Existence of algebraic vortex spirals and ill-posedness of inviscid flow (Part II)

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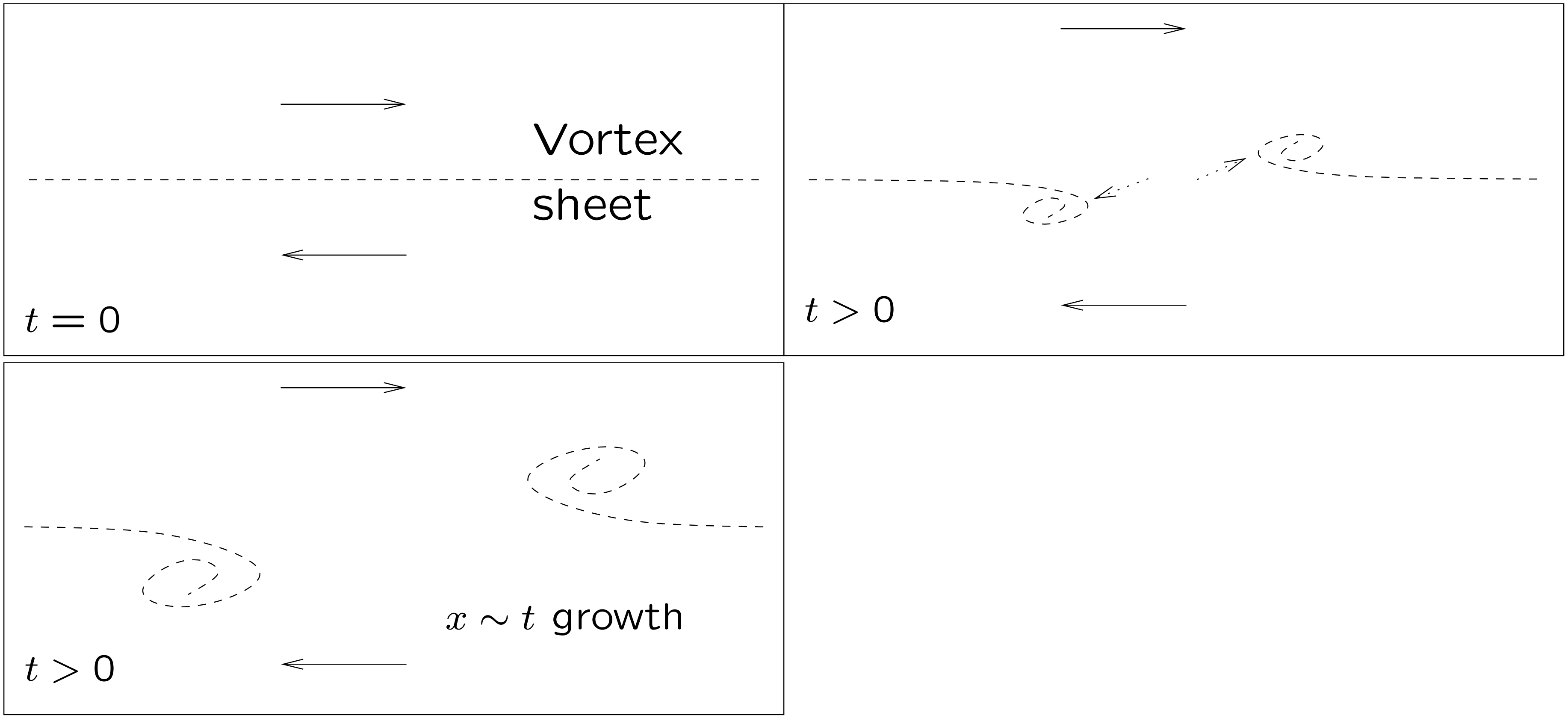
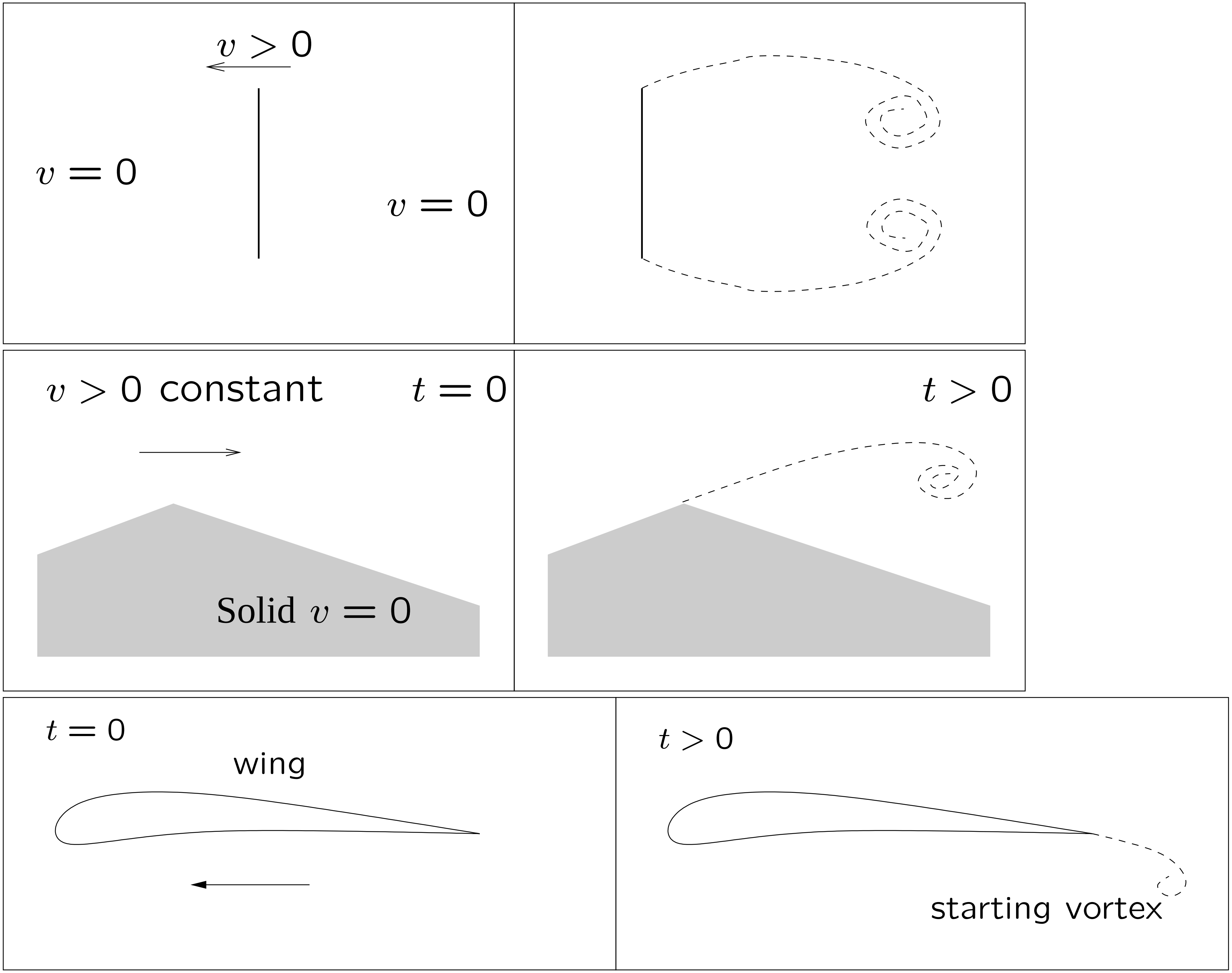
Pullin (1989) separated sheet



Self-similar:

$$v(t,x) = v(\frac{x}{t}) \quad , \quad \pi(t,x) = \pi(\frac{x}{t})$$

"Coffee spoon" experiment



Pressure

Given smooth $v = (v^1, ..., v^d)$ with $\nabla \cdot v = 0$, how to find pressure π:

$$v_t + v \cdot \nabla v + \nabla \pi = 0 \quad ?$$

$$v_t^k + v^j \partial_j v^k + \partial_k \pi = 0 \quad (k = 1, ..., d), \qquad \sum_{k=1}^{d} \partial_k v^k = 0$$

Take $\nabla \cdot$ of this equation, i.e. $\sum_{k=1}^{d} \partial_k$:

$$0 = \underbrace{\sum_k \partial_t \partial_k v^k}_{= \partial_t \nabla \cdot v = 0} + \sum_{j,k} \partial_k (v^j \partial_j v^k) + \underbrace{\sum_k \partial_k (\partial_k \pi)}_{= \Delta \pi}$$

$$0 = \sum_j v^j \partial_j \underbrace{\sum_k \partial_k v^k}_{= \nabla \cdot v = 0} + \sum_{j,k} \partial_k v^j \partial_j v^k + \Delta \pi$$

$$0 = \mathrm{tr}\left((\nabla v)^2 \right) + \Delta \pi$$

$\rightsquigarrow$ have to solve a Poisson problem

Vorticity formulation: (assuming smooth flow)

$$\frac{\partial}{\partial t}\vec{v} + \vec{v}\cdot\nabla\vec{v} + \nabla\pi = 0, \qquad \nabla\cdot\vec{v} = 0$$

$$v^1_t + v^1 v^1_1 + v^2 v^1_2 + \pi_1 = 0 \quad , \quad v^2_t + v^1 v^2_1 + v^2 v^2_2 + \pi_2 = 0 \quad , \quad v^1_1 + v^2_2 = 0$$

$$\begin{aligned}
0 = & v^1_{2t} + v^1_2 v^1_1 + v^1 v^1_{12} + v^2_2 v^1_2 + v^2 v^1_{22} + \pi_{12} \\
& - v^2_{1t} - v^1_1 v^2_1 - v^1 v^2_{11} - v^2_1 v^2_2 - v^2 v^2_{12} - \pi_{12} \\
& = (v^1_2 - v^2_1)_t + v^1_1(v^1_2 - v^2_1) + v^1(v^1_{12} - v^2_{11}) + v^2_2(v^1_2 - v^2_1) + v^2(v^1_{22} - v^2_{12}) \\
& = \underbrace{(v^1_2 - v^2_1)}_{=:\omega}{}_t + \underbrace{(v^1_1 + v^2_2)}_{=0}(v^1_2 - v^2_1) + v^1(v^1_2 - v^2_1)_1 + v^2(v^1_2 - v^2_1)_2 \\
& = \omega_t + v^1\omega_1 + v^2\omega_2 = \omega_t + v\cdot\nabla\omega
\end{aligned}$$

ω constant along streamlines ($(t, \vec{x})$ integral curves of $(1, \vec{v})$).

$\omega = 0$ at $t = 0 \quad \Rightarrow \quad 0 = \omega = \nabla\times\vec{v}$ for all $t \geq 0$

Then $\vec{v} = \nabla\phi$ for scalar ϕ, and

$$\Delta\phi = \nabla\cdot\vec{v} = 0.$$

Complex variables

Assume $\omega = 0$ in some region.

v 2-d harmonic $\rightsquigarrow$ more conveniently discussed with complex analysis

$$0 = \omega = \frac{\partial}{\partial x} v^y - \frac{\partial}{\partial y} v^x \tag{1}$$

$$0 = \nabla \cdot \vec{v} = \frac{\partial}{\partial x} v^x + \frac{\partial}{\partial y} v^y \tag{2}$$

With

$$Z = x + iy,$$

Cauchy-Riemann equations for complex velocity

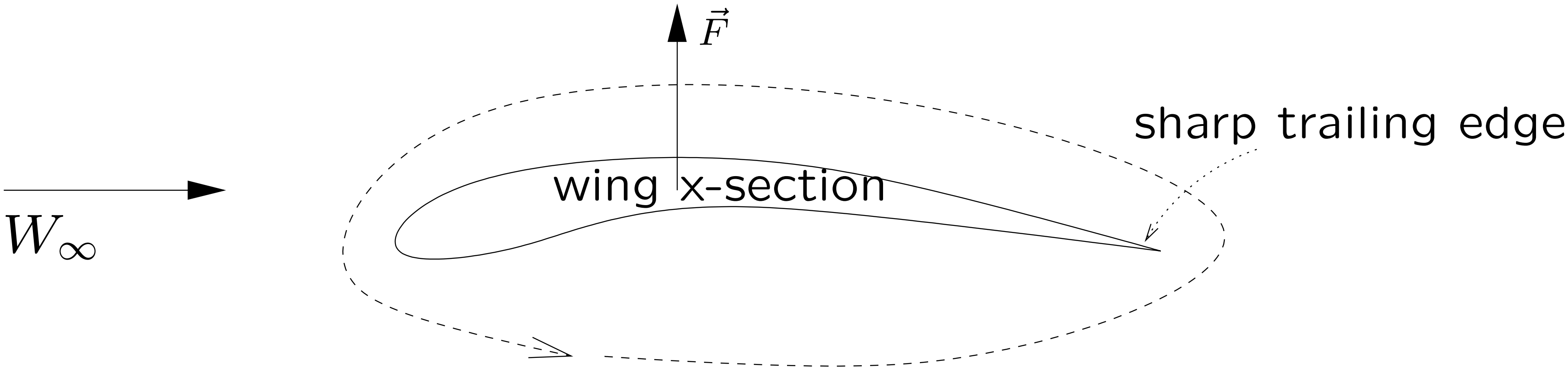
$$W = v^x - iv^y.$$

Seek holomorphic $Z \mapsto W(Z)$.

Given W holomorphic in Ω, $v = (\operatorname{Re} W, -\operatorname{Im} W)$ is an irrotational incompressible Euler solution in Ω (with appropriate π).

Force on aircraft wing: Laurent series

$$W(Z) = W_\infty + \frac{\Gamma}{2\pi i} Z^{-1} + a_2 Z^{-2} + \ldots$$



$$\Gamma = \oint_C v \cdot ds$$

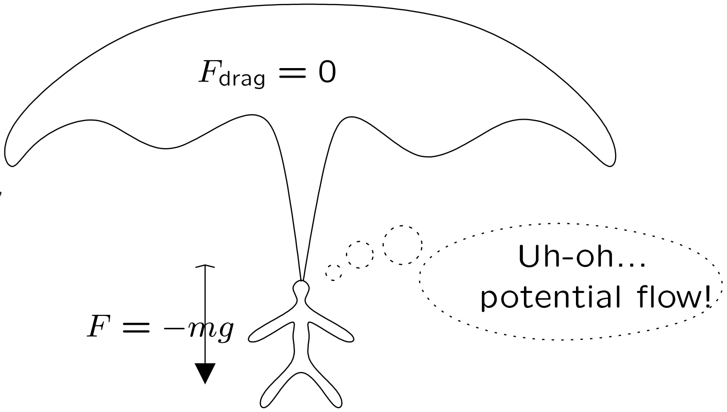
$$(F^x, F^y) = \int_{\text{body surface}} \pi \ \vec{n}_{\text{inner}} \ ds, \qquad F^x - iF^y = -\frac{i\varrho}{2} \Gamma W_\infty.$$

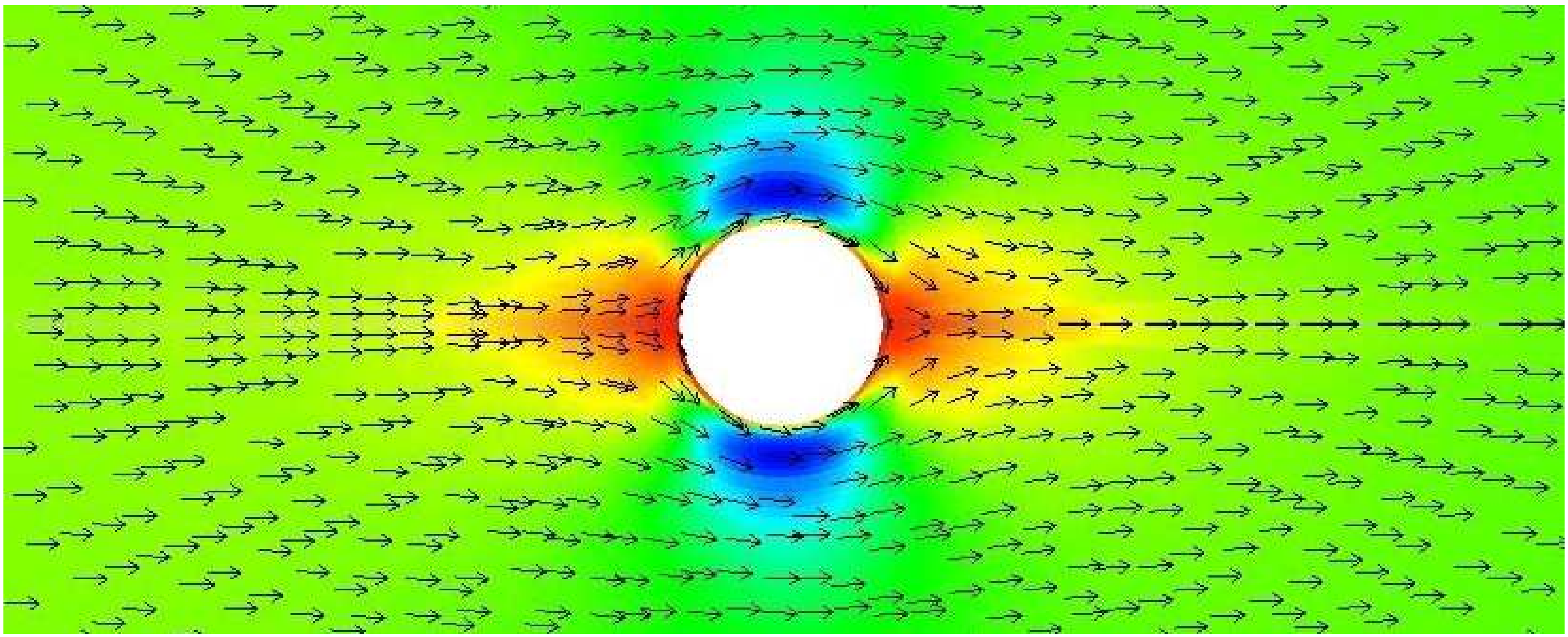
Kutta-Joukowski: choose unique Γ so that $W = 0$ at trailing edge.
Result: approximation for correct lift (force perpendicular to $\vec{v}_\infty$).

d'Alembert paradox: no drag (force in $\vec{v}_\infty$ direction). Drag generation very complex, requires some viscosity, boundary layer separation, vortex shedding, carrying energy away in wake.

d'Alembert paradox

Zero ω, zero Γ (symmetry): no force on body



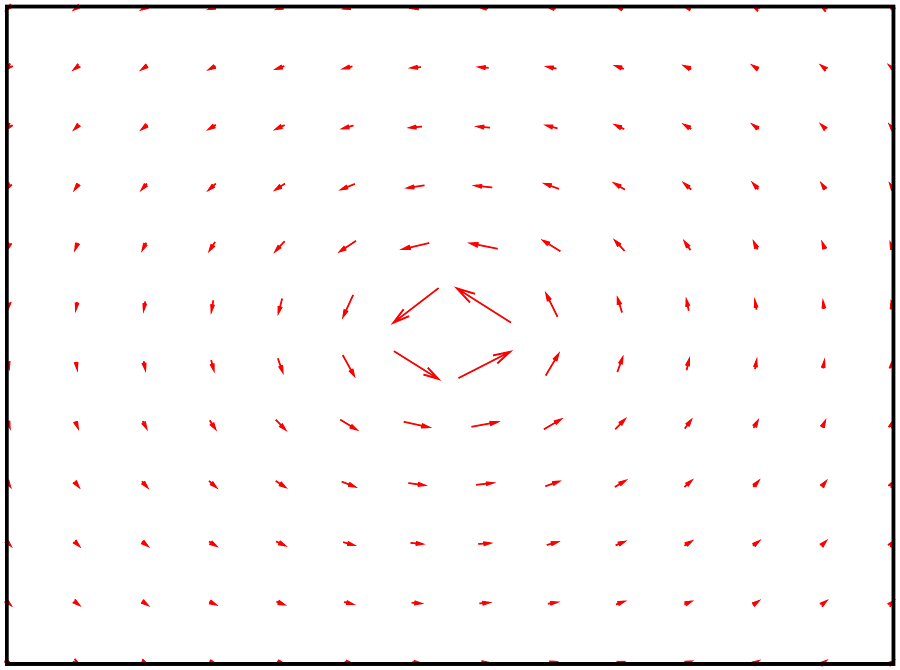


$\Rightarrow$ vorticity ω crucial for most basic questions, such as lift and drag

Point vortices

$$W(Z) = \frac{\Gamma}{2\pi i(Z - Z')}$$

$\sim \frac{1}{r}$ velocity field induced by vortex of circulation Γ located in Z'.



$$\Gamma = \oint_C v \cdot ds$$

where C is any simple contour passing around Z' counterclockwise.

For $k = 1, ..., N$:

$$\frac{\partial}{\partial t} Z_n = \partial_t (x_n + i y_n) = (v_n^x - i v_n^y)^* = \left(\sum_{k \neq n} \frac{\Gamma_k}{2\pi i(Z_n - Z_k)} \right)^*$$

Like gravitational N body problem, but “forces” angular, not radial.

Vortex sheets: ω concentrated on smooth curve.

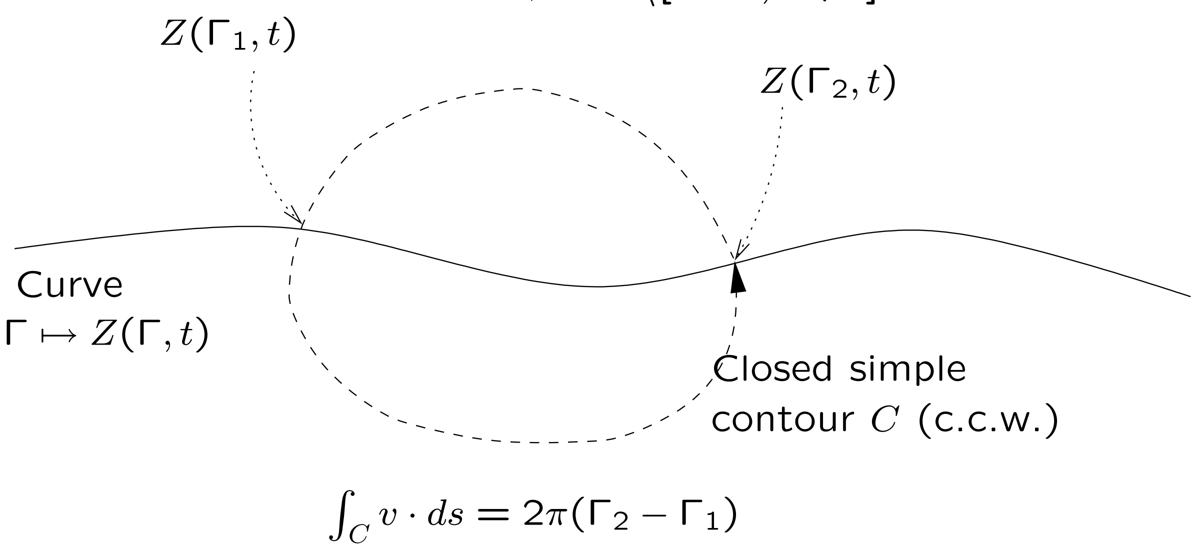
Consider composed of small vortices $d\Gamma'$.

$$\frac{\partial}{\partial t} Z_n(t) = \left(\sum_{k \neq n} \frac{\Gamma_k}{2\pi i (Z_n(t) - Z_k(t))} \right)^* \qquad (k, n = 1, ..., N)$$

$\rightsquigarrow$ Birkhoff-Rott equation:

$$\frac{\partial}{\partial t} Z(\Gamma, t) = W^* := \left(\text{p.v.} \int_{-\infty}^{\infty} \frac{d\Gamma'}{2\pi i (Z(\Gamma, t) - Z(\Gamma', t))} \right)^*$$

where p.v. $\int := \lim_{\epsilon \downarrow 0} \int_{\mathbb{R} \setminus [\Gamma - \epsilon, \Gamma + \epsilon]}$.



$t \mapsto Z(t, \Gamma)$ for fixed Γ is supposed to follow particle paths (arithmetic average of v on each side).

Birkhoff-Rott $\leftrightarrow$ incompressible Euler

For sufficiently smooth Z (neglect question of $\Gamma \to \pm\infty$ here):
define v off-sheet by

$$W(x+iy) = v^x - iv^y = \int_G \frac{d\Gamma'}{x+iy-Z(t,\Gamma')} \quad (x+iy \notin Z(t,G)),$$

then v is weak solution of incompressible Euler equations

$$v_t + \nabla \cdot (v \otimes v) + \nabla\pi = 0 \quad , \quad \nabla \cdot v = 0$$

I.e. for test functions ϕ,

$$\int\int \vec{v}\ \partial_t\phi + (\vec{v}\cdot\nabla\phi)\vec{v} + \pi\nabla\cdot\vec{v}\ dx\ dt = 0 \quad , \quad \int \nabla\phi\cdot v\ dx = 0$$

Proof:

1. off-sheet: v, π smooth irrotational incompressible Euler solution.
2. on sheet: verify Rankine-Hugoniot conditions.
3. in singular points (spiral centers): v does not blow up too fast.

Jump conditions for incompressible Euler discontinuities

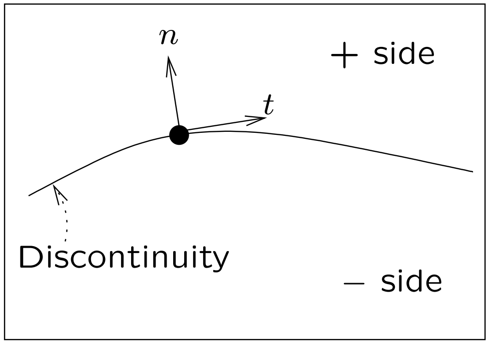
Generally, in any point on discontinuity with speed σ and normal n:

$$0 = U_t + \nabla \cdot f(U) \quad \rightsquigarrow \quad \sigma[U] = [f(U)] \cdot n, \quad [A] = \text{jump of } A.$$

(Integrate over volume and consider boundary integrals.)

For incompressible Euler: $\nabla \cdot v = 0 \rightsquigarrow$

$$0 = [v] \cdot n = [v \cdot n]$$



$v_t + \nabla \cdot (v \otimes v) + \nabla \pi = 0 \rightsquigarrow$

$\sigma[v] = [(v \cdot n)v] + [\pi]n$ (several components)

$$n\cdot \quad \Rightarrow \quad \sigma[v \cdot n] = [(v \cdot n)^2] + [\pi] \quad \Rightarrow \quad [\pi] = 0$$

When solving $\Delta\pi = -\mathrm{tr}((\nabla v)^2)$ on each side, impose "π continuous" as coupling condition on sheet.

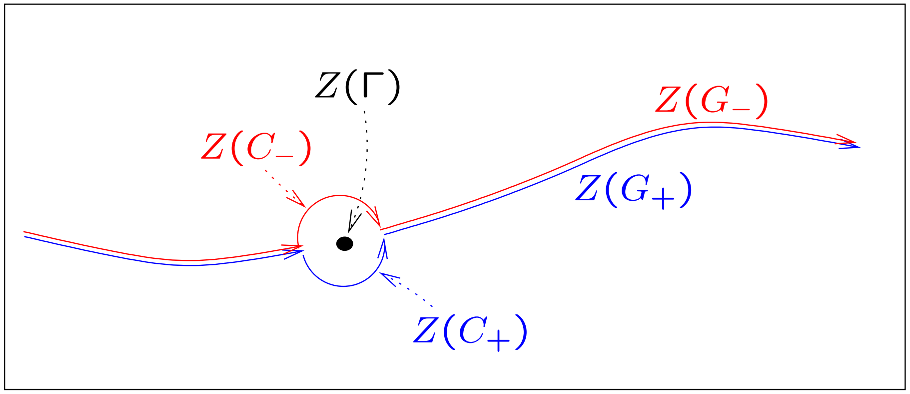
$$t\cdot \quad \Rightarrow \quad \sigma[v\cdot t] = [(v\cdot n)(v\cdot t)] = [v\cdot t](v\cdot n) \quad \Rightarrow \quad \sigma = v_+ \cdot n = v_- \cdot n$$

(unless $[v \cdot t] = 0$ which means no jumps in any variable).

Principal-value integrals: If $R(s)$ may have singularity at $s = 0$, $a < 0 < b$:

$$\text{p.v.} \int_a^b R(s)ds := \lim_{\epsilon \downarrow 0} \left(\int_a^{-\epsilon} + \int_\epsilon^b \right) R(s)ds$$

$\mathbb{R} \supset G \ni \Gamma \mapsto Z(\Gamma) \in \mathbb{C}$ injective ($Z_\Gamma \neq 0$), real-analytic
$\rightsquigarrow$ holomorphic extension to neighbourhood U of $\mathbb{R}$ in $\mathbb{C}$.



$$W_\pm := \frac{1}{2\pi i} \int_{G_\pm} \frac{d\Gamma'}{Z(\Gamma) - Z(\Gamma')}$$

W_+ limit of W in $Z(\Gamma)$ on upper side, W_- limit on lower side.

C_- radius ϵ upper half-circle around Γ, C_+ lower half-circle.
No $\int$ change when shifting contours without crossing Γ !

$$\frac{1}{Z(\Gamma) - Z(\Gamma')} = \frac{1}{Z_\Gamma(\Gamma)(\Gamma - \Gamma') - \frac{1}{2} Z_{\Gamma\Gamma}(\Gamma)(\Gamma - \Gamma')^2 + \ldots}$$

$$= \frac{1}{Z_\Gamma(\Gamma)} \cdot \frac{1}{\Gamma - \Gamma'} \cdot \frac{1}{1 + O(\Gamma - \Gamma')}$$

$$\overset{|\Gamma - \Gamma'| \ll 1}{=} \frac{1}{Z_\Gamma(\Gamma)} \cdot \frac{1}{\Gamma - \Gamma'} \cdot \Big(1 + O(\Gamma - \Gamma')\Big) = \frac{1}{Z_\Gamma(\Gamma)} \cdot \frac{1}{\Gamma - \Gamma'} + O(1)$$

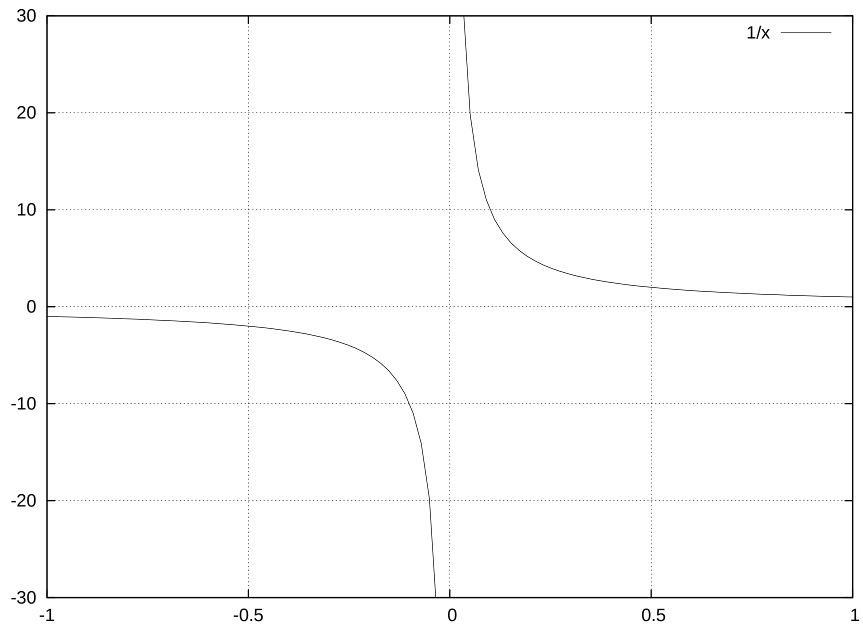
Simple pole in principal-value integral

$$\int_{\Gamma+\epsilon}^{\Gamma+1} \frac{d\Gamma'}{\Gamma-\Gamma'} = \left[\log(\Gamma'-\Gamma)\right]_{\Gamma'=\Gamma+\epsilon}^{\Gamma+1} = \log 1 - \log \epsilon$$

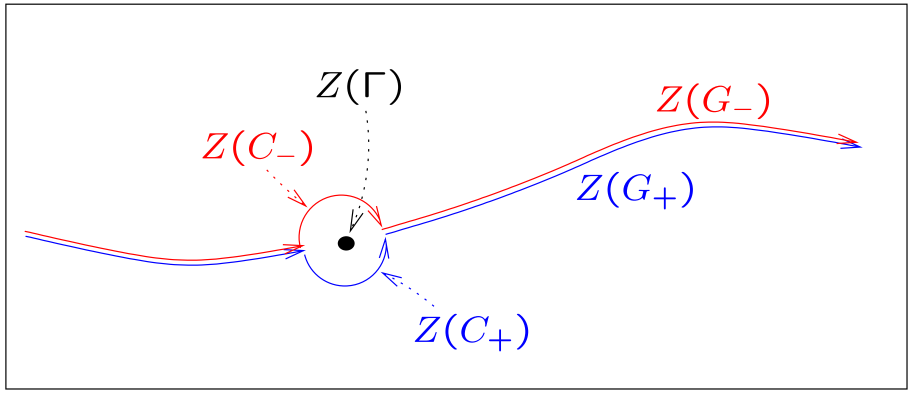
$$\int_{\Gamma-1}^{\Gamma-\epsilon} \frac{d\Gamma'}{\Gamma-\Gamma'} = \left[\log(\Gamma-\Gamma')\right]_{\Gamma'=\Gamma-1}^{\Gamma-\epsilon} = \log \epsilon - \log 1$$

$$\lim_{\epsilon\downarrow 0} \int_{[\Gamma-1,\Gamma+1]\setminus[\Gamma-\epsilon,\Gamma+\epsilon]} \frac{1}{Z_\Gamma(\Gamma)} \cdot \frac{1}{\Gamma-\Gamma'} d\Gamma' = 0$$

($[\Gamma-a, \Gamma+b]$ yields $\neq 0$ value depending on $a, b \in (0, \infty)$, but still convergent.)



Contour integrals



C_- upper half-circle (clockwise) from $\Gamma + \epsilon$ to $\Gamma - \epsilon$

$$\int_{C_-} \frac{1}{\Gamma - \Gamma'} d\Gamma' = \int_{\pi}^{0} \frac{1}{\Gamma - (\Gamma + \epsilon e^{i\theta})} \frac{d(\Gamma + \epsilon e^{i\theta})}{d\theta} d\theta = -\int_{\pi}^{0} \frac{1}{\epsilon e^{i\theta}} i \epsilon e^{i\theta} d\theta = \pi i$$

$$\lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{C_-} \frac{d\Gamma'}{Z(\Gamma) - Z(\Gamma')} = \frac{1}{2} \frac{1}{Z_\Gamma(\Gamma)}$$

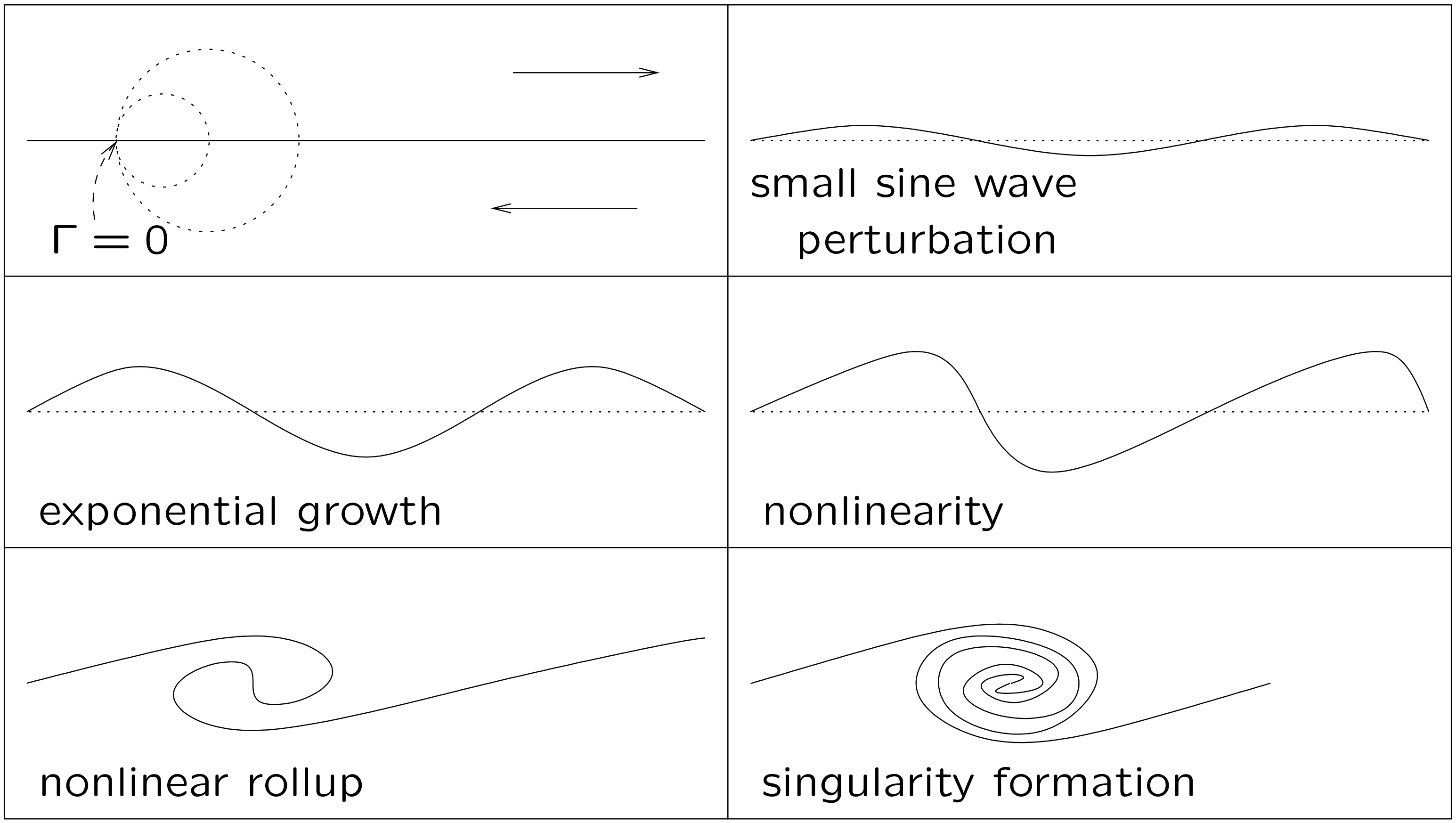
$$\lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{C_+} \frac{d\Gamma'}{Z(\Gamma) - Z(\Gamma')} = -\frac{1}{2} \frac{1}{Z_\Gamma(\Gamma)}$$

$$\lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \text{p.v.} \int_{[-\epsilon, \epsilon]} \frac{d\Gamma'}{Z(\Gamma) - Z(\Gamma')} d\Gamma' = 0.$$

Hence: complex velocity W has tangential jump from lower to upper side by

$$\frac{1}{Z_\Gamma(\Gamma)}$$

Kelvin-Helmholtz instability:



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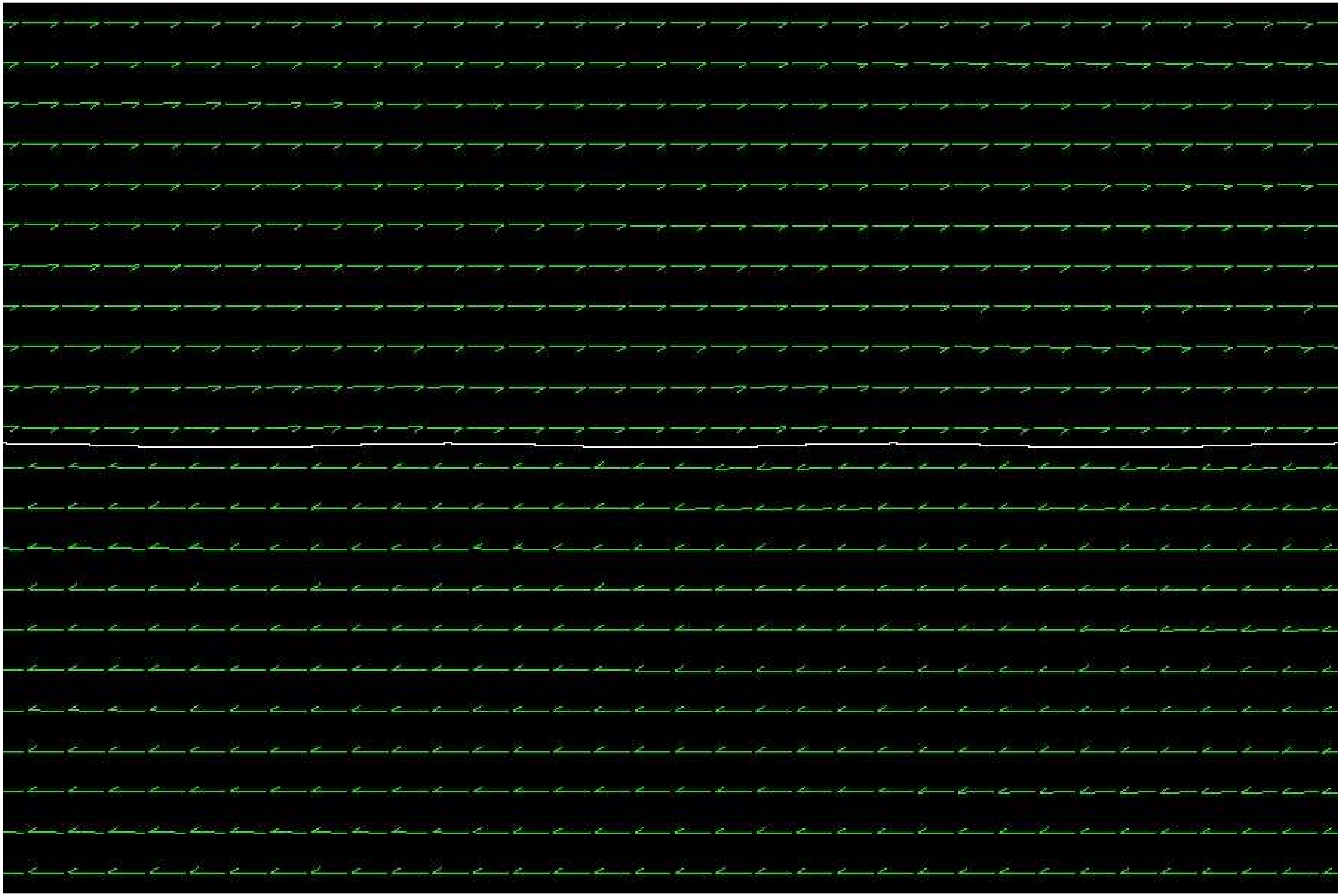
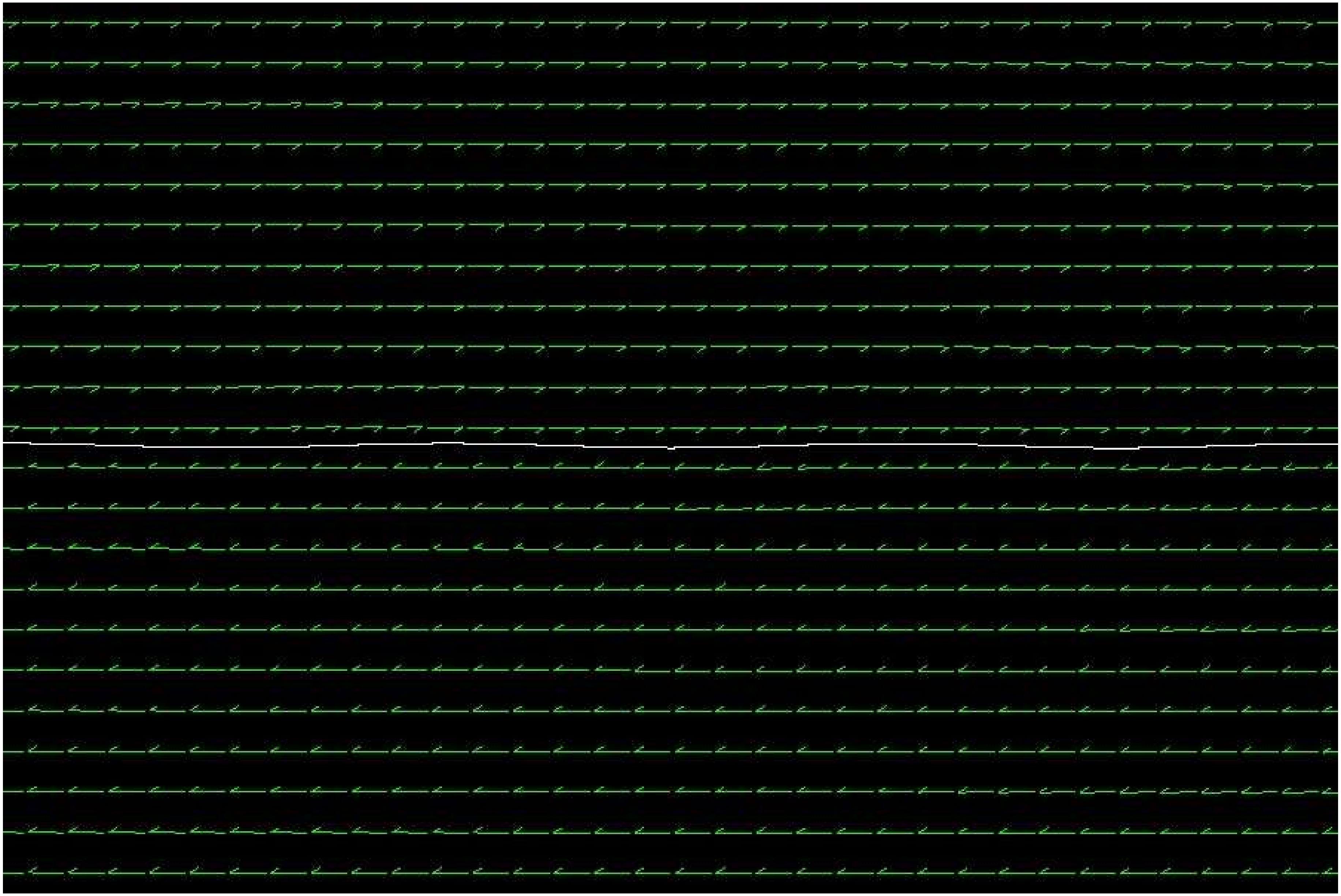
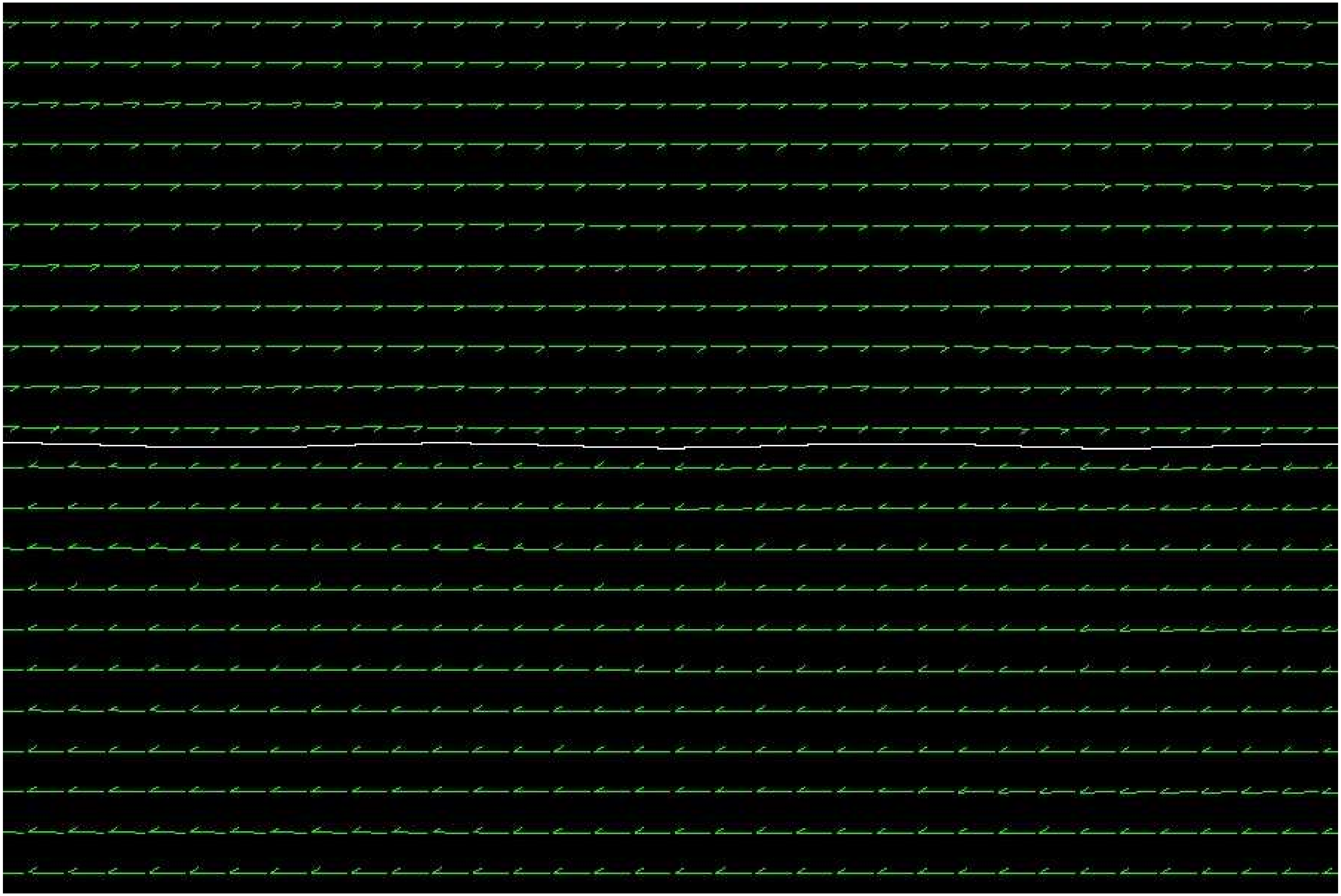
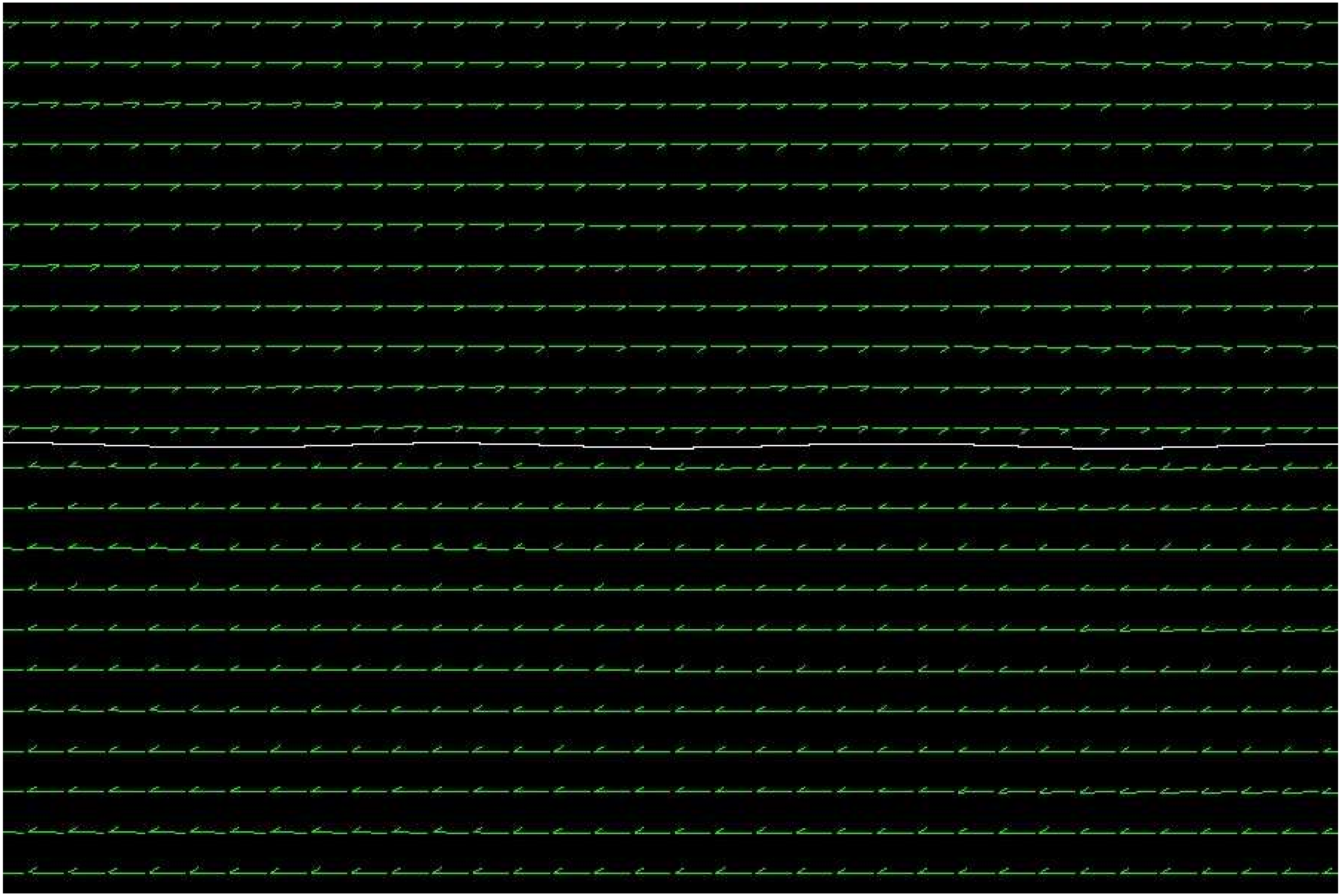
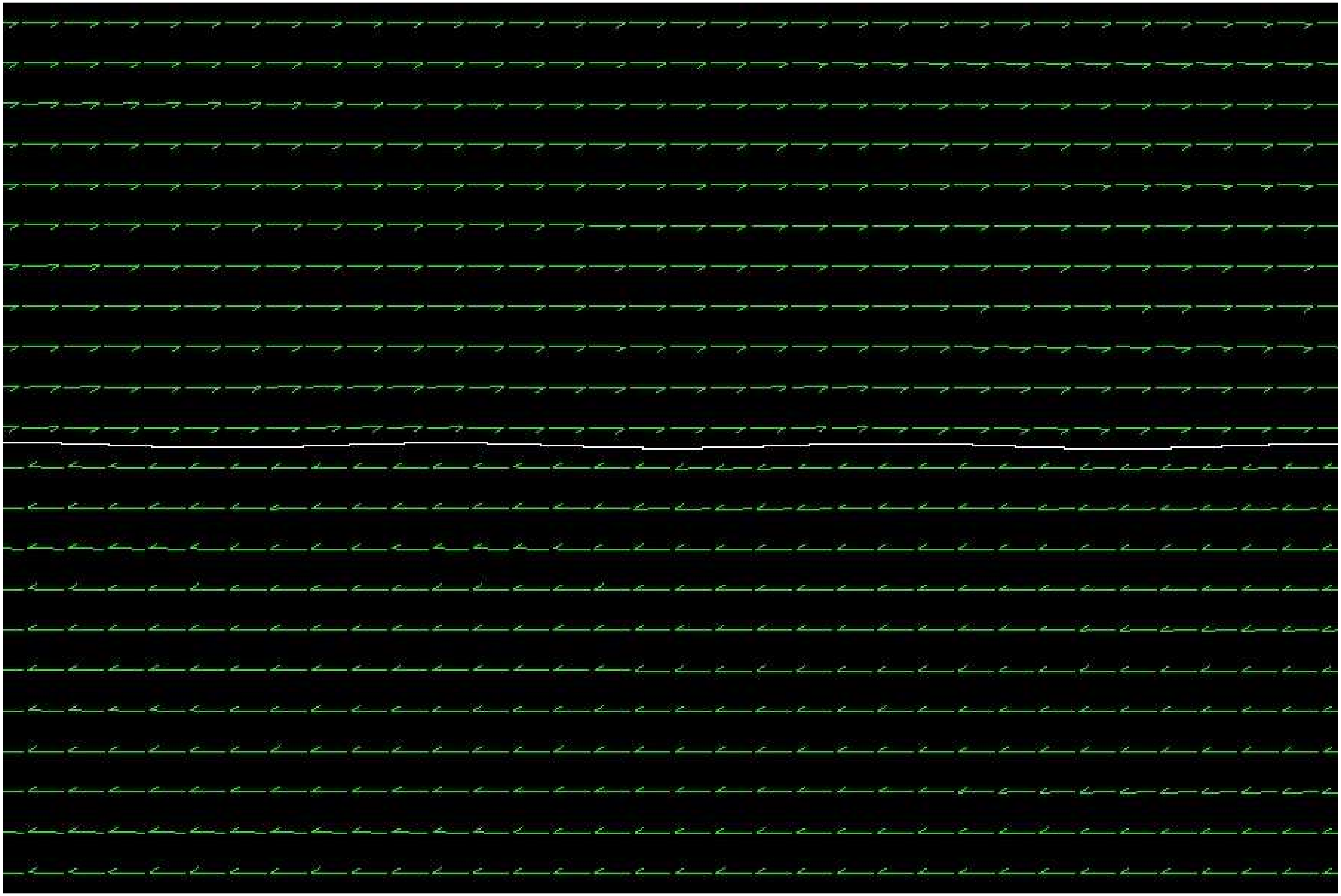
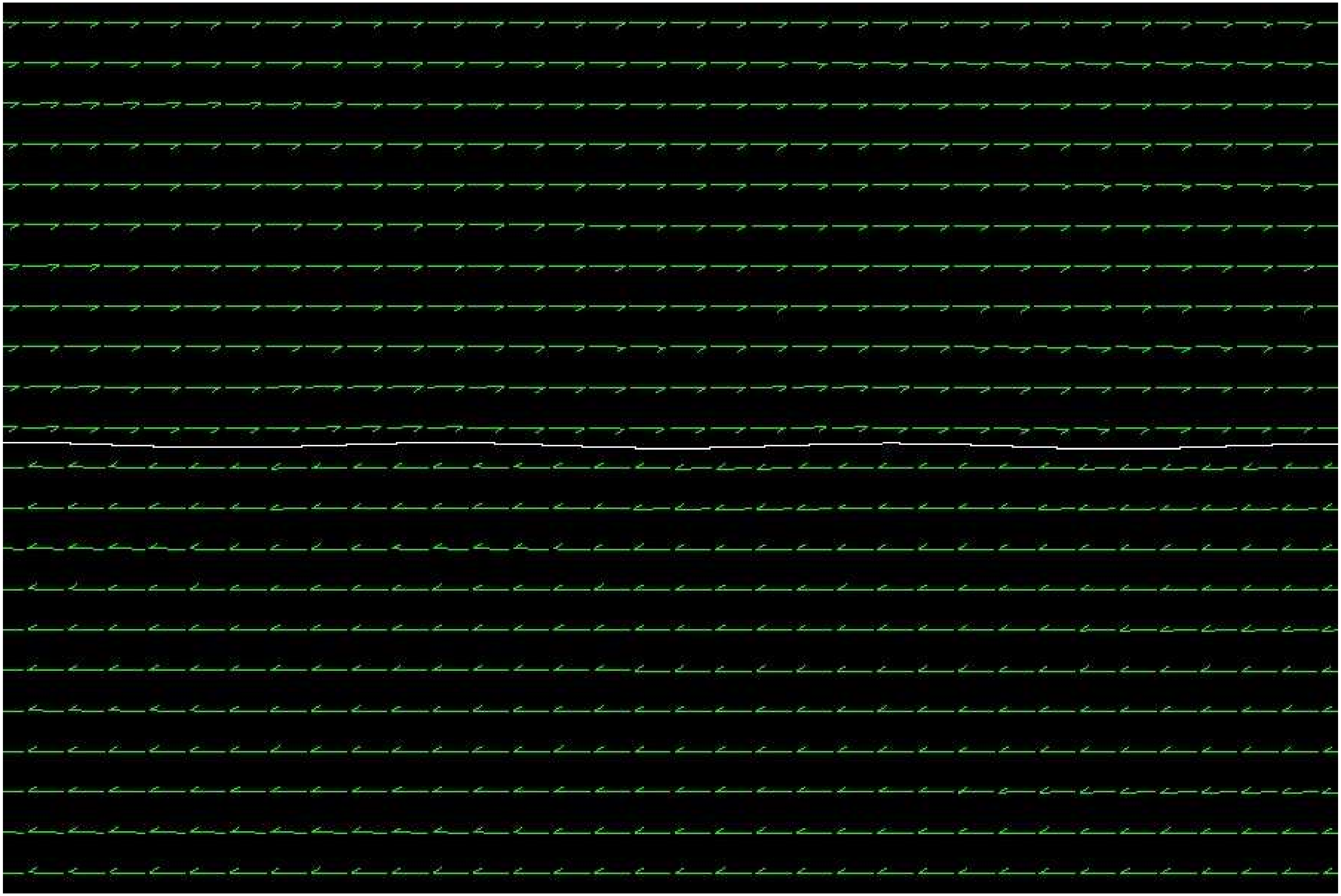


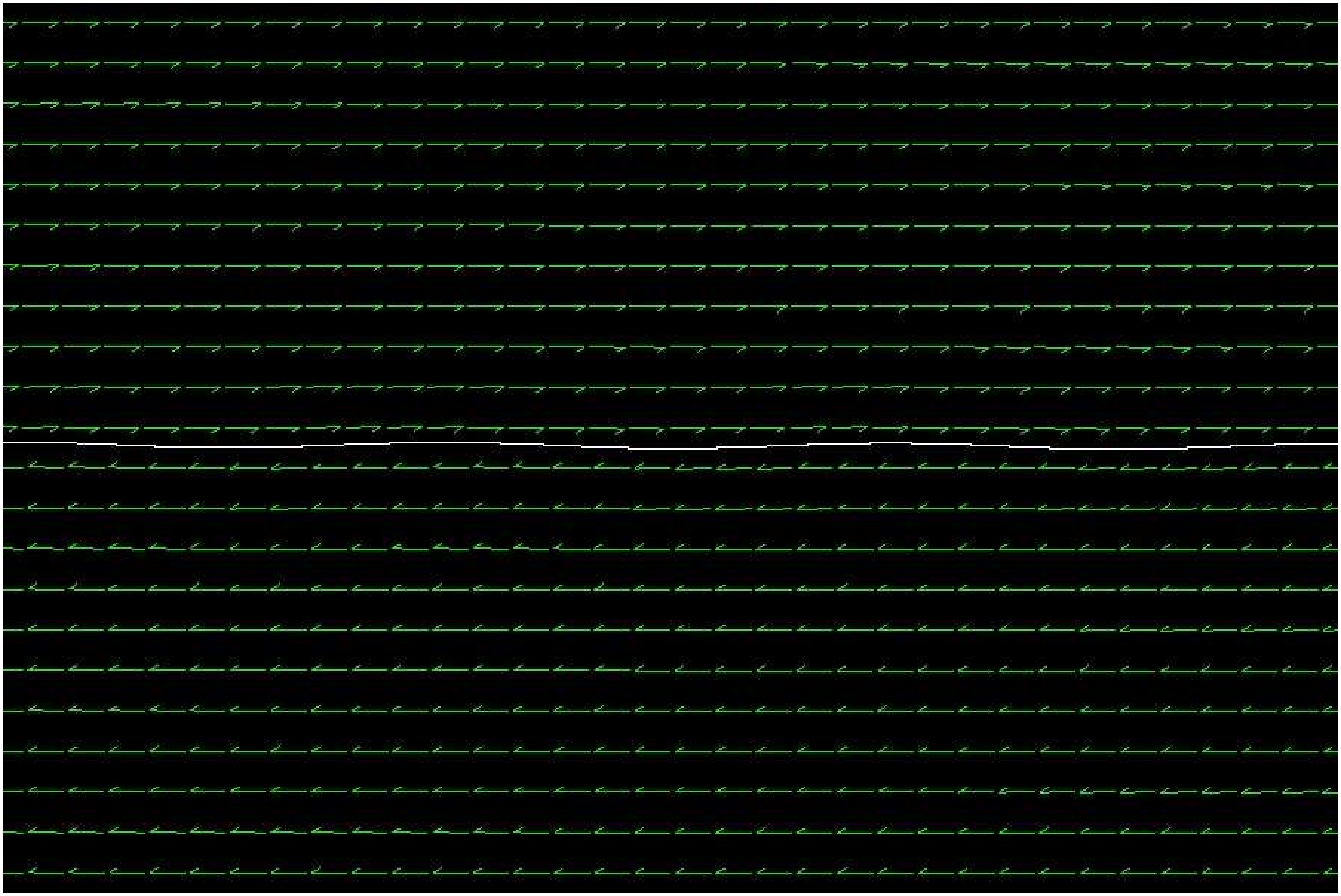


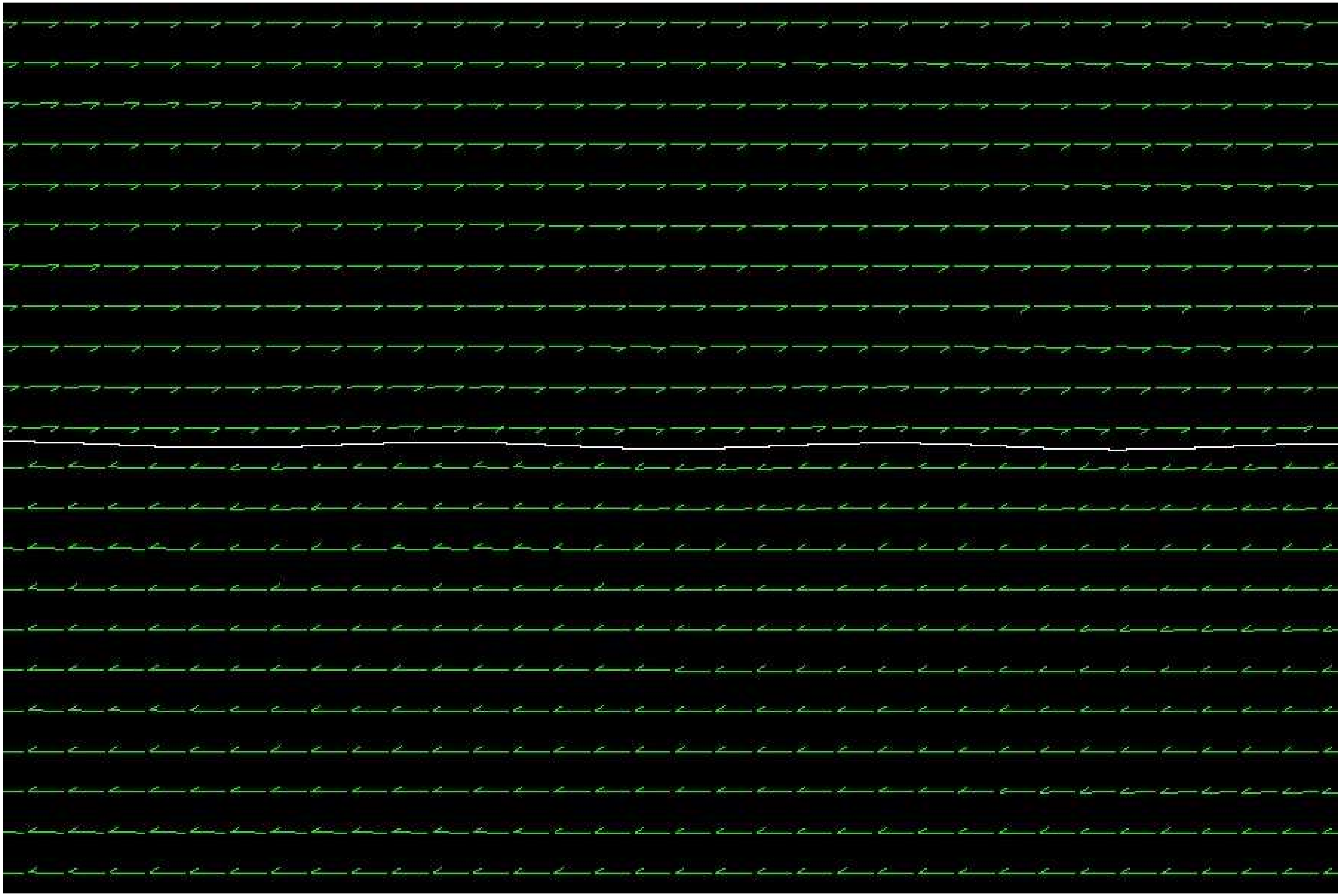


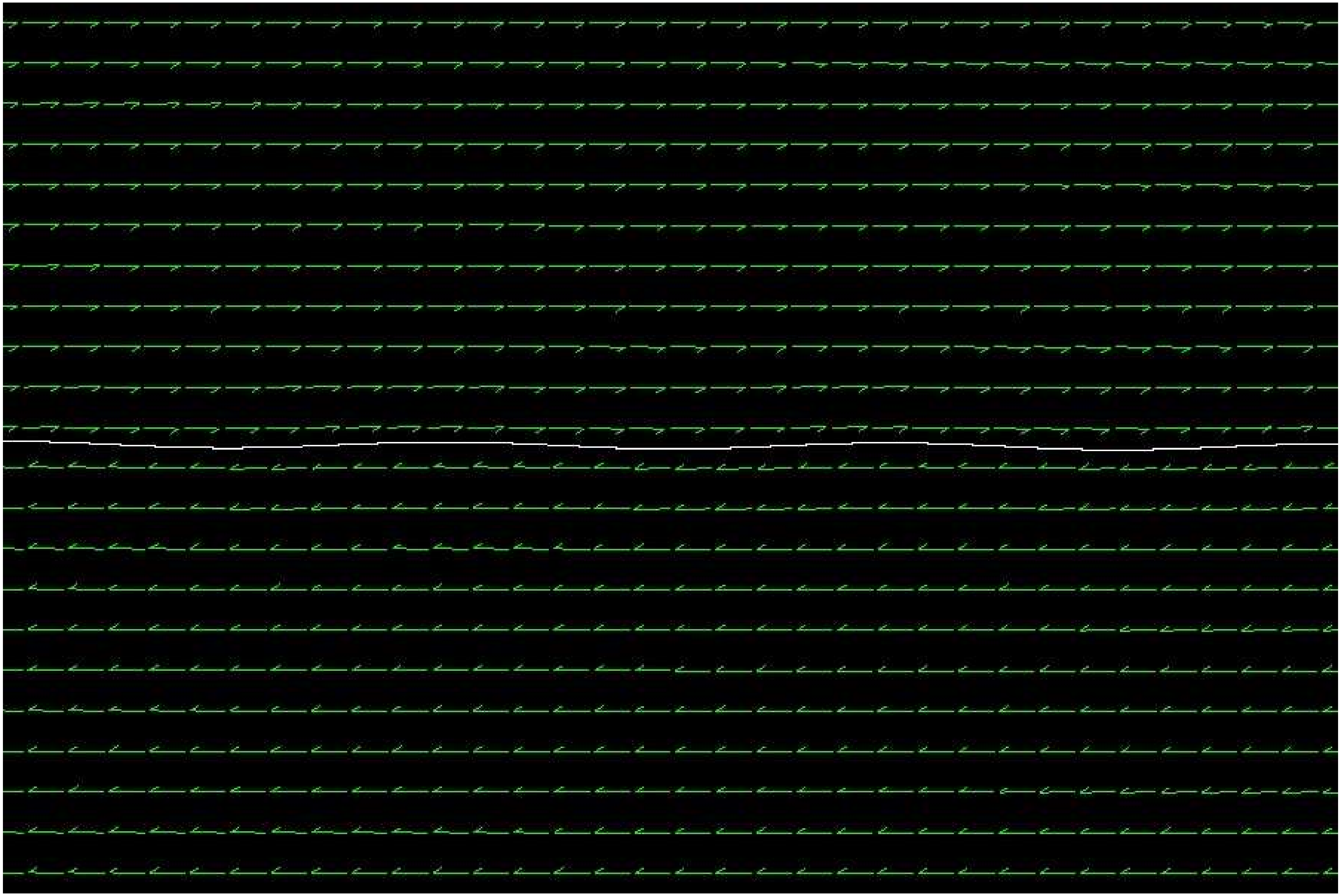


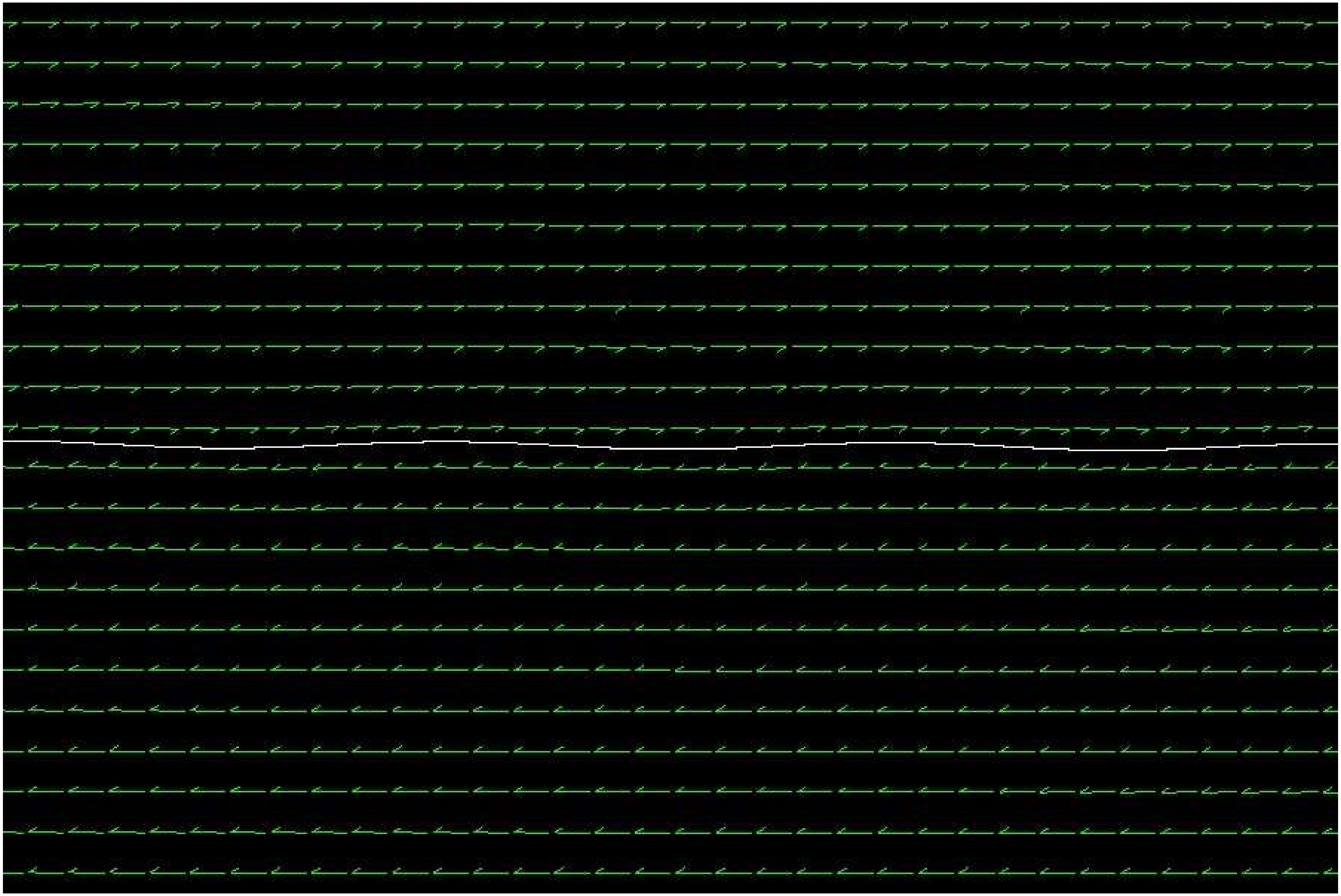


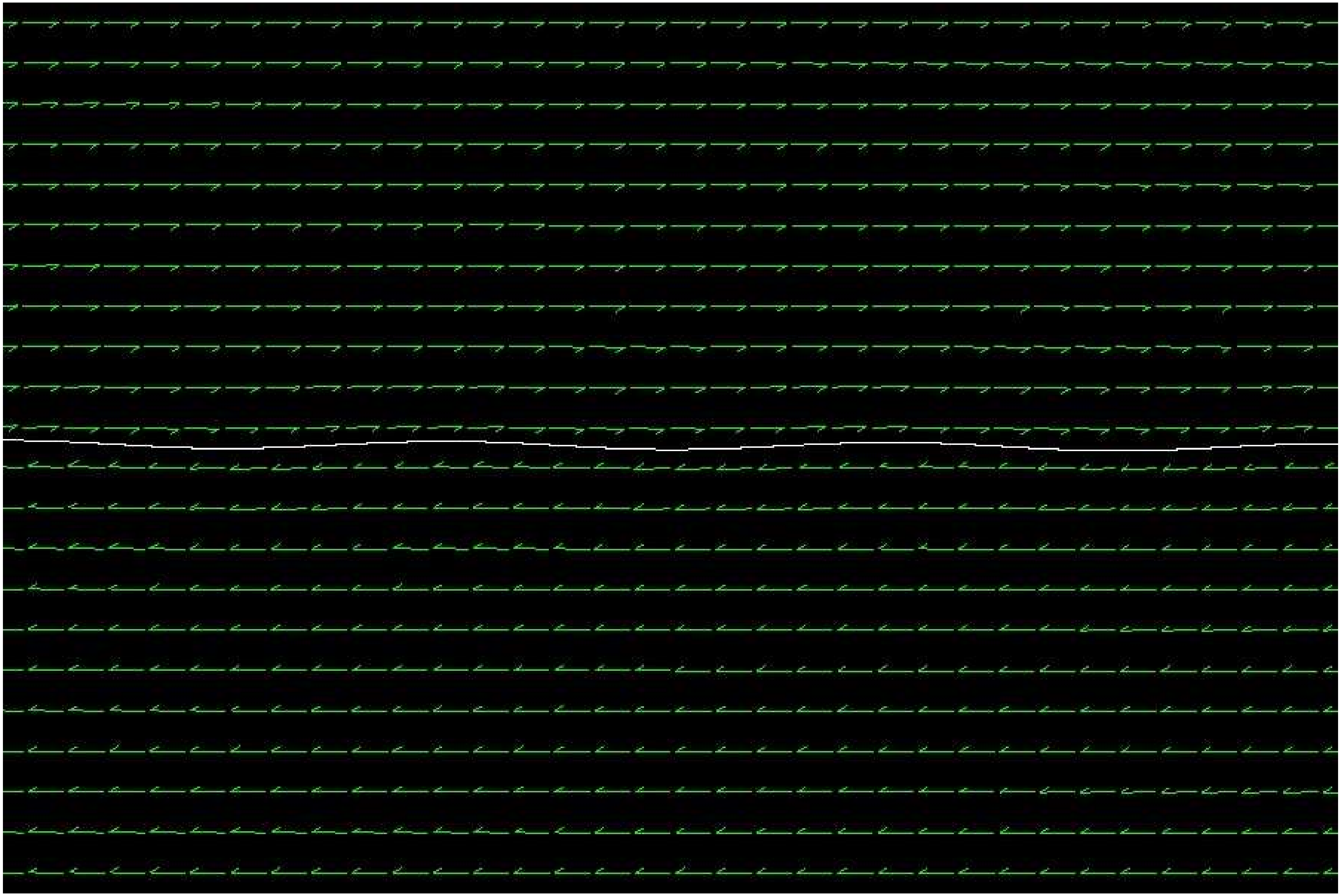


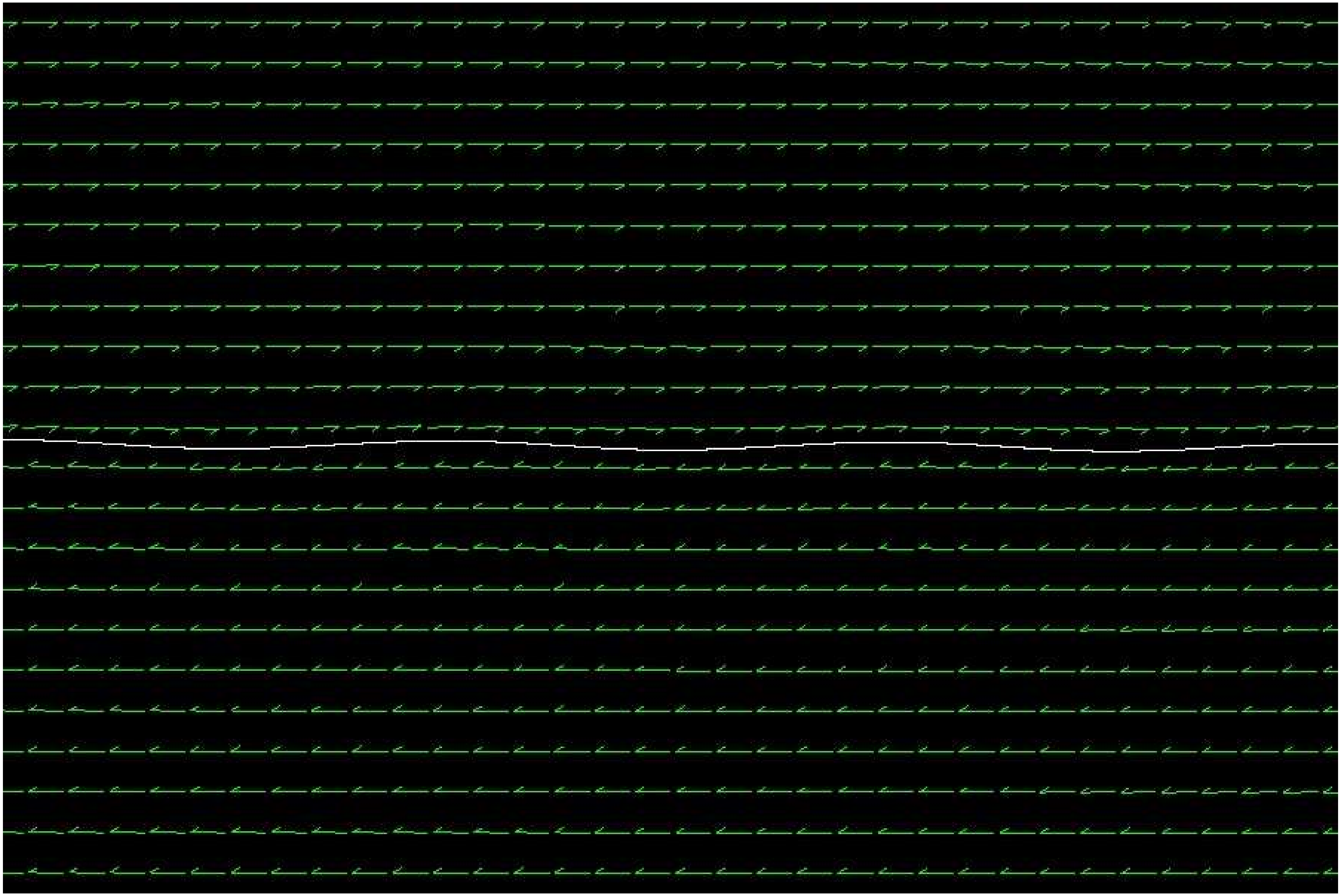


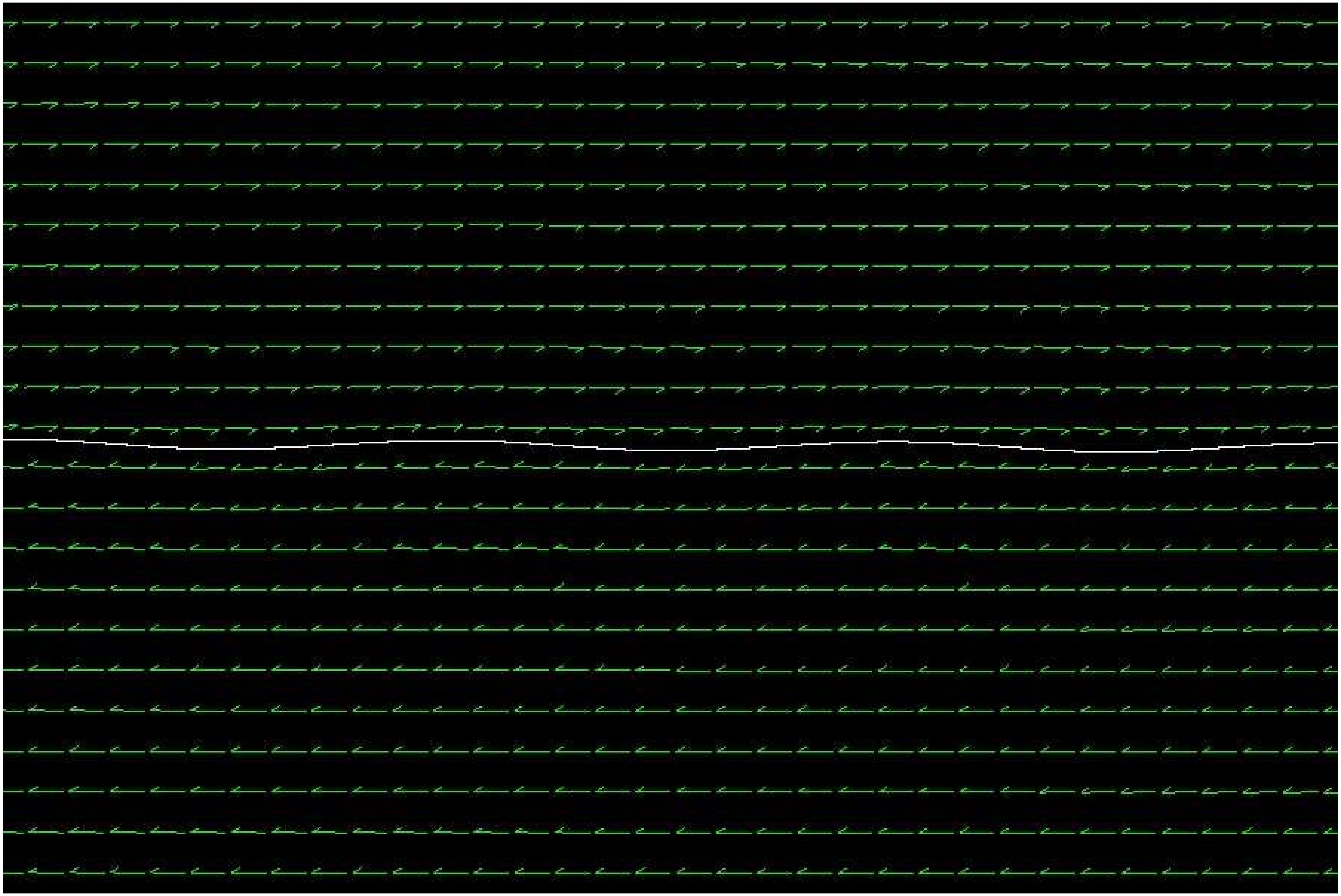


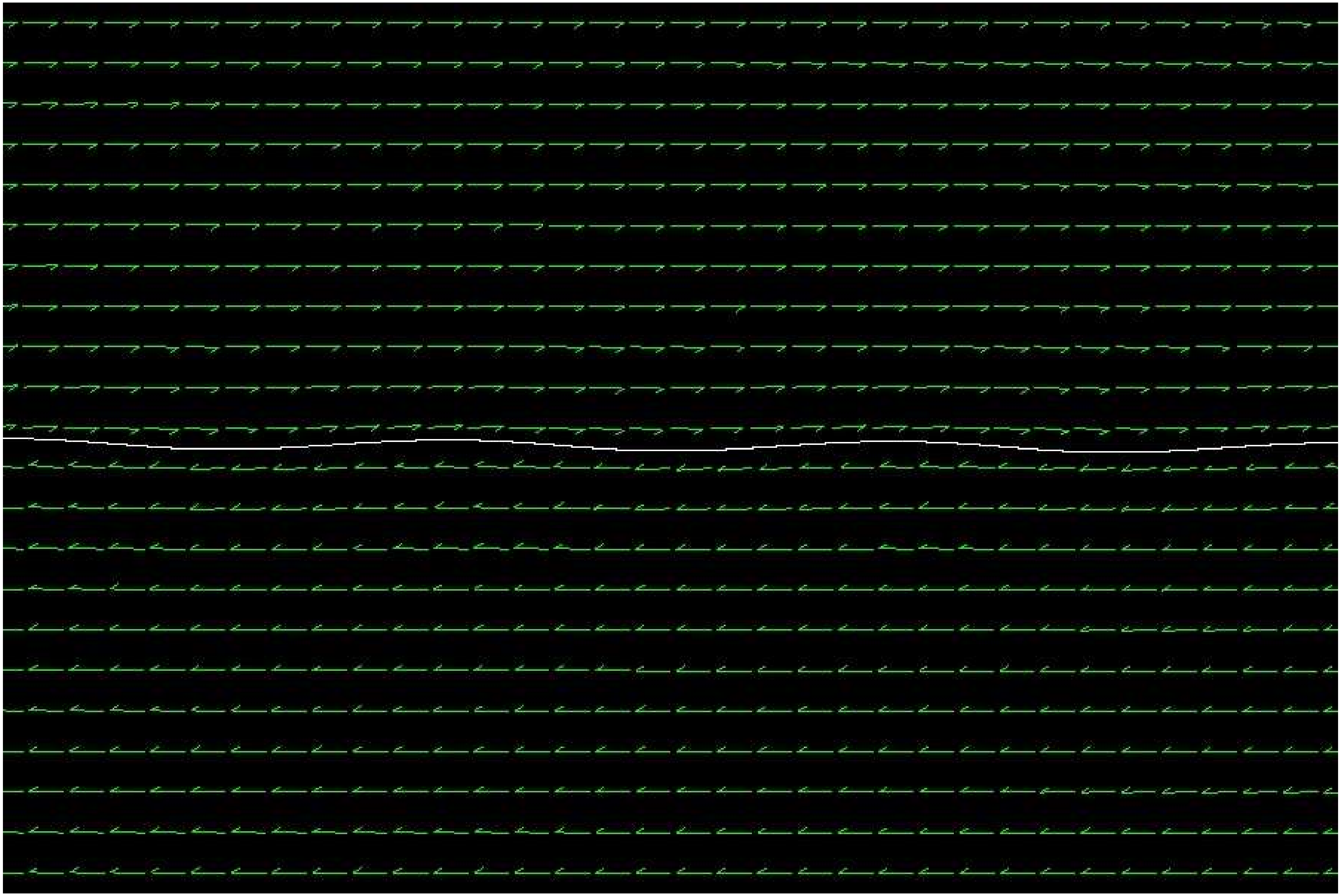


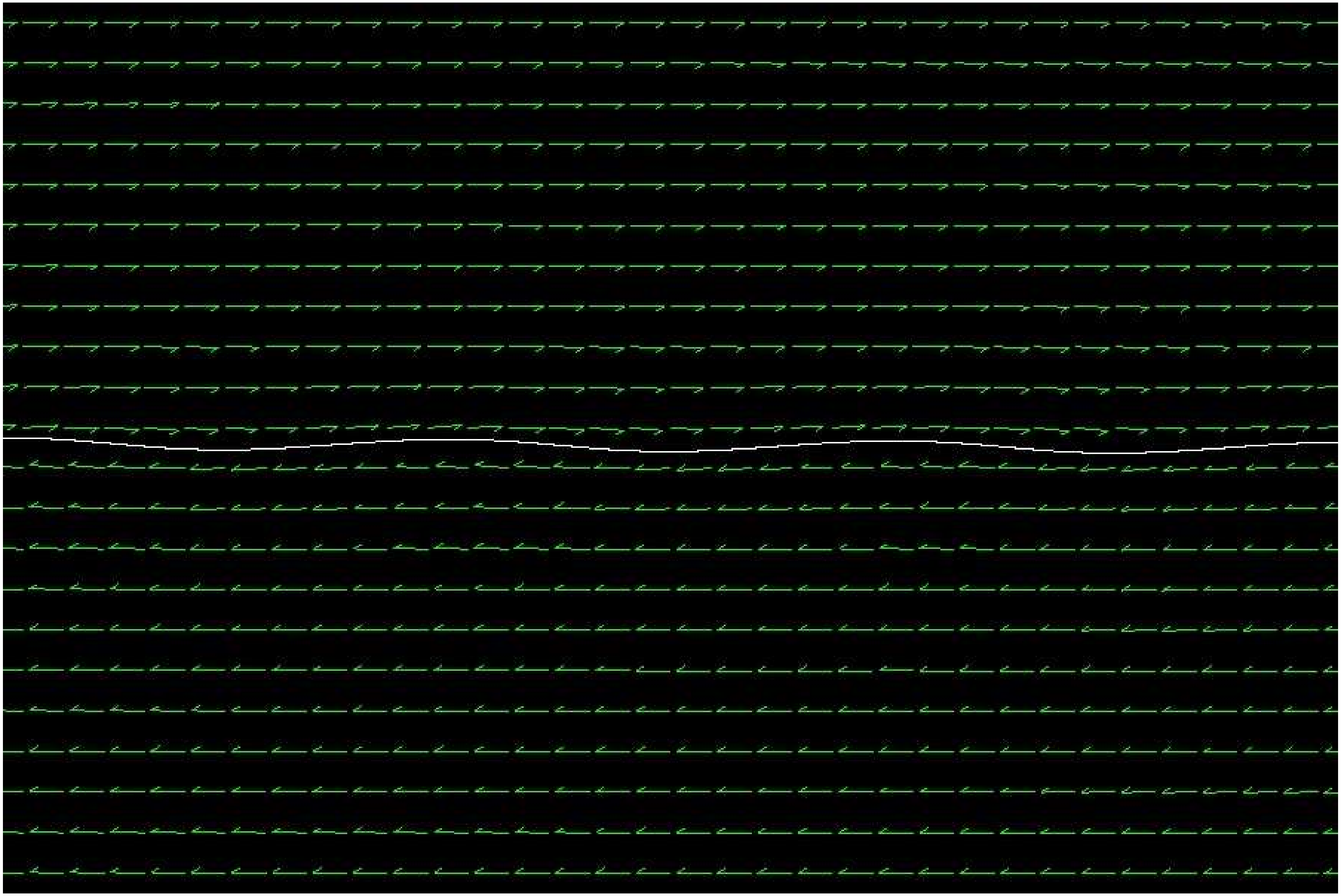


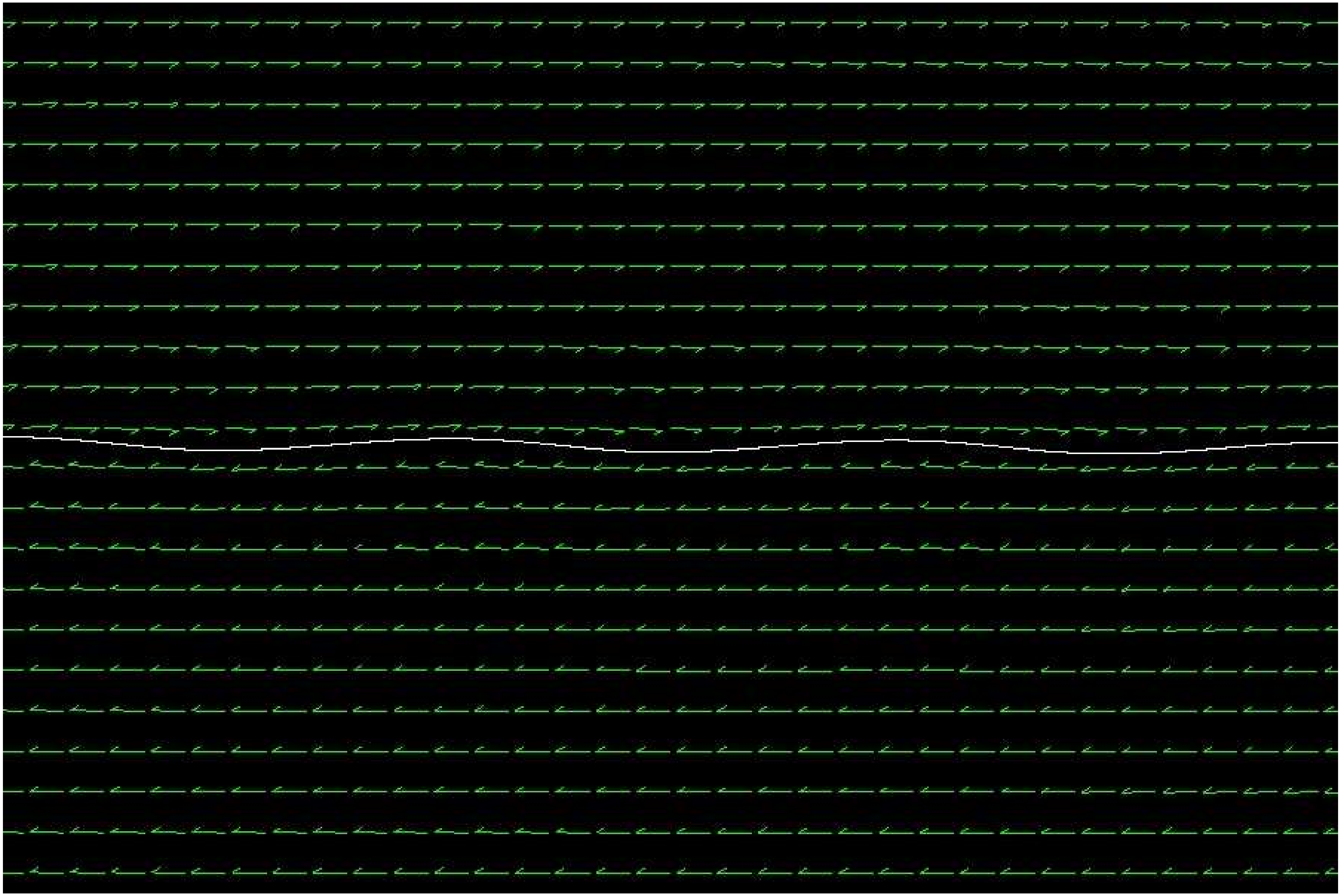


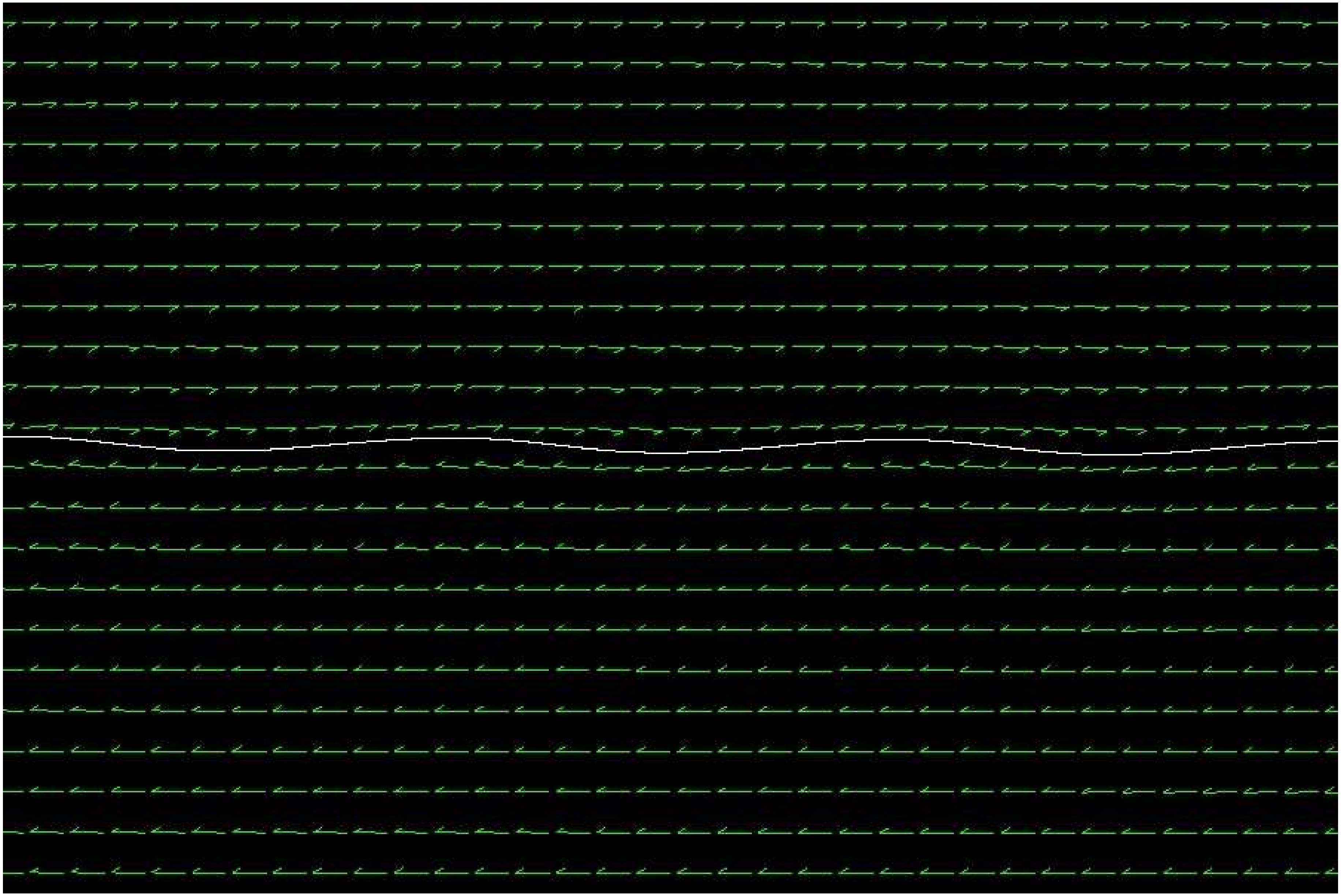


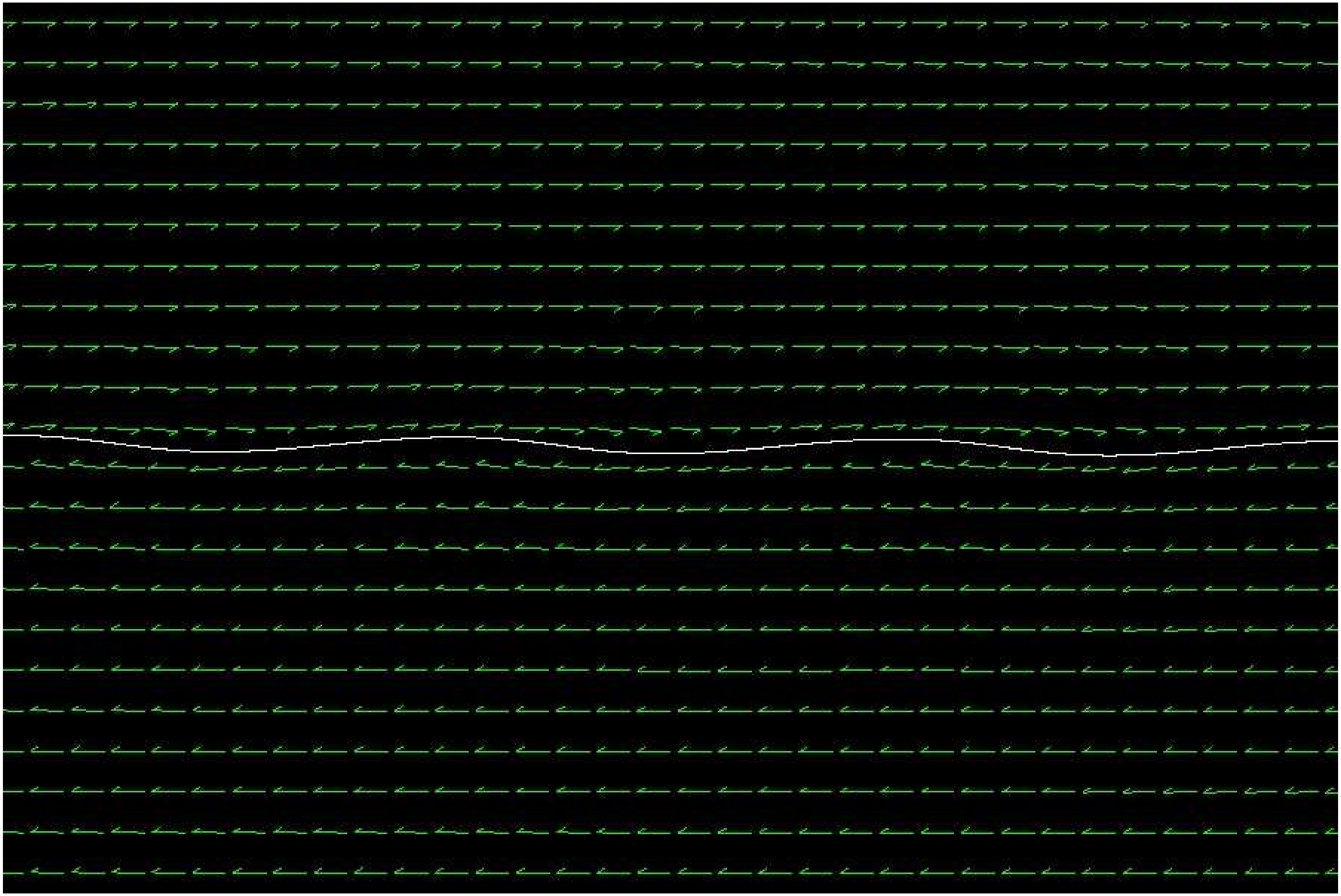


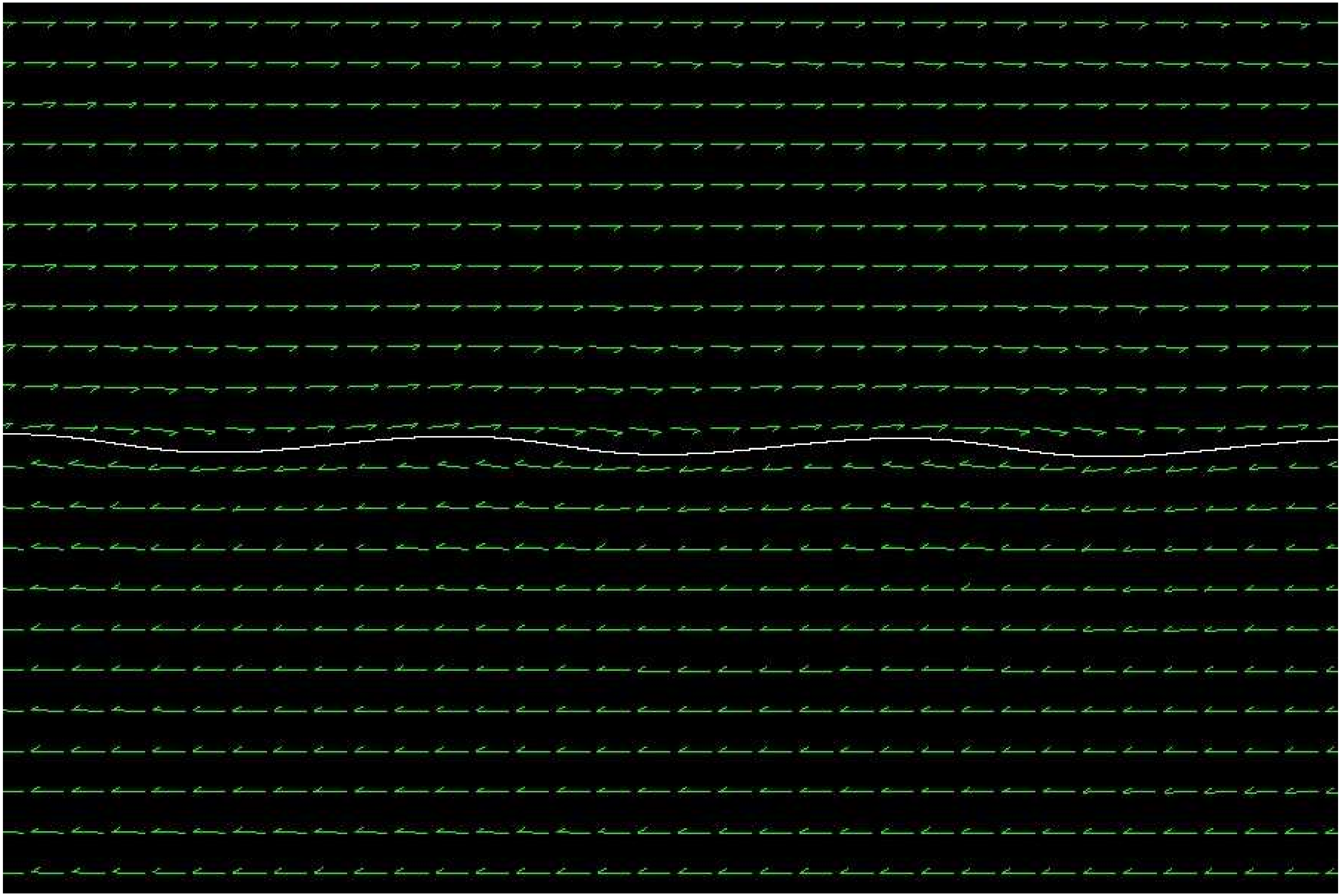


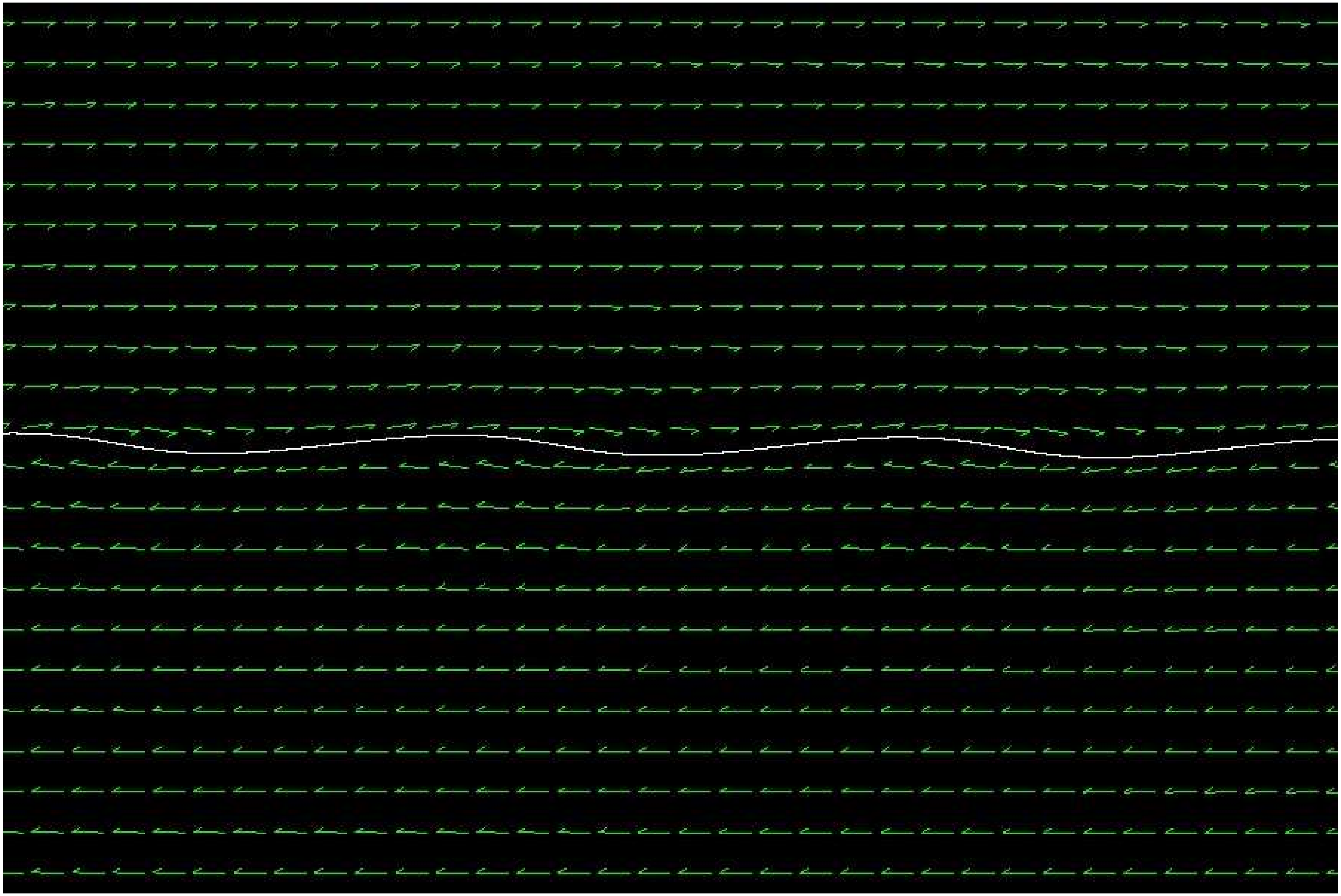


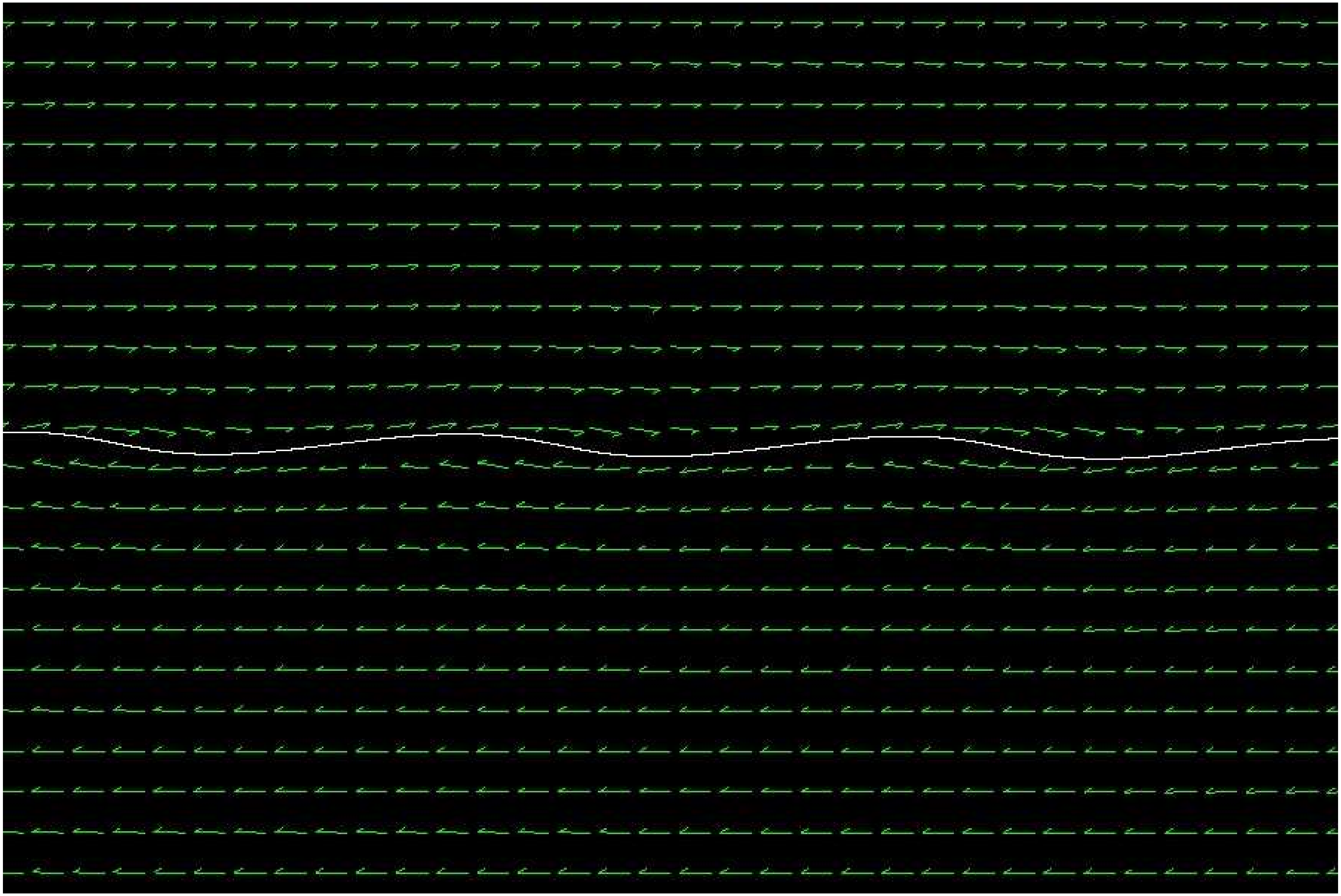


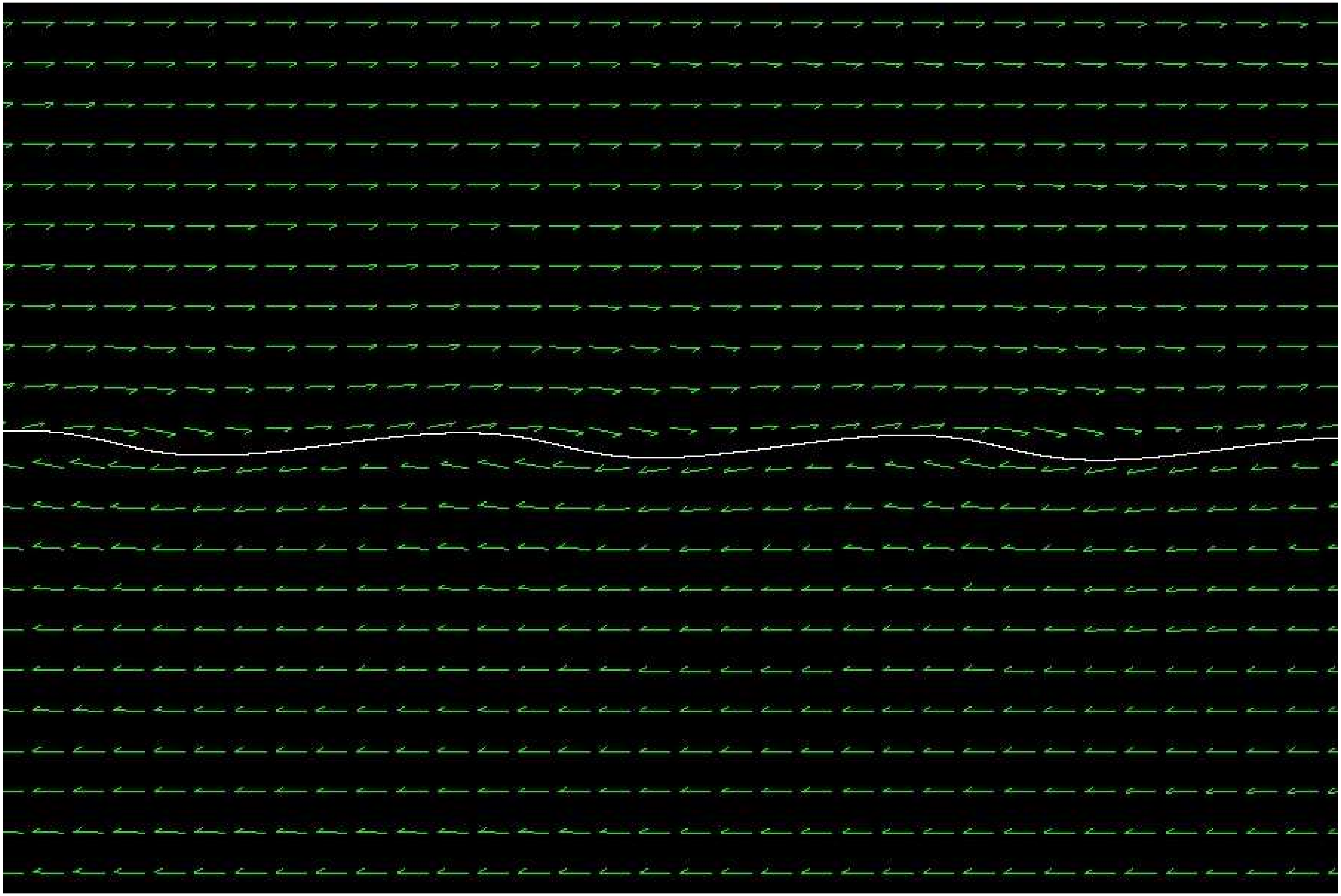


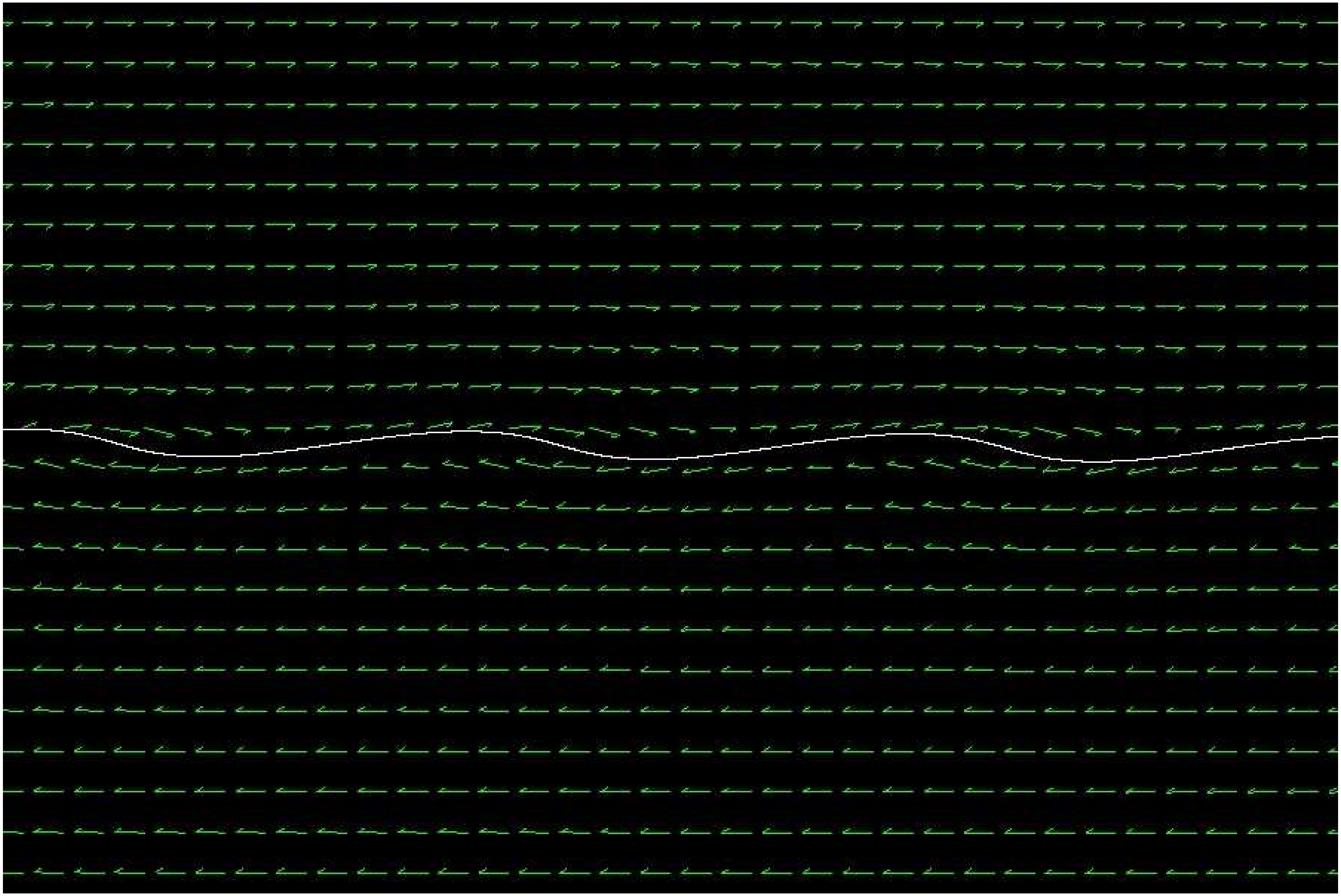


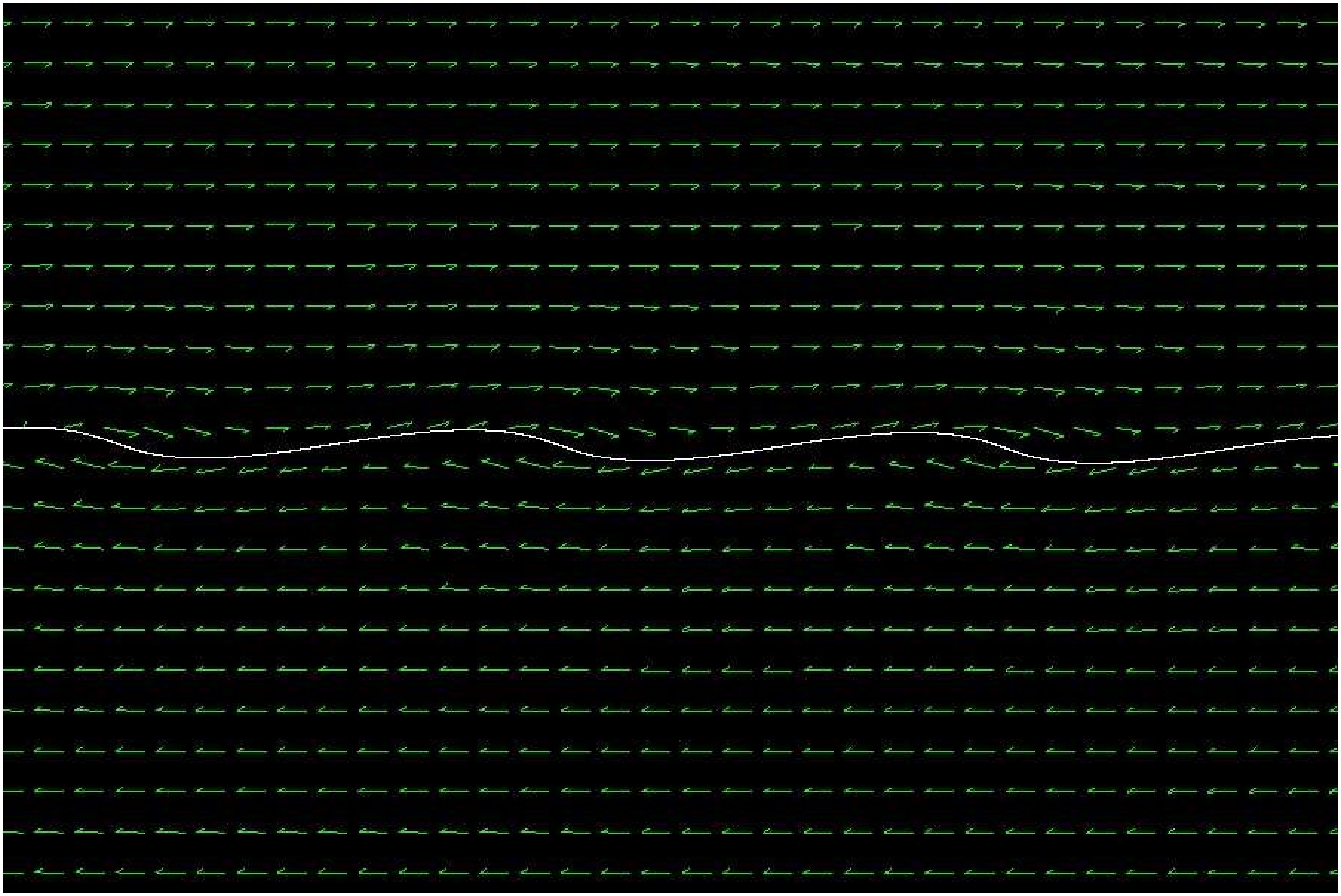


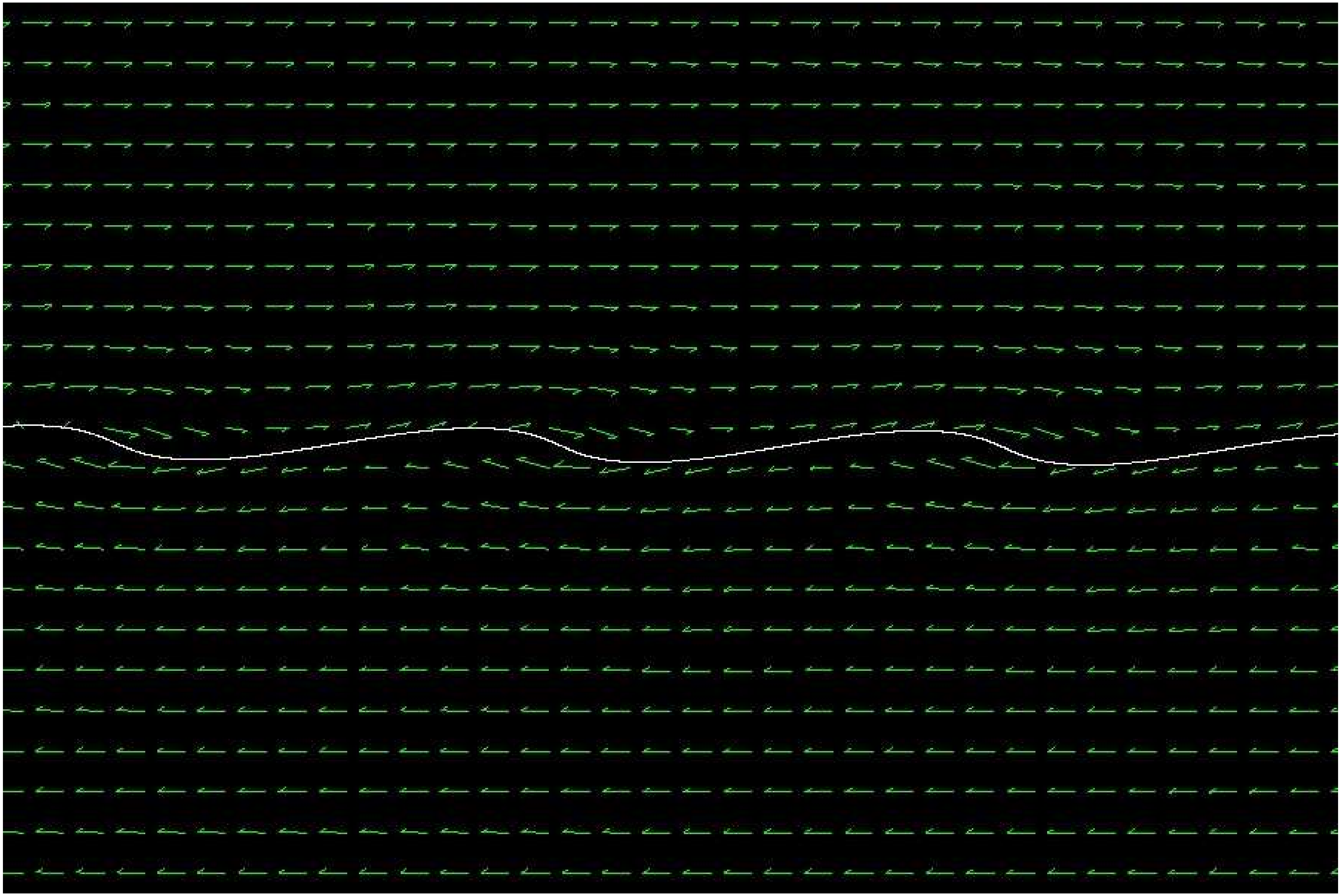


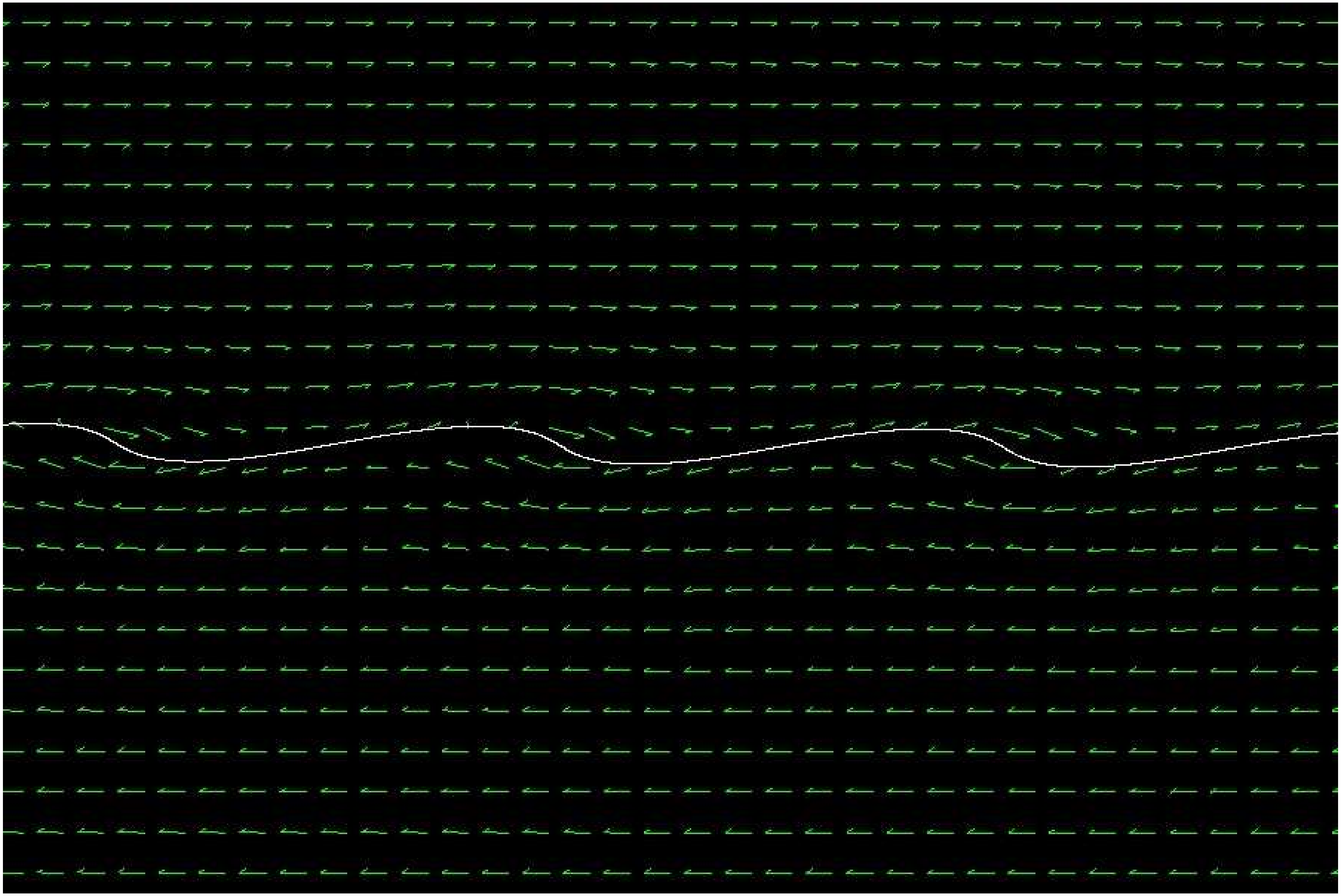


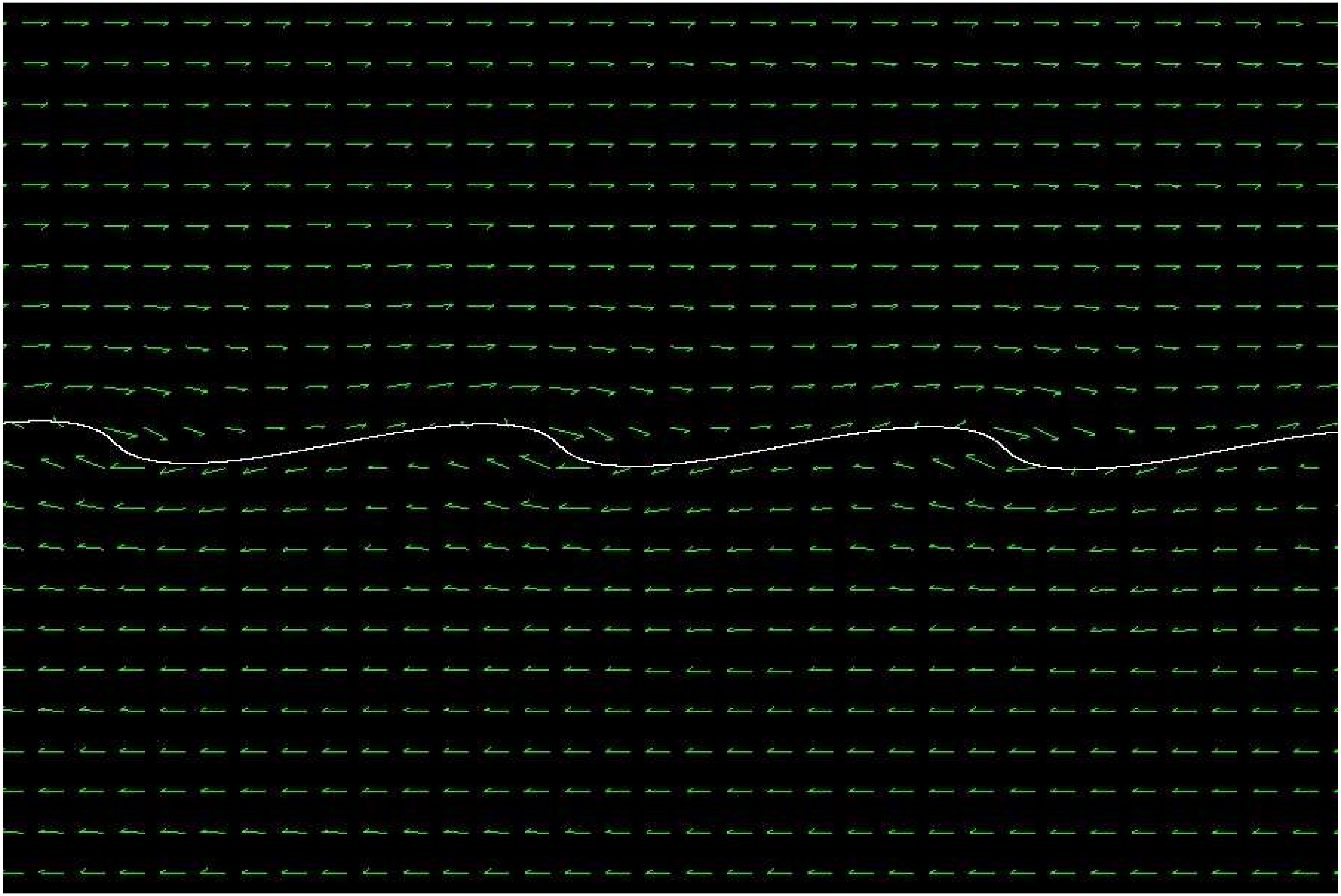


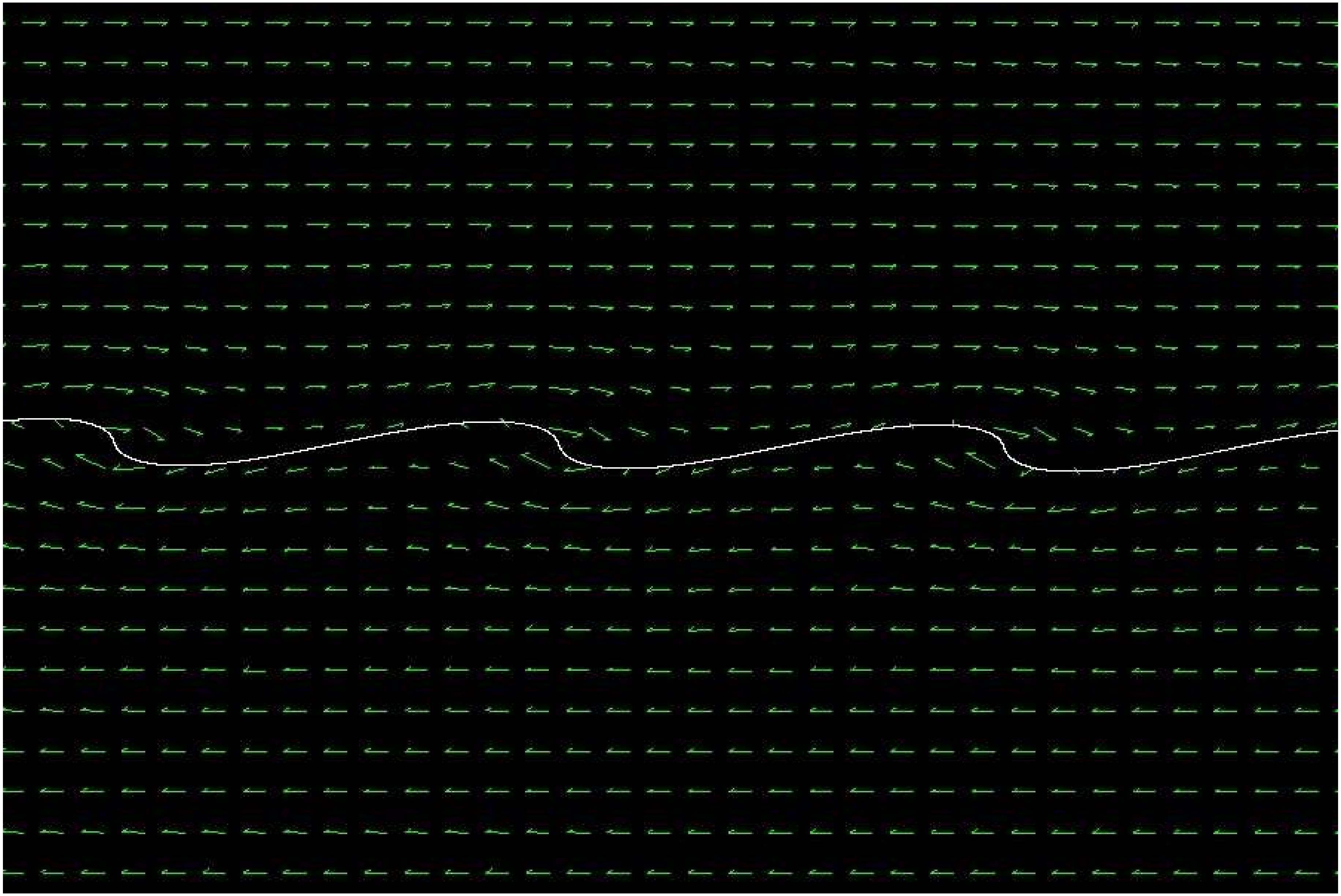


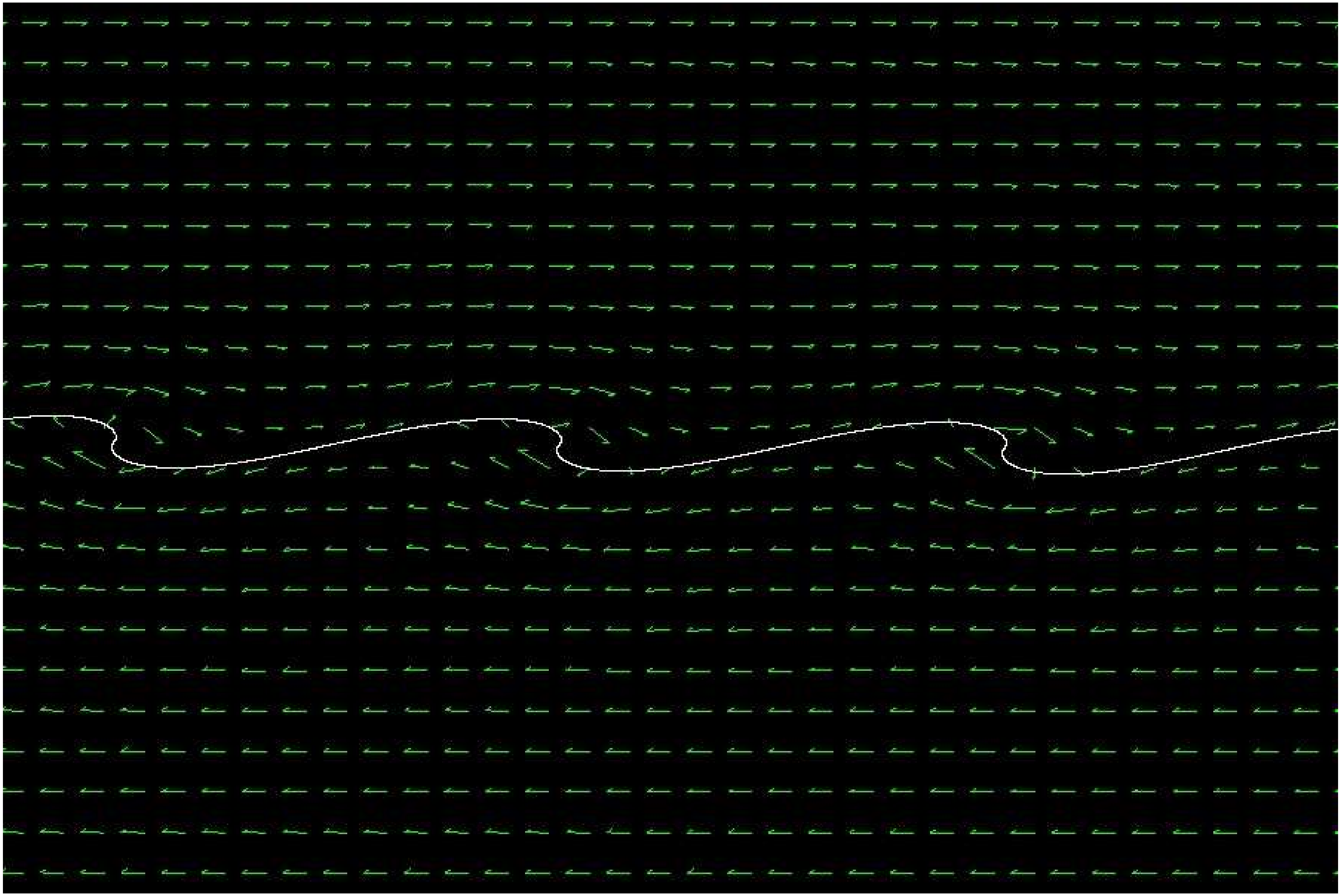


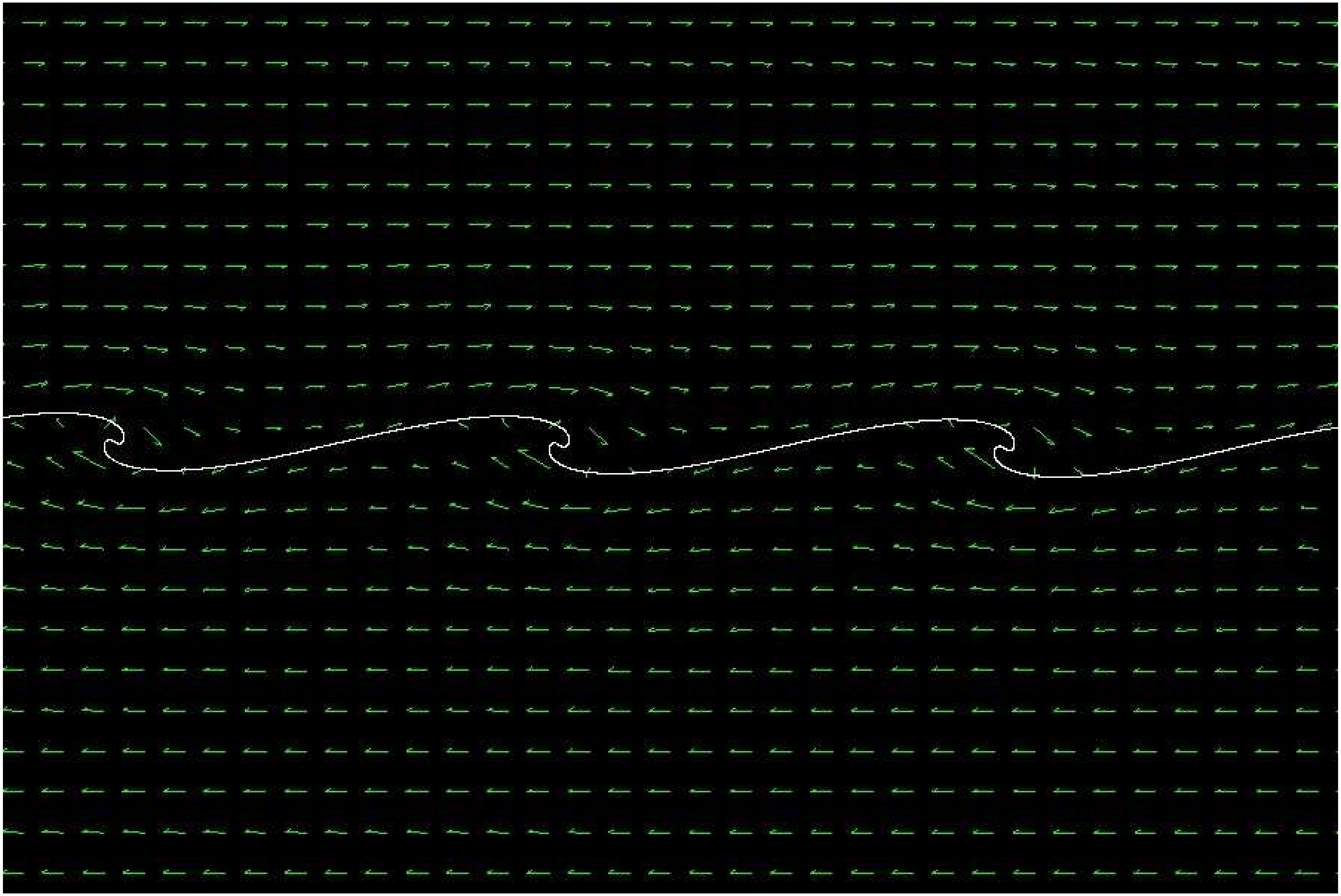


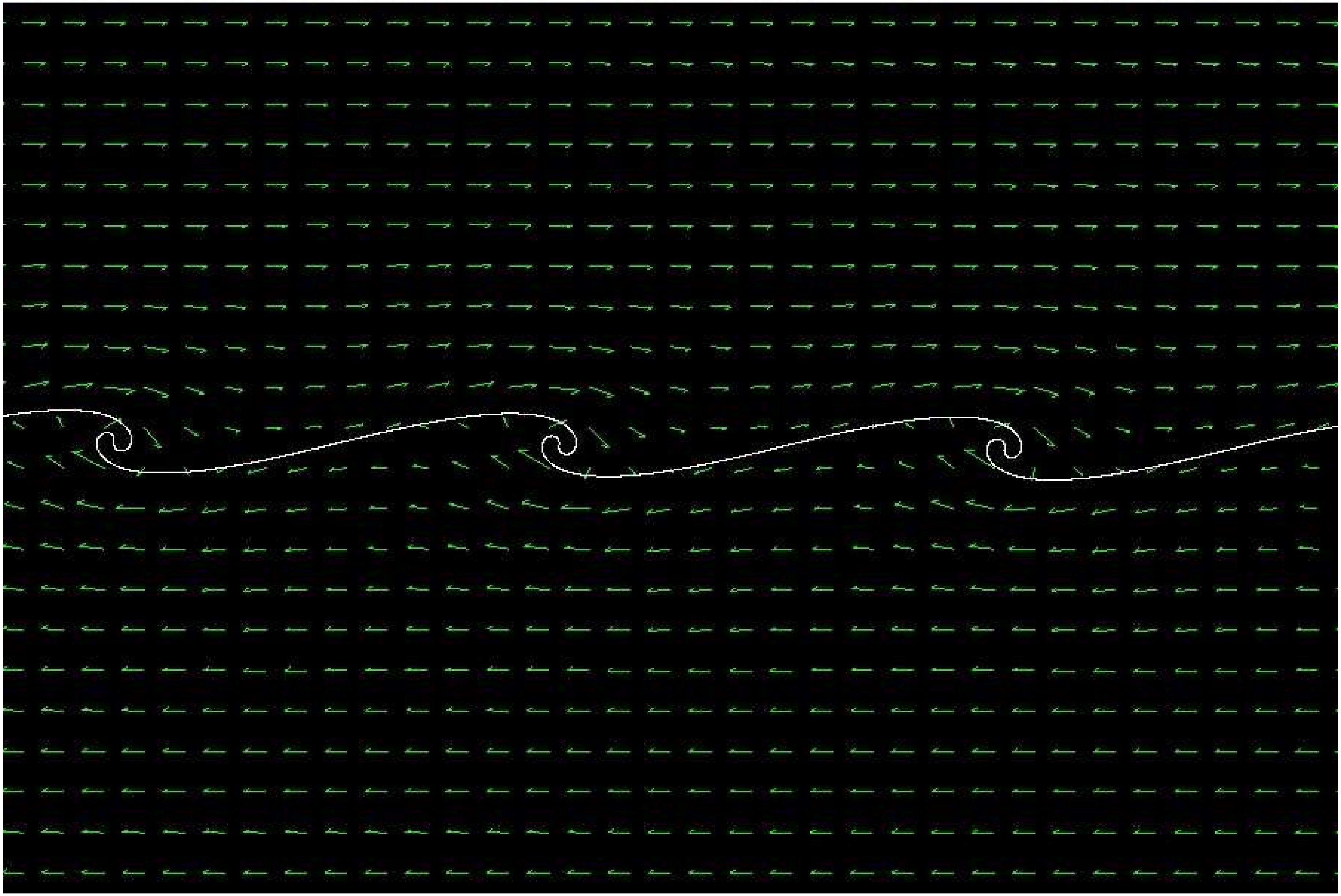


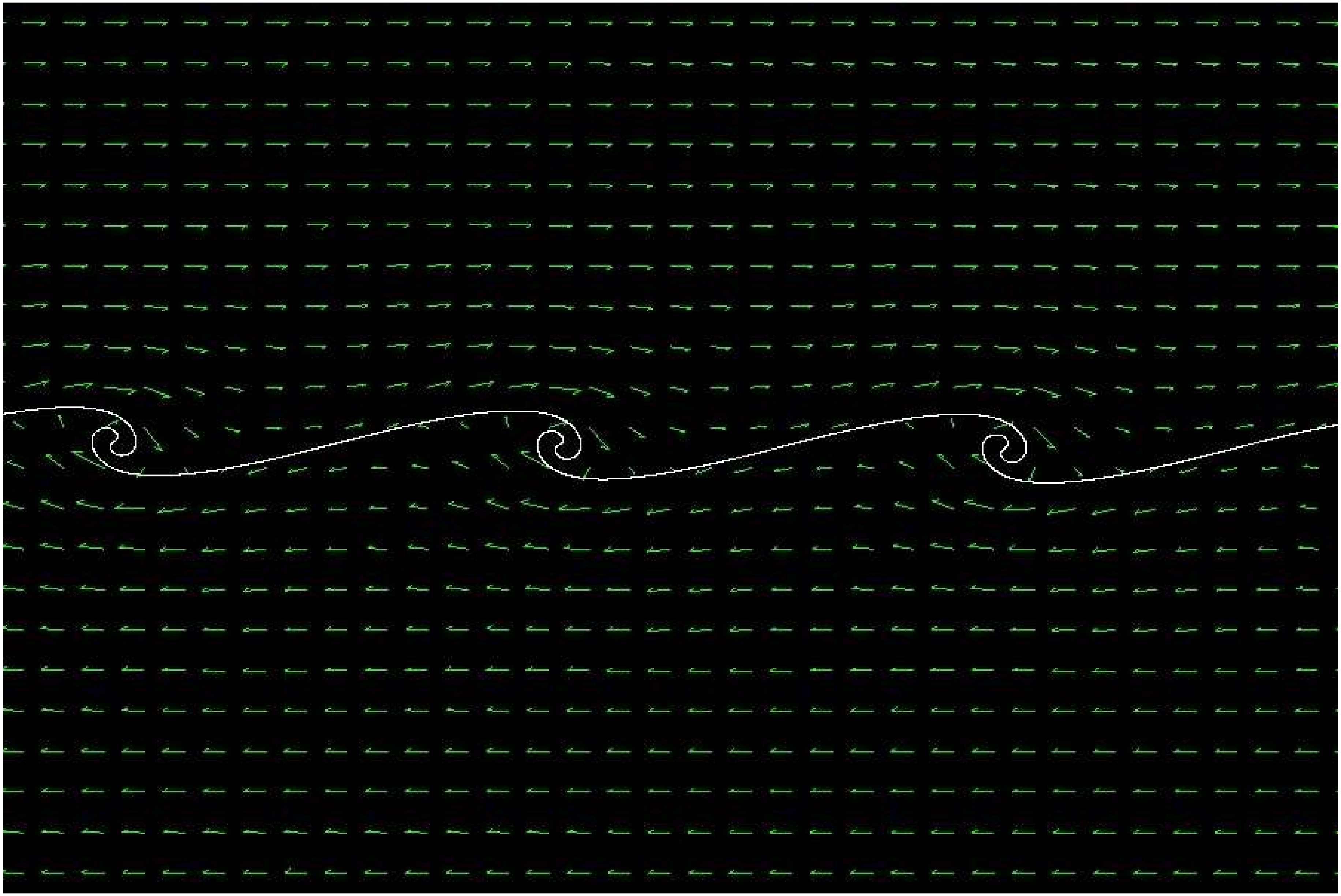


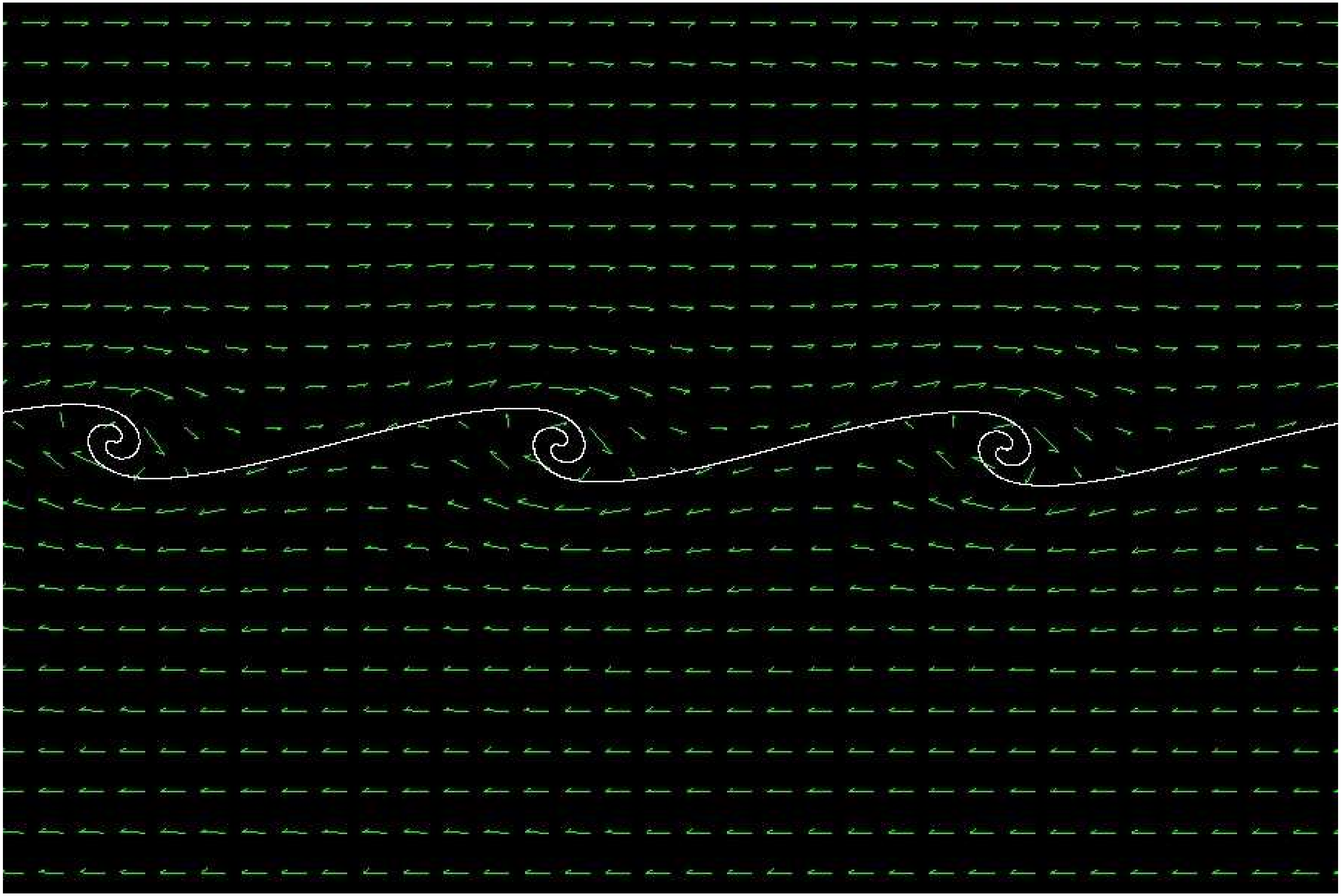


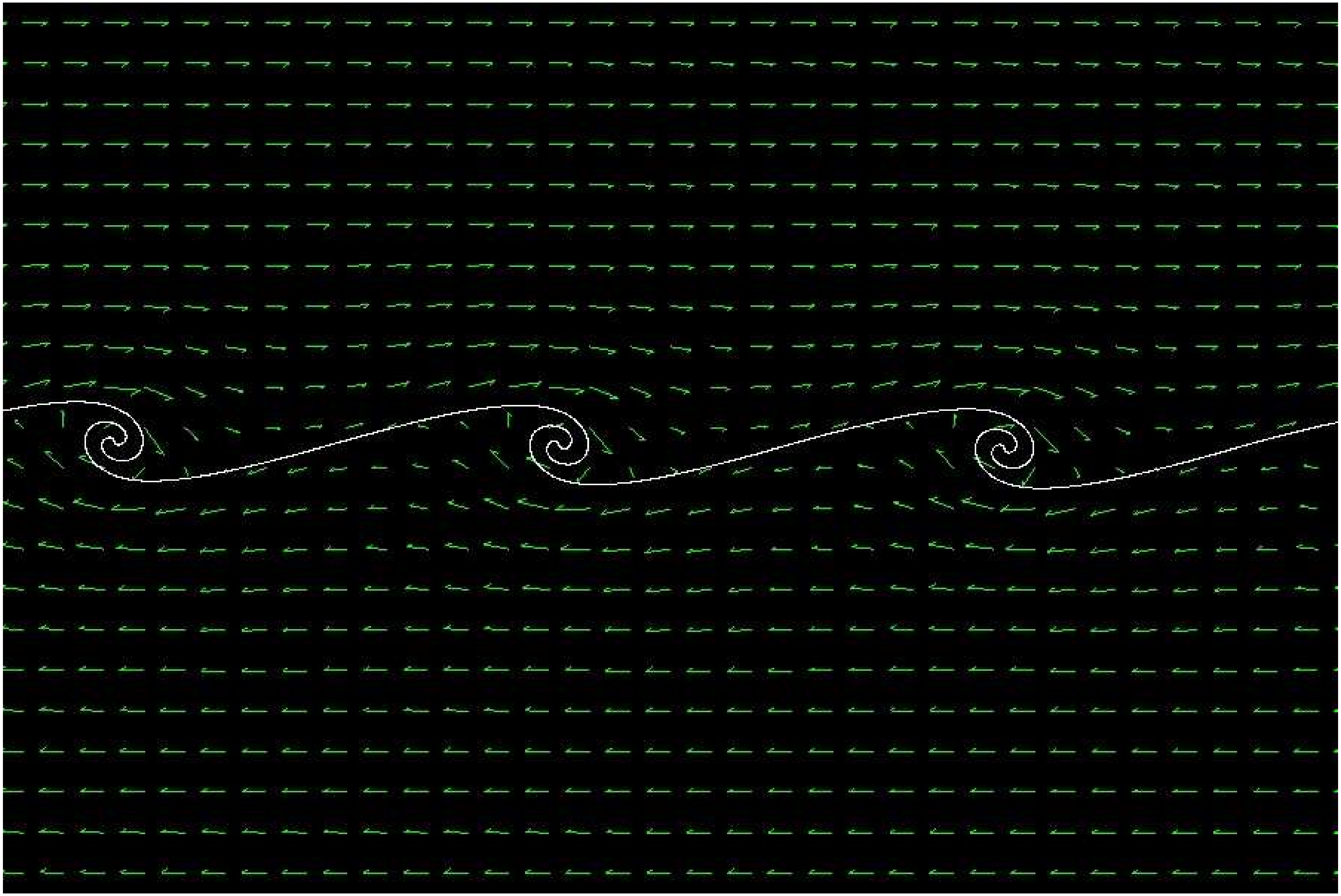


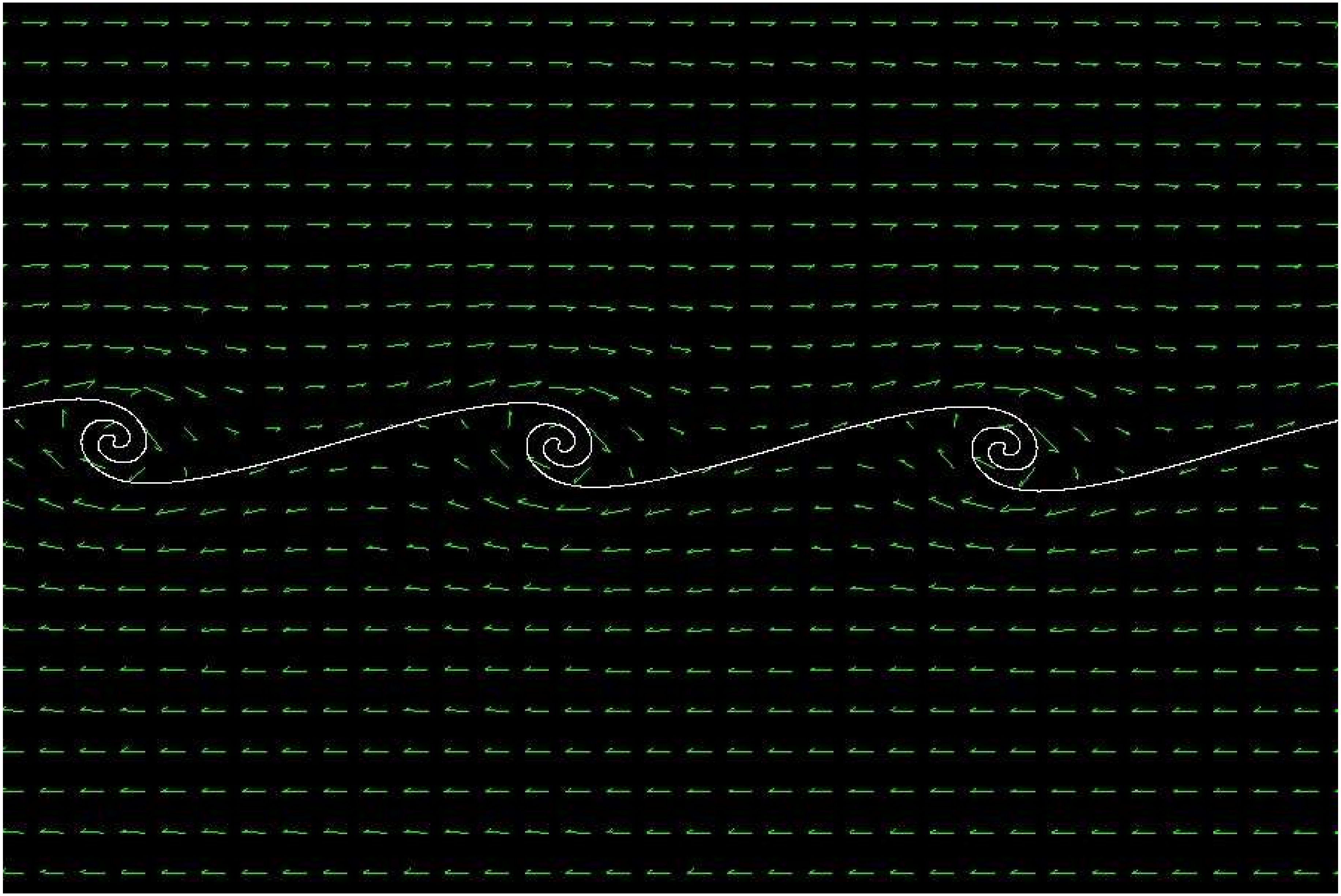


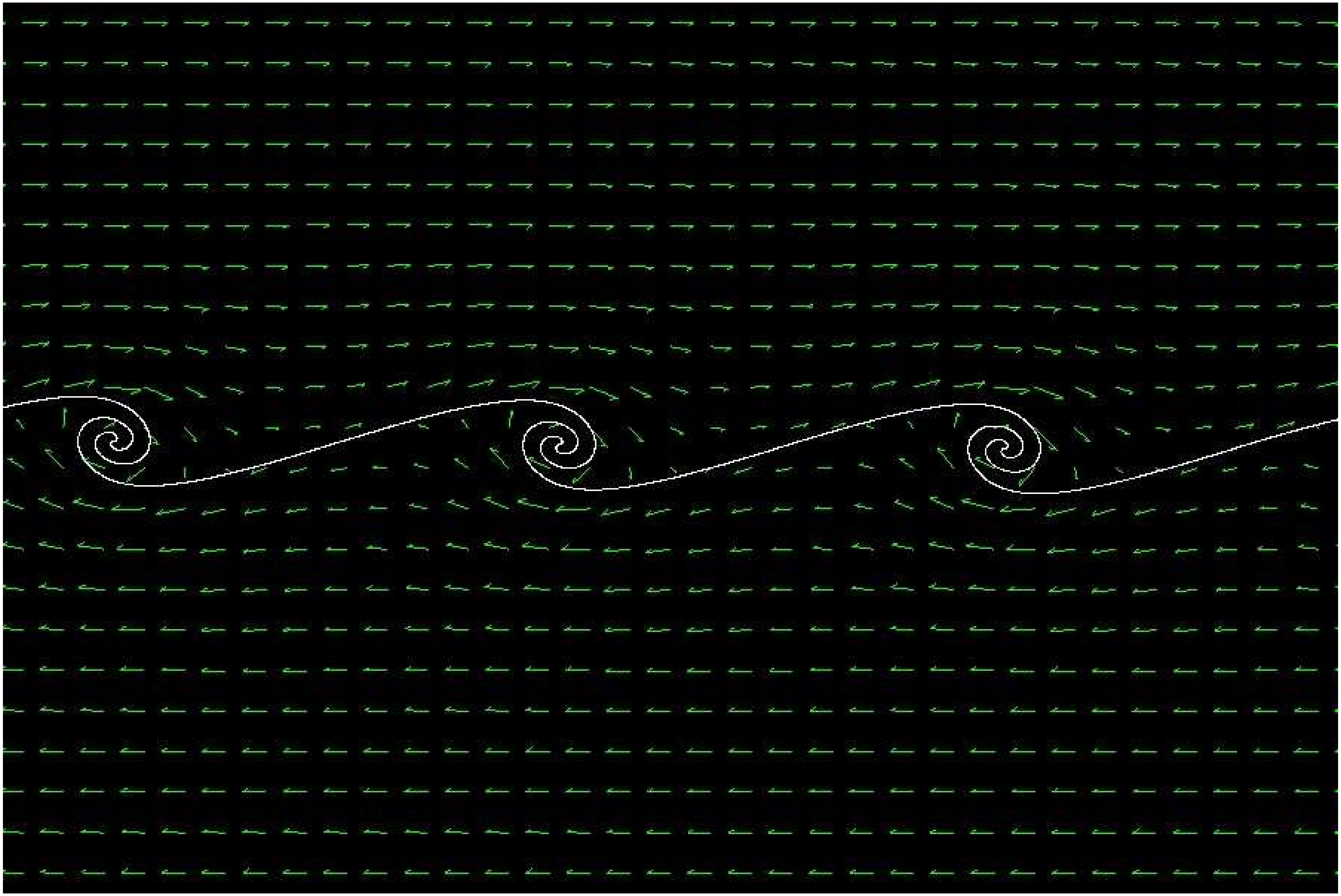


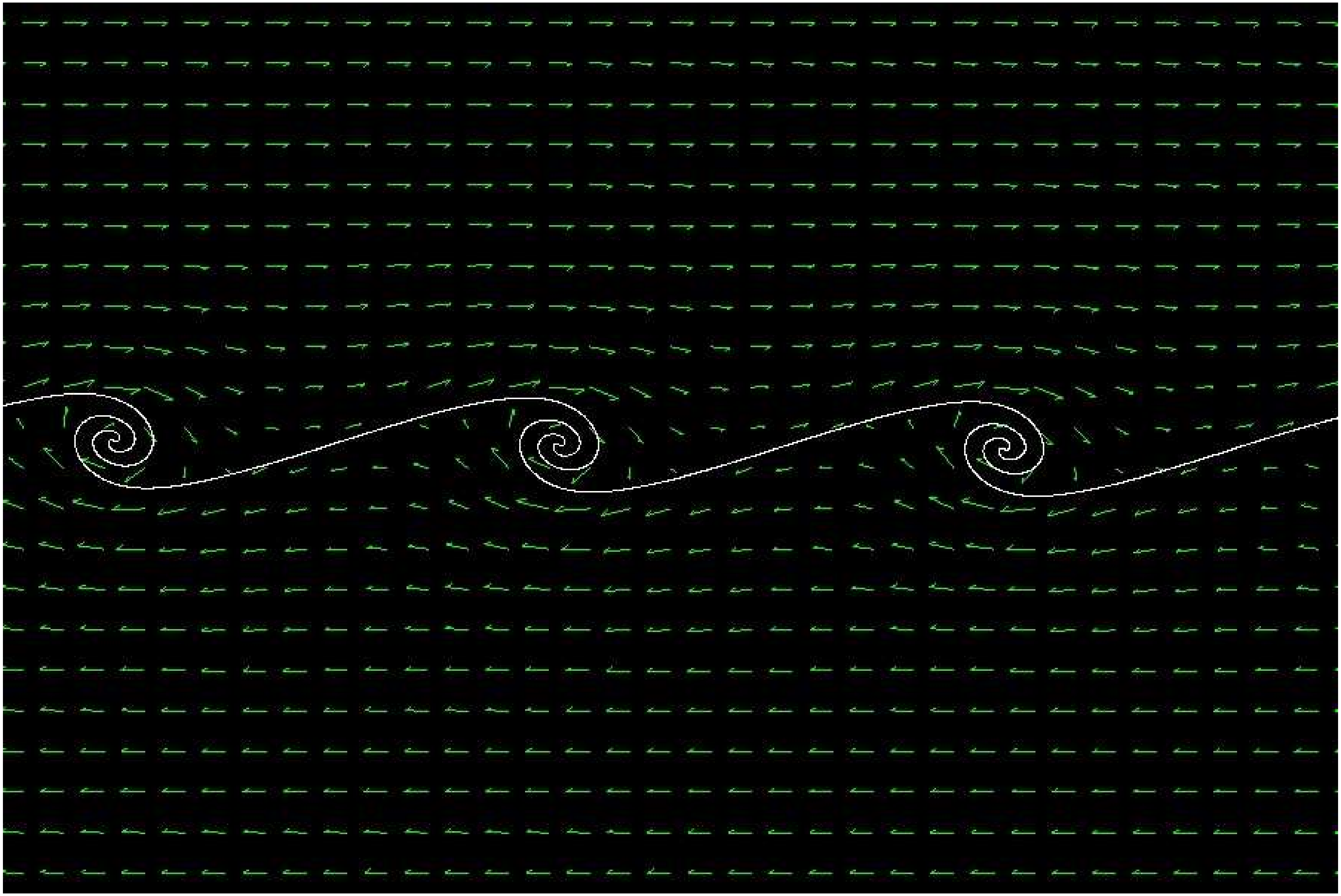


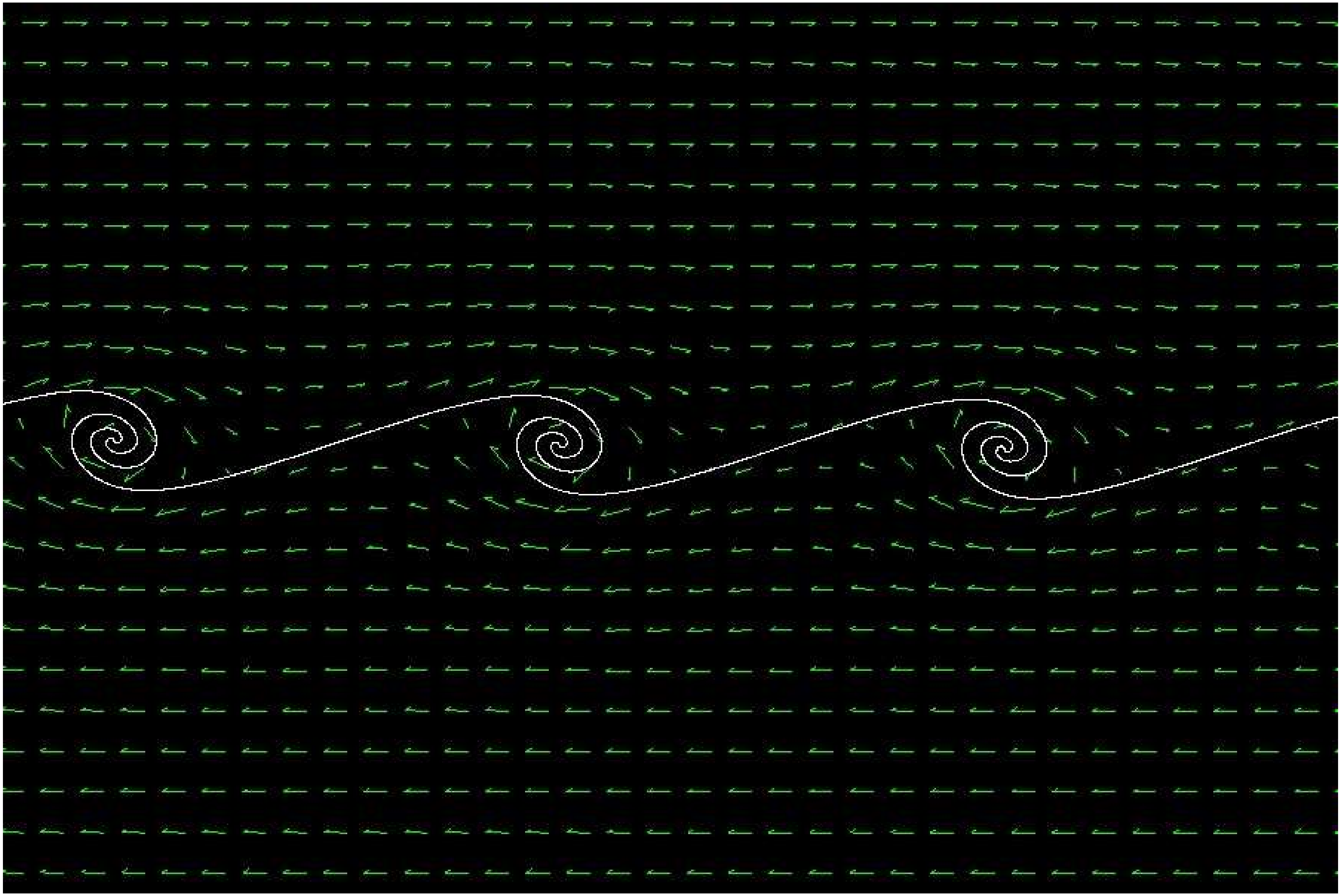


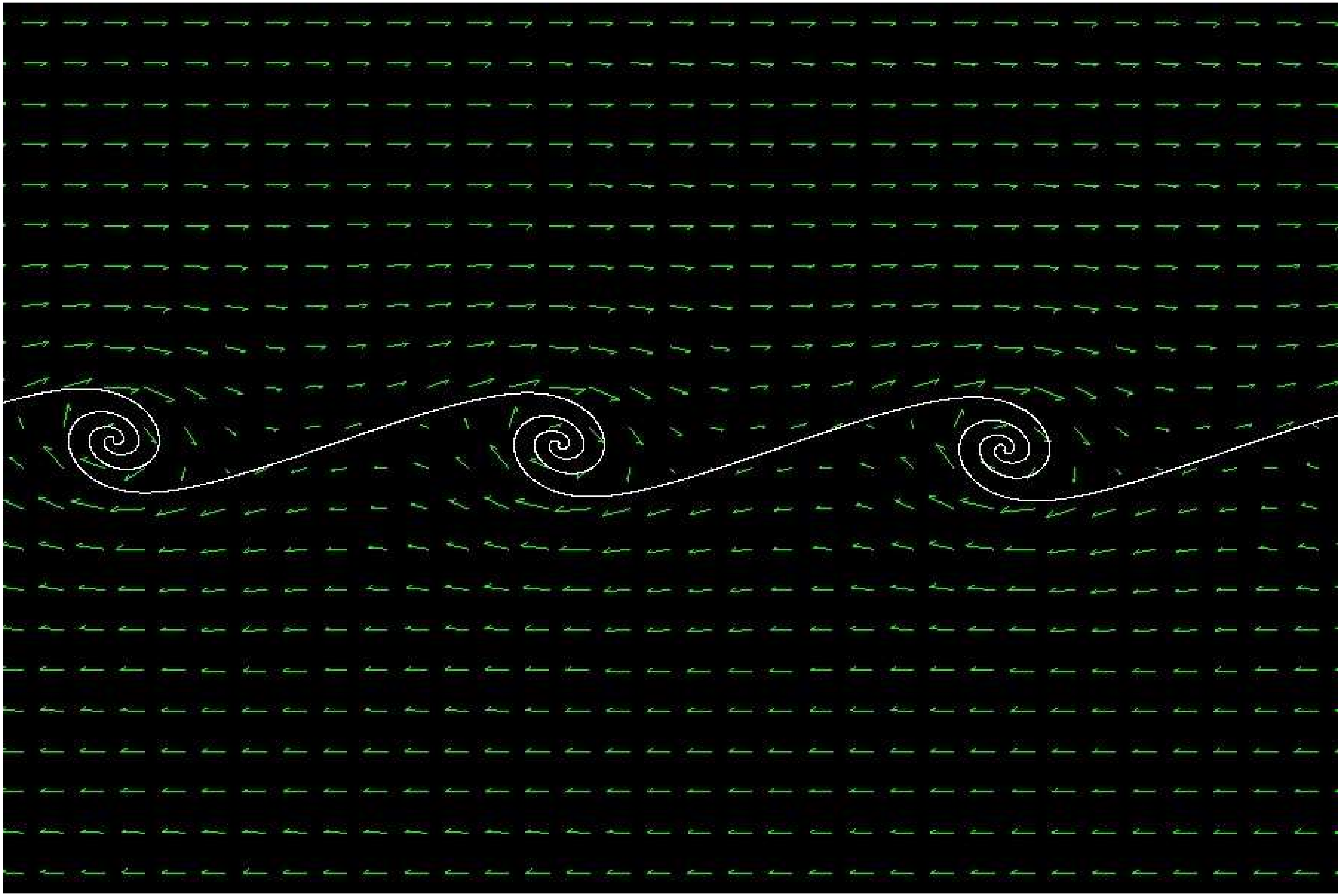


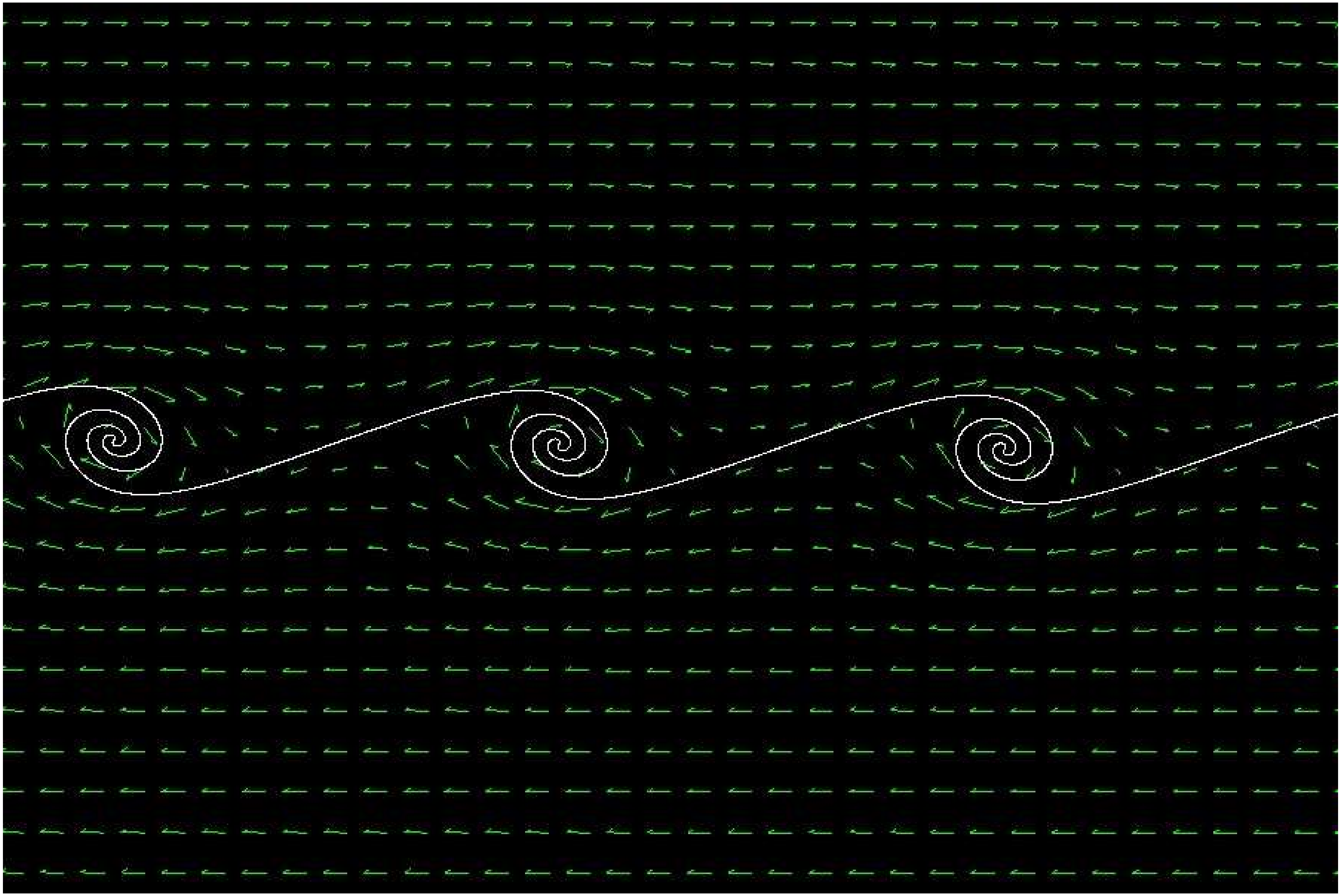


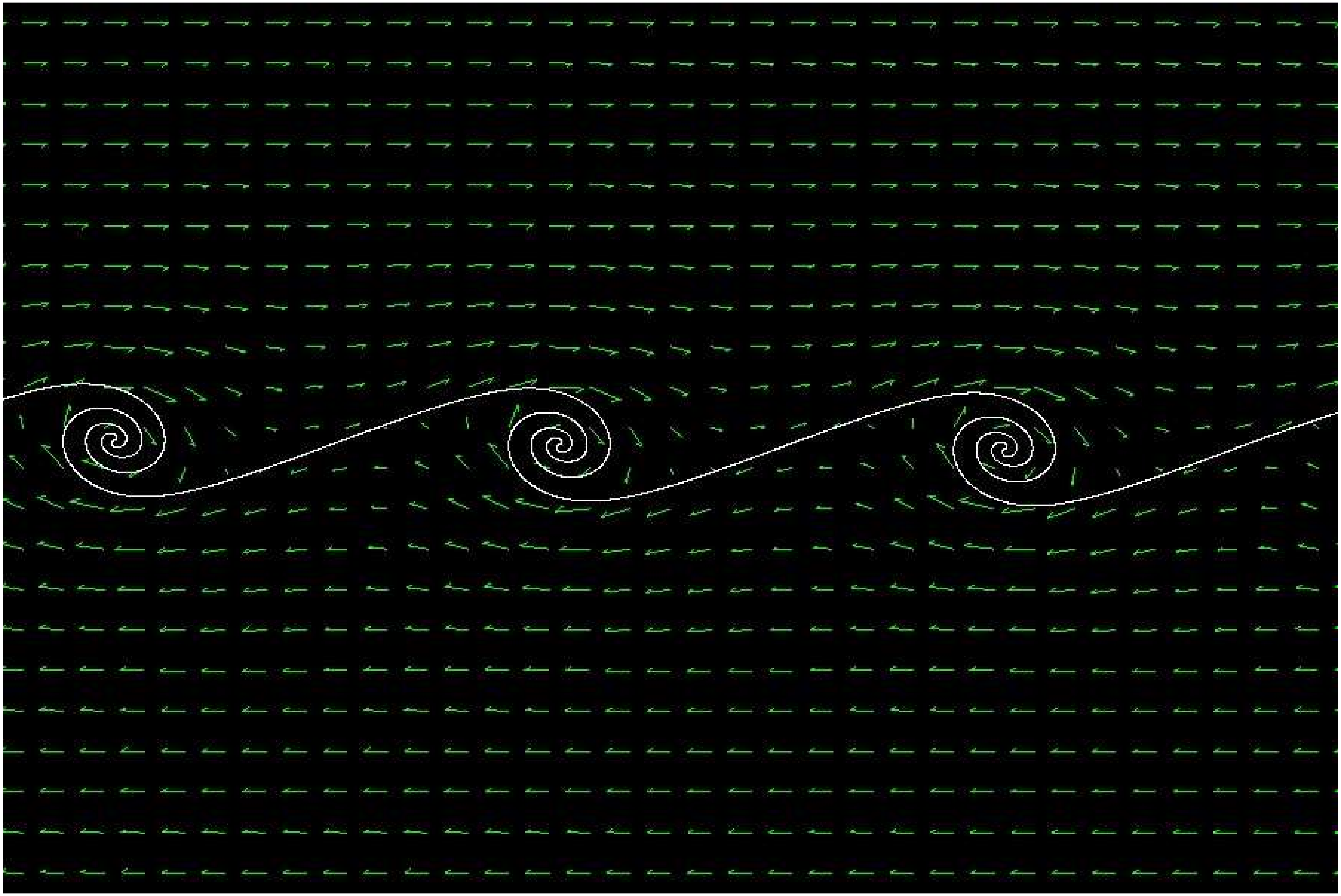


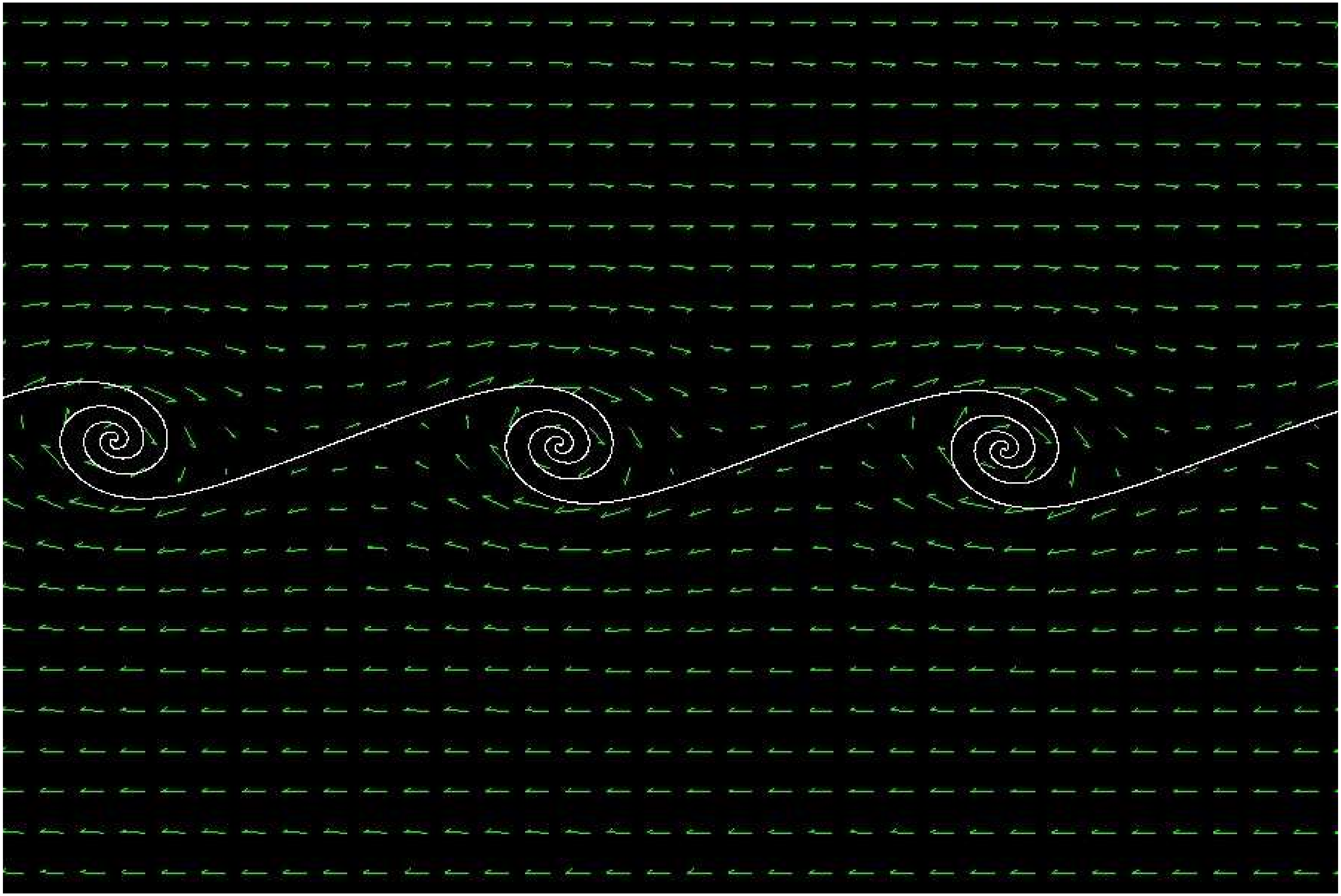


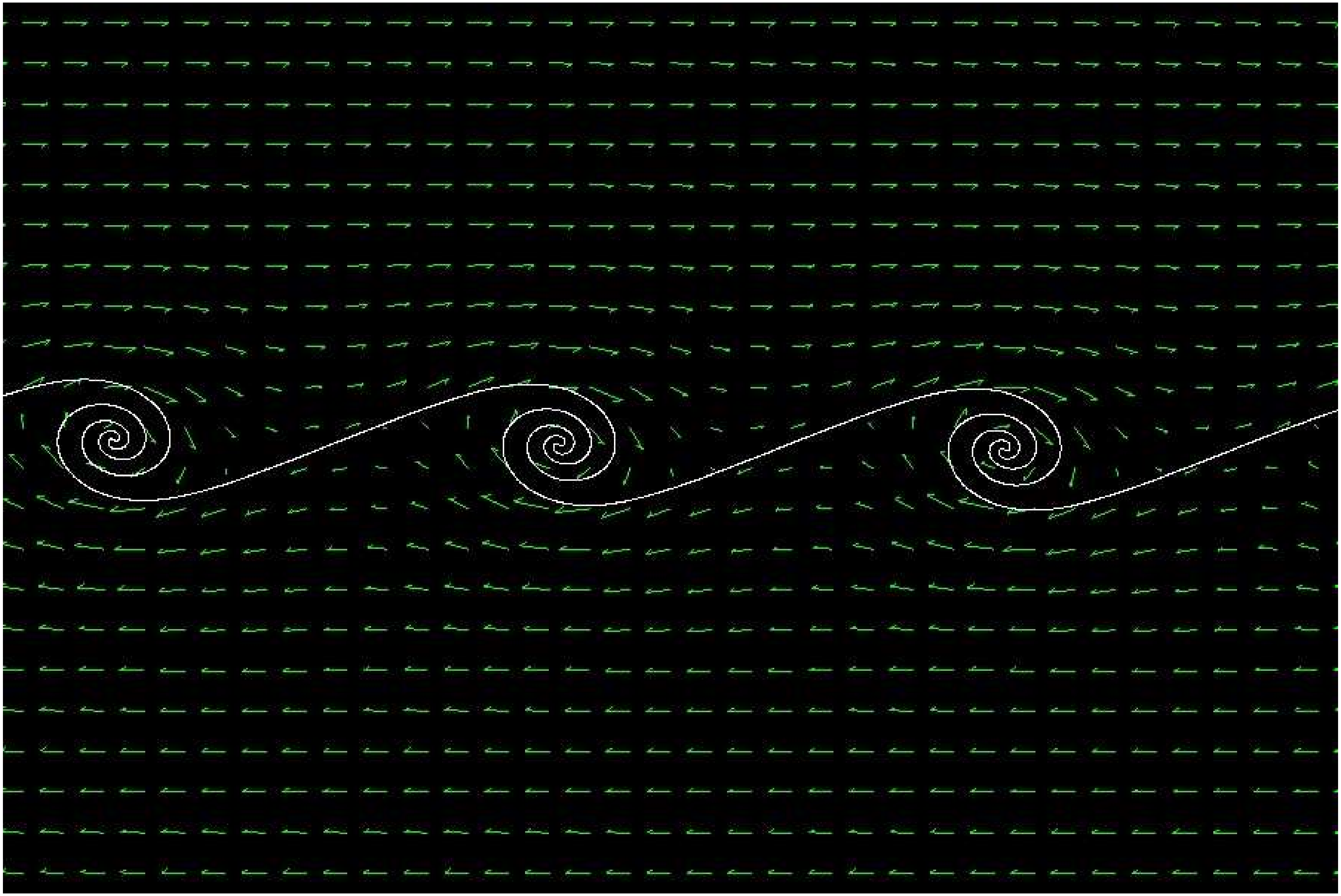


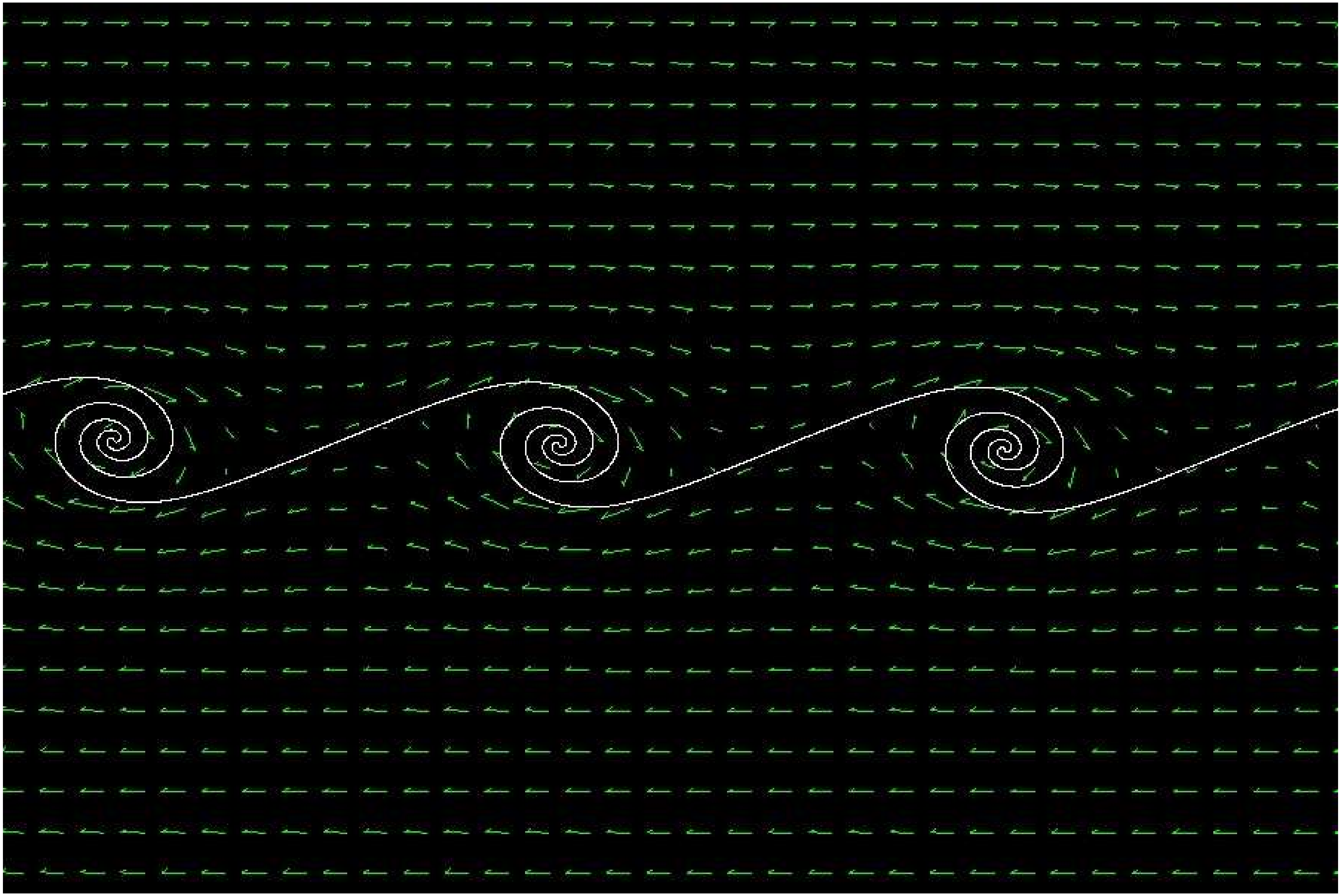


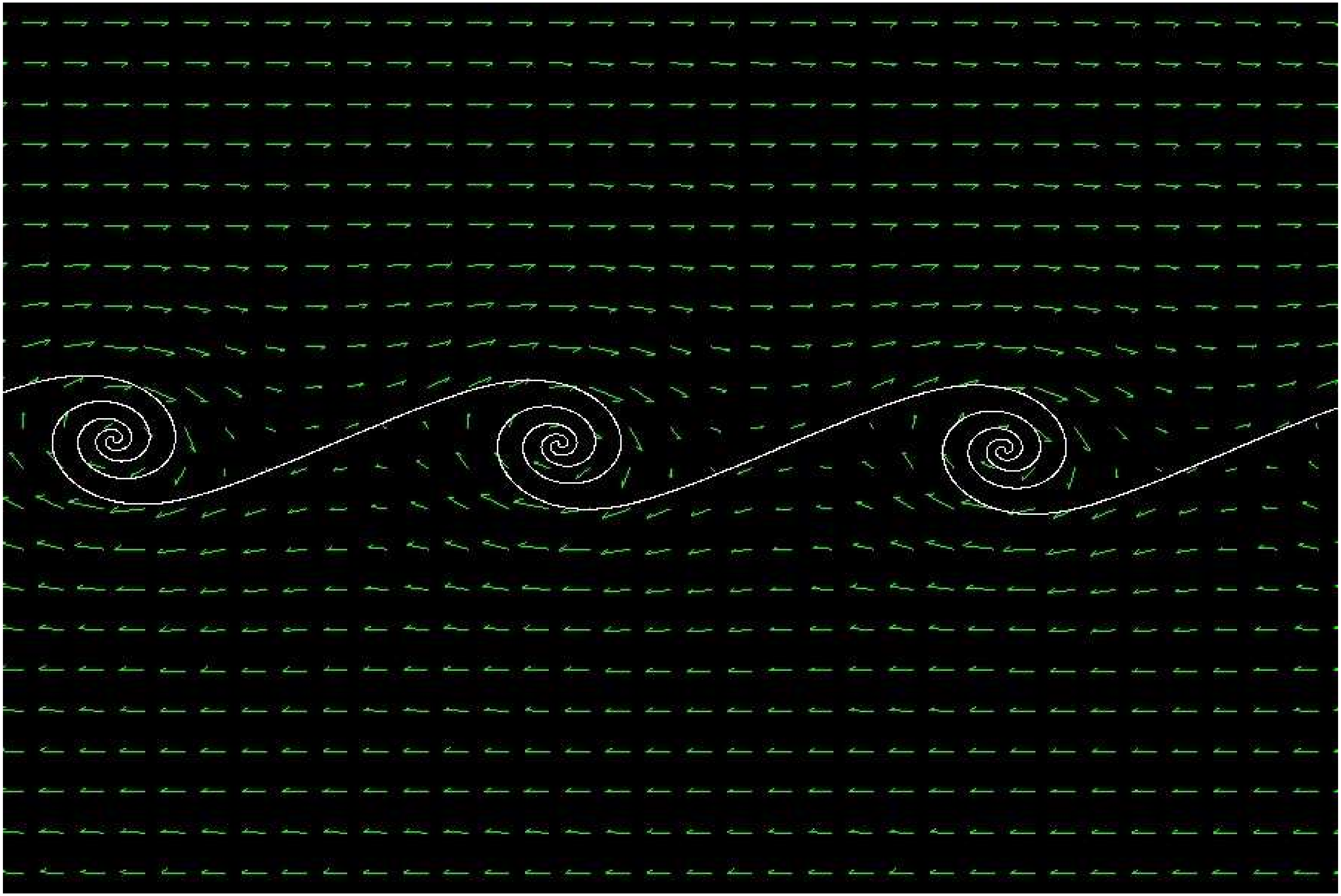


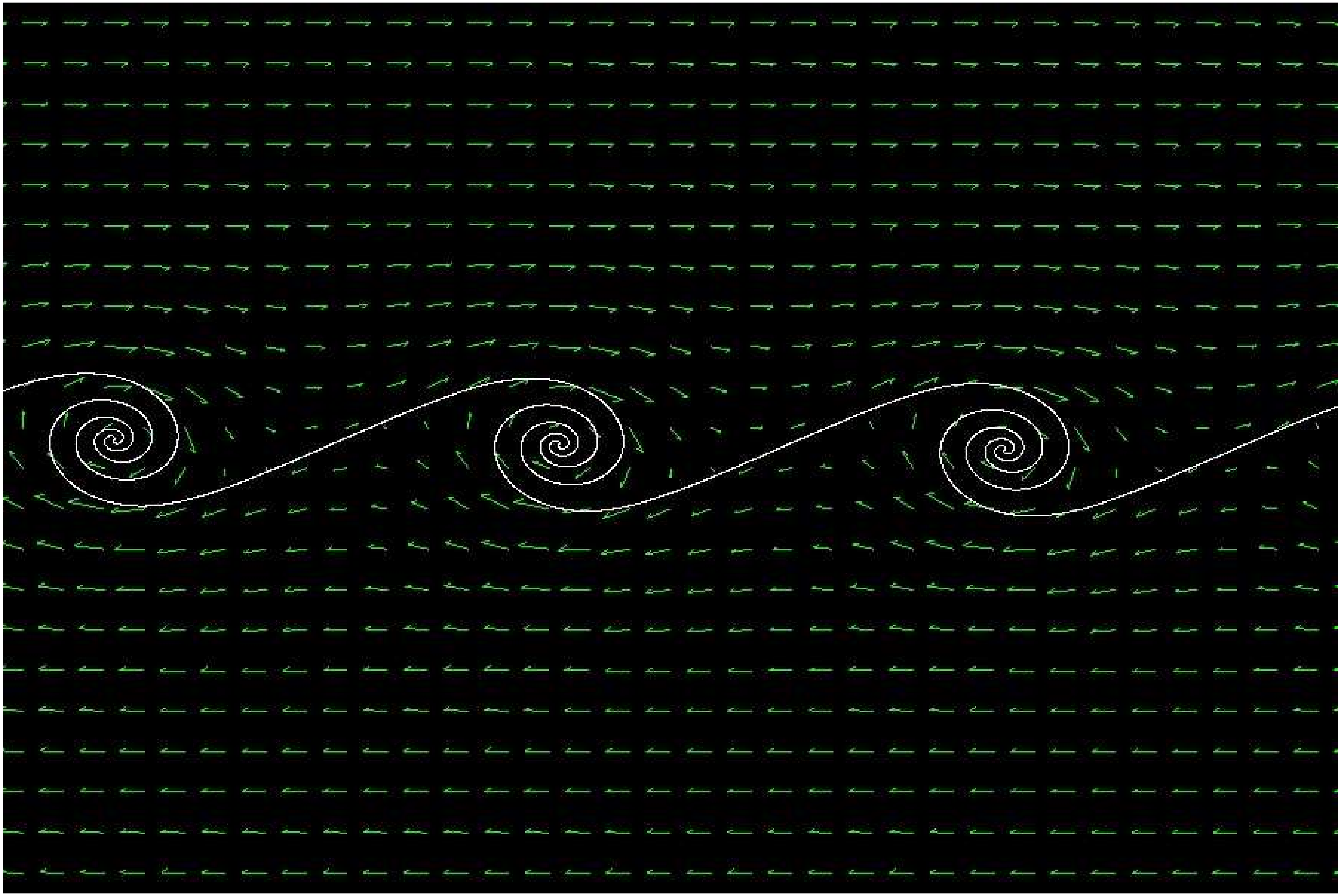


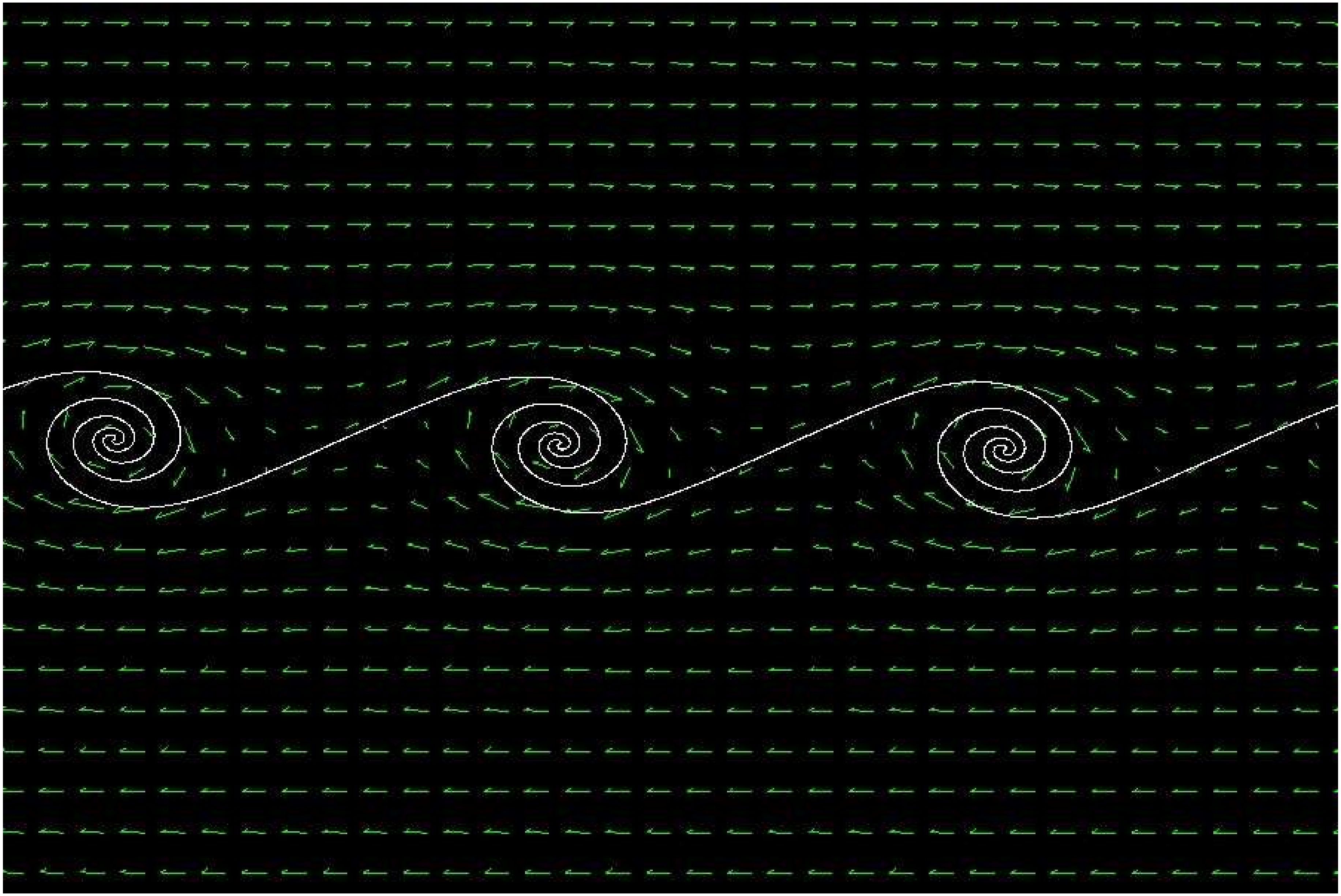


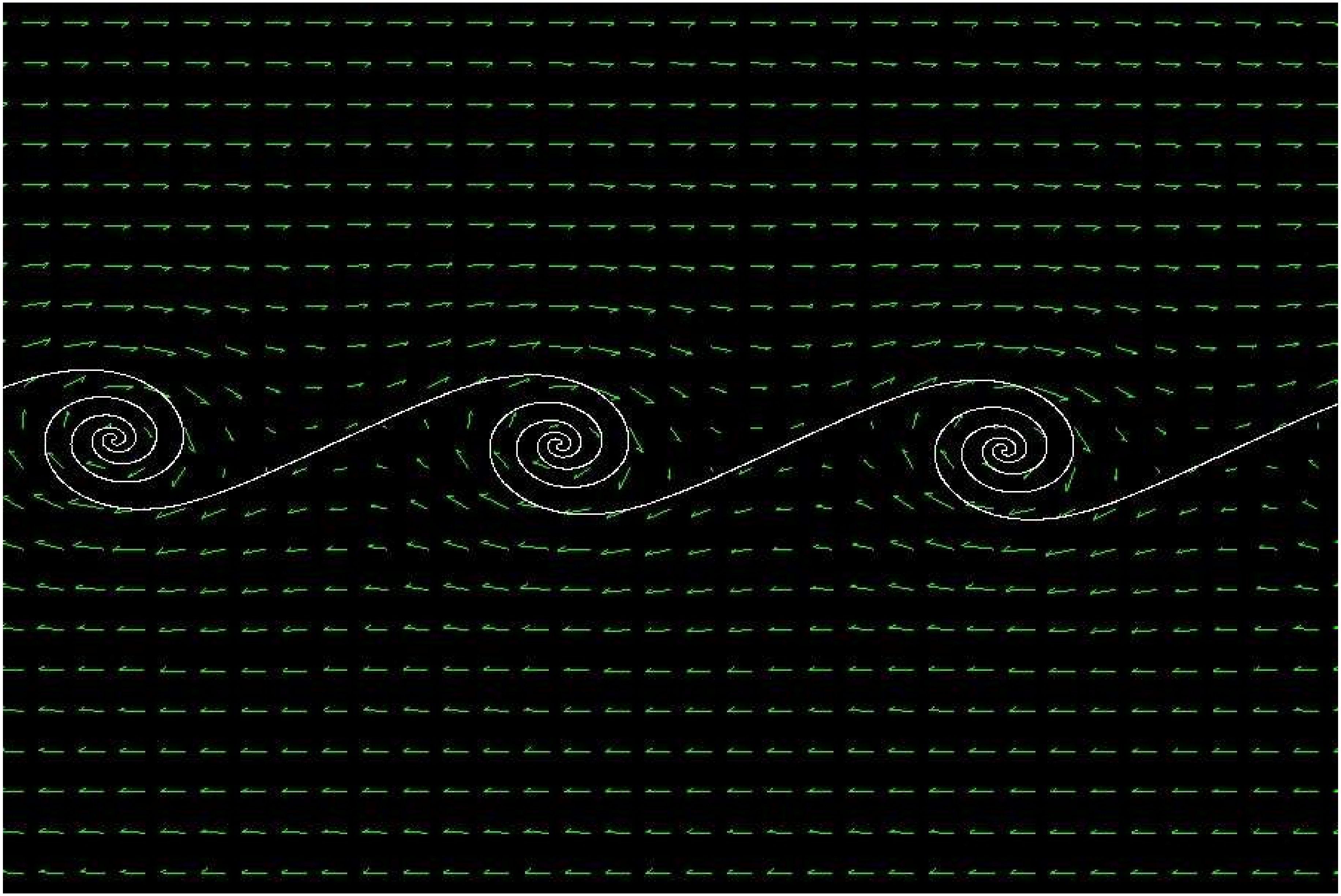


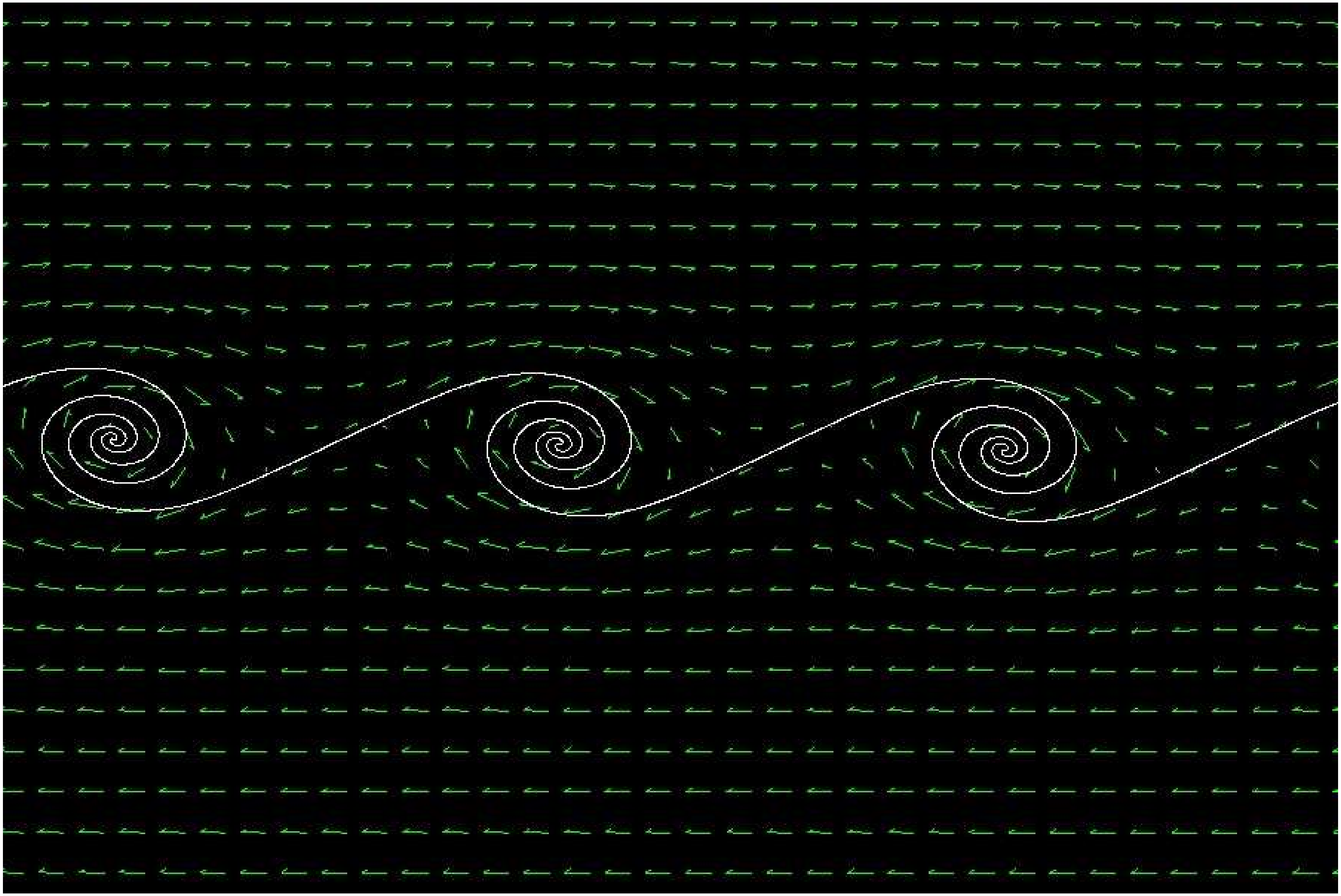


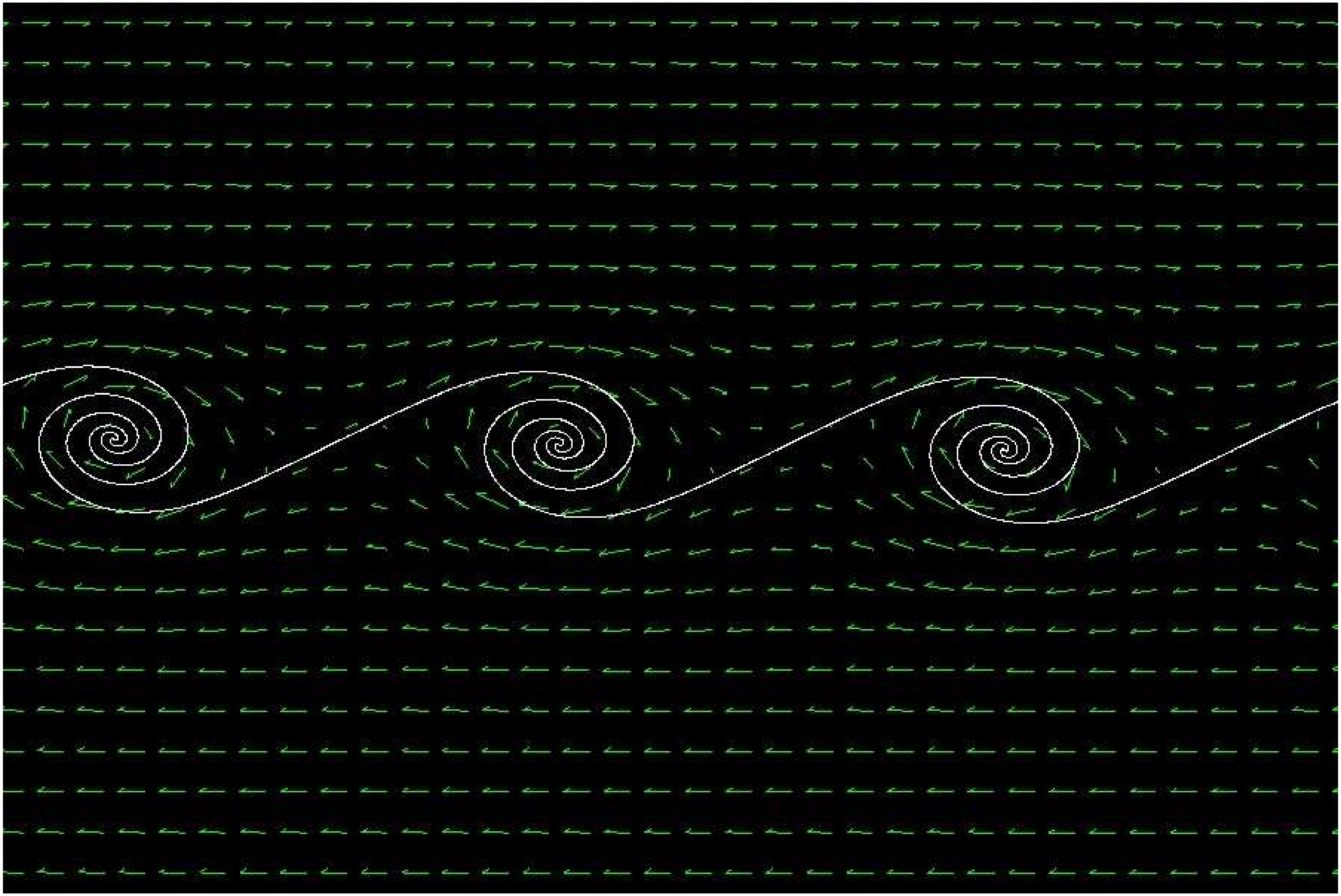




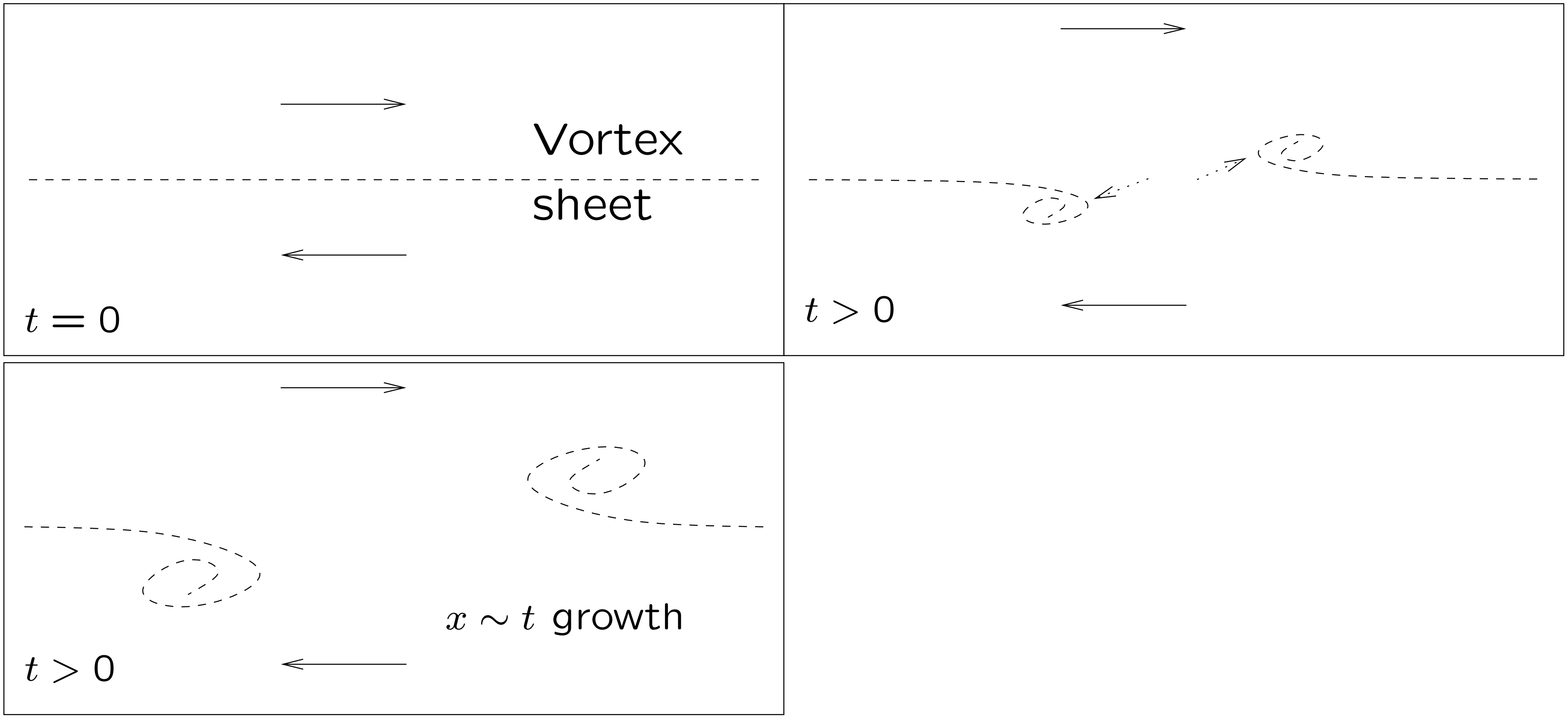








Pullin (1989) separated sheet



Self-similar:

$$v(t,x) = v(\frac{x}{t}) \quad , \quad \pi(t,x) = \pi(\frac{x}{t})$$

Self-similar vortex sheets

Let $\frac{1}{2} < \mu < \infty$ ($x \sim t^{\mu}$; similarity exponent; $\mu = 1$ most important).

$$Z(\Gamma, t) = t^{\mu} z(\gamma), \qquad \gamma = t^{-\alpha}\Gamma.$$

$\gamma \in \mathbb{R}$ is Γ at $t = 1$, and then $z(\gamma)$ is $Z(\Gamma, t)$ with $t = 1$.

$$\frac{\partial}{\partial t} Z = t^{\mu}(-\alpha) t^{-1} t^{-\alpha} \Gamma \frac{\partial}{\partial \gamma} z + \mu t^{\mu-1} z = t^{\mu-1}\left(-\alpha\gamma \frac{\partial}{\partial \gamma} z + \mu z\right),$$

$$\text{p.v.} \int \frac{d\Gamma'}{Z(\Gamma, t) - Z(\Gamma', t)} = \text{p.v.} \int \frac{t^{\alpha} d\gamma'}{t^{\mu}[z(\gamma) - z(\gamma')]}$$

Comparison: need

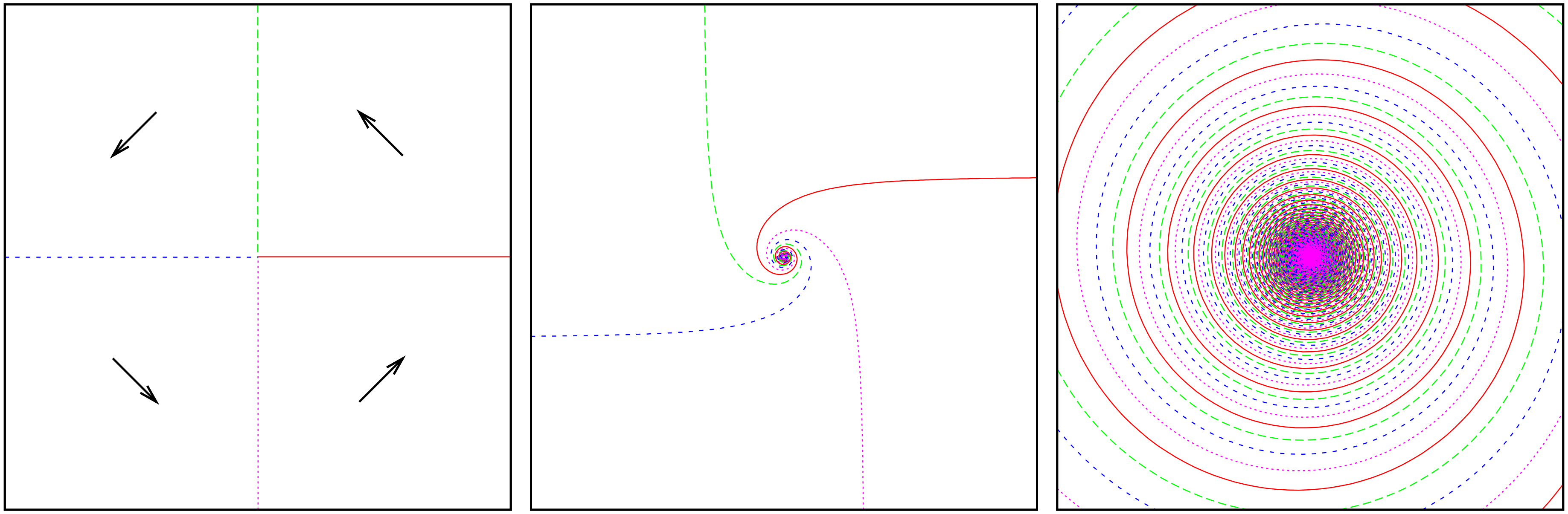
$$\mu - 1 = \alpha - \mu \qquad \Rightarrow \qquad \alpha = 2\mu - 1.$$

Self-similar Birkhoff-Rott equation:

$$(1 - 2\mu)\gamma \frac{\partial}{\partial \gamma} z + \mu z = W^* = \left(\frac{1}{2\pi i} \text{p.v.} \int_{\mathbb{R}} \frac{d\gamma'}{z(\gamma) - z(\gamma')} \right)^* .$$

ODE, integral equation $\Rightarrow$ like a boundary-value problem

Multi-branched rollup Example: $N = 4$.



Result discussed here: existence for N sufficiently large (but finite).

[E., submitted to *Arch. Rat. Mech. Anal.*]

N large: (1) N^{-1} yields a form of "smallness", (2) symmetry eliminates unpleasant terms

Key difficulty: spiral center very dense, corrections must have good decay, must not self-intersect [▷ m1N4ell.vs]

Birkhoff-Rott for multiple branches

Sheets $p = 0, ..., N-1$ at equal angles $\frac{2\pi}{N}$, $N \in \mathbb{N}$.

$z(\gamma)$ contour for sheet 0 $\rightsquigarrow$ sheet p is $u^p z(\gamma)$ with $u = \exp \frac{2\pi i}{N}$.

Redefine Γ as circulation for all branches combined
$\Rightarrow$ each individual branch generates $\frac{\Gamma}{N}$

$$(1-2\mu)\gamma \frac{\partial}{\partial \gamma} z(\gamma) + \mu z(\gamma) = W^* = \left(\frac{1}{2\pi i} \text{ p.v.} \int_{\mathbb{R}} \underbrace{\frac{1}{N} \sum_{p=0}^{N-1}}_{=: A_p} \frac{d\gamma'}{z(\gamma) - u^p z(\gamma')} \right)^*.$$

Approach: express problem as

$F(z) = 0$ $\qquad$ (F nonlinear C^1 operator; $z, F(z) \in$ Banach spaces)

Solve by quasi-Newton iteration

$$z \leftarrow K(z) := z - A^{-1}F(z)$$

A^{-1} "approximate inverse" for $F'(z)$.

$$|K(z)-K(w)| = |z-w-A^{-1}(F(z)-F(w))| = \Big|A^{-1}\Big(A(z-w)-(F(z)-F(w)\Big)\Big|$$

$$\leq |A^{-1}|\Big|A(z-w) - F'(z)(z-w) + F'(z)(z-w) - (F(z) - F(w))\Big|$$

$$\leq |A^{-1}|\Big(|A - F'(z)||z-w| + |F(w) - F(z) - F'(z)(w-z)|\Big)$$

$$= |A - F'(z)|O(z-w) + o(z-w)$$

If $A \approx F'(z)$ and $|z-w|$ small, then for some $0 \leq L < 1$

$$\leq L|z-w|,$$

Iteration map K is uniform contraction, Banach fixed point theorem: converges to fixed point $z = K(z)$ so that $F(z) = 0$.

Simpler example: steady sheet

$$0 \overset{!}{=} \partial Z_t = W^* - \text{given } W_e^*$$

$$W = \frac{1}{2\pi i}\text{p.v.}\int_{\mathbb{R}} \frac{d\Gamma'}{Z(\Gamma) - Z(\Gamma')} \overset{!}{=} W_e.$$

Linearize:

$$-\frac{1}{2\pi i}\text{p.v.}\int_{\mathbb{R}} \frac{\tilde{Z}(\Gamma) - \tilde{Z}(\Gamma')}{(Z(\Gamma) - Z(\Gamma'))^2} d\Gamma'.$$

For straight uniform sheet $Z(\Gamma) = \Gamma$:

$$-\frac{1}{2\pi i}\text{p.v.}\int_{\mathbb{R}} \frac{\tilde{Z}(\Gamma) - \tilde{Z}(\Gamma')}{(\Gamma - \Gamma')^2} d\Gamma' =: L\tilde{Z}(\Gamma)$$

Fourier transform $\tilde{Z} = \exp(i\xi\Gamma)$:

$$(L\tilde{Z})^\wedge(\xi) = C|\xi|\tilde{Z}^\wedge(\xi).$$

Nice multiplier operator $L : \dot{H}^s \to \dot{H}^{s-1}$: Hilbert transform $\circ\ \partial$.

Can invert (Fourier multiplier $|\xi|^{-1}$) modulo some polynomials.

No dynamic instabilities in steady/self-similar problem.

Self-similarity strong constraint; eliminates unstable modes.

Why are self-similar vortex sheets better behaved?

Kelvin-Helmholtz instability: analogous to inverse heat equation

$$u_t = -\Delta u$$

or Cauchy problem for Laplace equation

$$u_{tt} = -\Delta u.$$

Many Fourier modes yield exponential growth in t:

$$u^\wedge_{tt}(t,\xi) = |\xi|^2 u^\wedge(t,\xi)$$

$$\rightsquigarrow \quad u^\wedge(t,\xi) = C_+ \exp(|\xi|t) + C_- \exp(-|\xi|t)$$

Consider steady solutions: $\Delta u = 0$. No problem!

Self-similar: change coordinates from (t,x) to $(t,\frac{x}{t})$,
then ansatz $u(t,x/t) = u(x/t)$: “quasi-steady”.