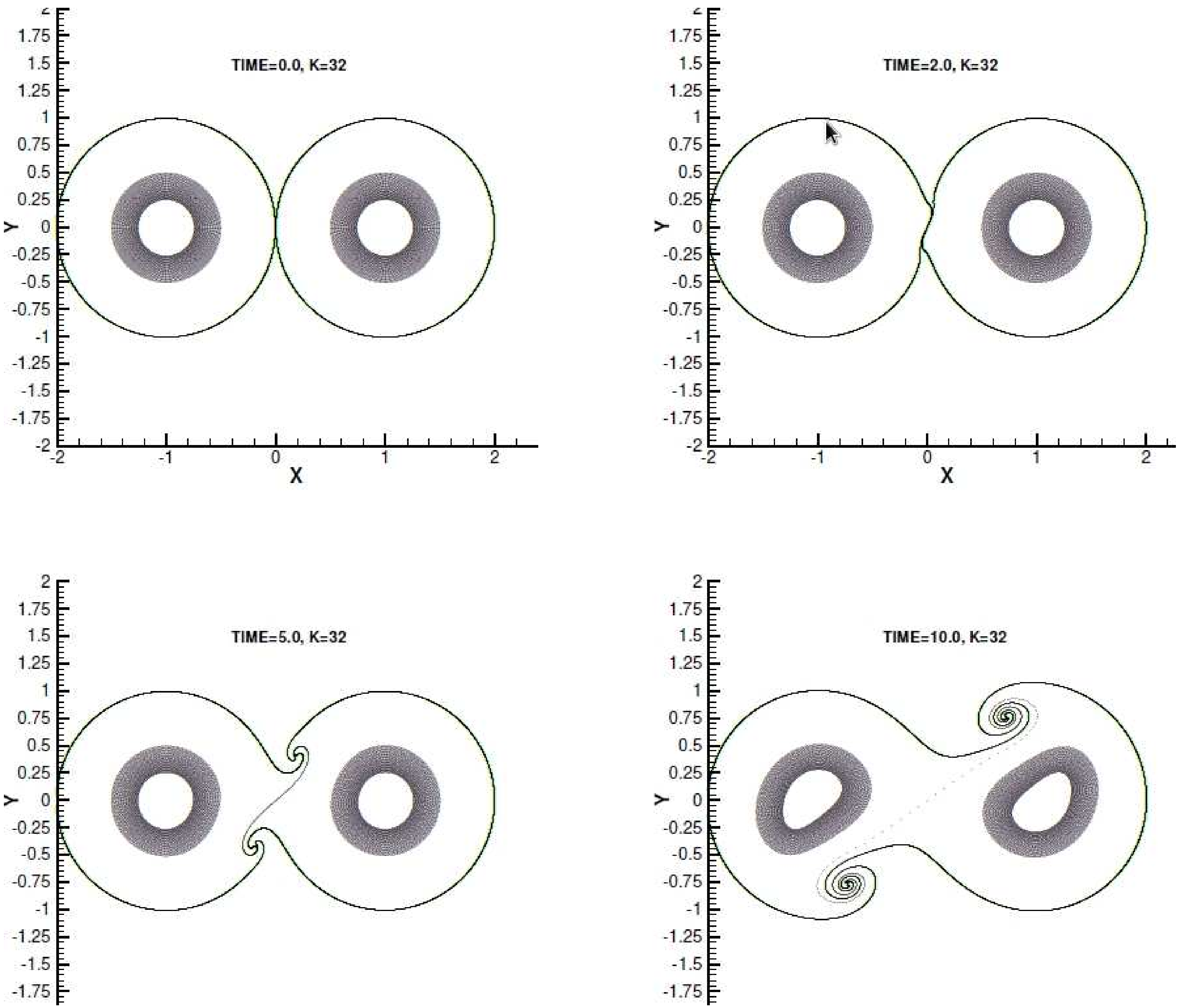


Non-uniqueness example for (incompressible) Euler.

My main research focus: get a rigorous proof.

[▷ flv]

Lopes/Lowengrub/Lopes/Zheng (2006)



Conjectures/conclusions

Navier-Stokes/Boltzmann/...: Near-instability. Consider

$$\epsilon \downarrow 0$$

(limit of zero heat conduction and viscosity μ/mean free path/...).
For each $\epsilon \gtrapprox 0$ have solution W_ϵ so that

$$d(U(0), W_\epsilon(0)) \to 0 \quad \text{but} \quad d(U(t), W_\epsilon(t)) \not\to 0 \qquad \text{as } \epsilon \downarrow 0$$

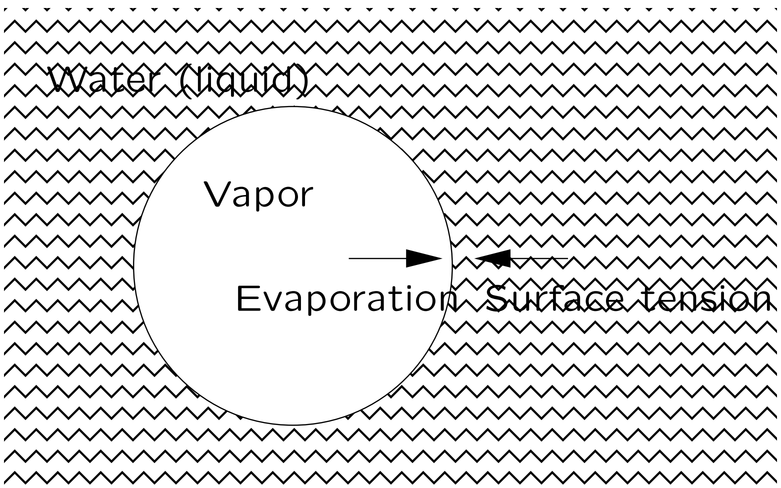
$$\Downarrow$$

Euler: Nonuniqueness: $\exists$ solution W_0 so that

$$d(U(0), W_0(0)) = 0 \quad \text{but} \quad d(U(t), W_0(t)) \neq 0.$$

(Near-)Instability — philosophical considerations

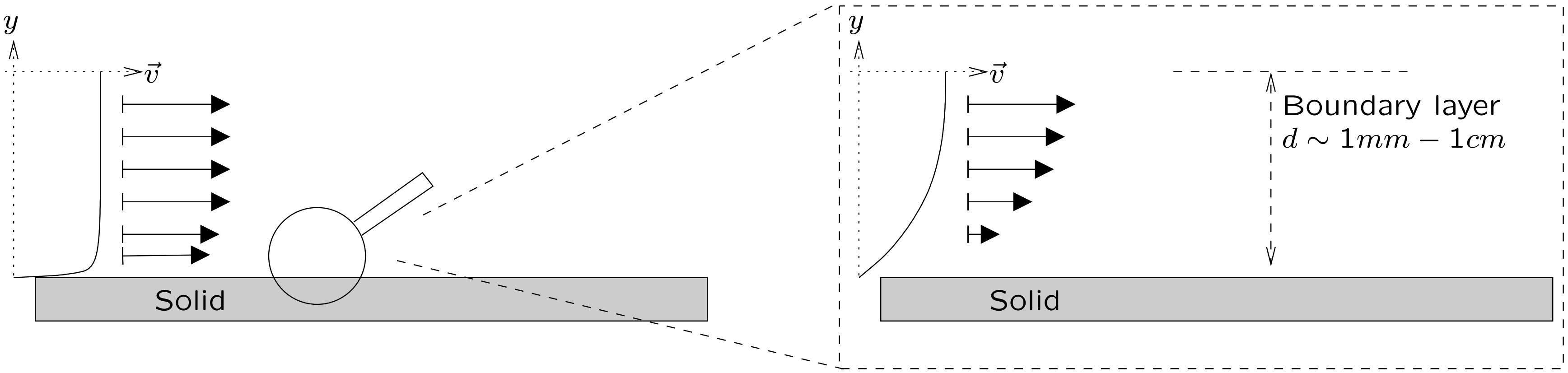
"Only stable solutions matter: unstable ones are destroyed by randomness/measurement errors."

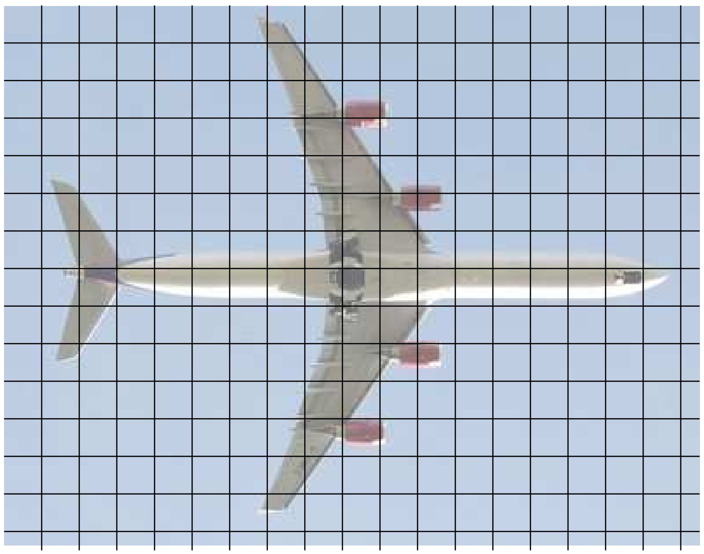


→ worst of all worlds: instabilities are sometimes triggered.

Paradox: turbulent flow may be easier to compute than laminar?
Source of randomness (?) triggers instabilities.

Numerics: why Euler?





Physical domain $\sim 10m$, boundary layer $\sim 1mm$, ratio 10^4

Three space dimensions $\rightsquigarrow$ 10^{12} grid cells

Plus: time stepping (CFL constraint $\Delta t \lesssim \Delta x$)

or: iteration to equilibrium (if any)

$\Rightarrow$ let's pray a coarse grid is enough

How to rescue Euler/large-Reynolds-number numerics?

Subgrid (turbulence) models? Extreme adaptivity? Anisotropic grids/front tracking?

1. Quantify instability, randomness
2. Obtain statistical averages
3. Will fail for some applications (forecasting hourly weather 100 days from now):

give up

The "unreasonable effectiveness of mathematics" (E. Wigner) ends here.

Modelling with differential equations requires that the space-time continuum limit is valid:
no propagation of errors from infinitely small to large scales.

My projects

1. Prove
a. existence of sheet separation as incompressible Euler solution,
b. generalize to compressible Euler,
c. then Navier-Stokes

Goal: find non-uniqueness examples that are

$ rigorously proven, and
☼ cannot be criticized as unphysical

(contain only physically observed features)

2. Vorticity is cause of non-uniqueness — try compressible potential flow? Conjecture: uniqueness, stability, existence at least for small data. (Admissibility condition?!)