Existence of algebraic vortex spirals and ill-posedness of inviscid flow (Part II)

Volker Elling

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Pullin (1989) separated sheet



Self-similar:

$$v(t,x) = v(\frac{x}{t})$$
, $\pi(t,x) = \pi(\frac{x}{t})$

"Coffee spoon" experiment



Pressure

Given smooth $v = (v^1, ..., v^d)$ with $\nabla \cdot v = 0$, how to find pressure π :

$$v_t + v \cdot \nabla v + \nabla \pi = 0 \quad ?$$

$$v_t^k + v^j \partial_j v^k + \partial_k \pi = 0$$
 $(k = 1, ..., d), \quad \sum_{k=1}^d \partial_k v^k = 0$

Take $\nabla \cdot$ of this equation, i.e. $\sum_{k=1}^{d} \partial_k$:

$$0 = \sum_{\substack{k \\ =\partial_t \nabla \cdot v = 0}} \partial_t \partial_k v^k + \sum_{j,k} \partial_k (v^j \partial_j v^k) + \underbrace{\sum_{\substack{k \\ =\Delta \pi}} \partial_k (\partial_k \pi)}_{=\Delta \pi}$$

$$0 = \sum_{j} v^{j} \partial_{j} \sum_{\substack{k \\ = \nabla \cdot v = 0}} \partial_{k} v^{k} + \sum_{j,k} \partial_{k} v^{j} \partial_{j} v^{k} + \Delta \pi$$

$$0 = \operatorname{tr}\left((\nabla v)^2\right) + \Delta \pi$$

→ have to solve a Poisson problem

Vorticity formulation: (assuming smooth flow)

$$\frac{\partial}{\partial t}\vec{v} + \vec{v} \cdot \nabla \vec{v} + \nabla \pi = 0, \qquad \nabla \cdot \vec{v} = 0$$

 $v_t^1 + v^1 v_1^1 + v^2 v_2^1 + \pi_1 = 0 \quad , \quad v_t^2 + v^1 v_1^2 + v^2 v_2^2 + \pi_2 = 0 \quad , \quad v_1^1 + v_2^2 = 0$

$$0 = v_{2t}^{1} + v_{2}^{1}v_{1}^{1} + v^{1}v_{12}^{1} + v_{2}^{2}v_{2}^{1} + v^{2}v_{22}^{1} + \pi_{12}$$

$$- v_{1t}^{2} - v_{1}^{1}v_{1}^{2} - v^{1}v_{11}^{2} - v_{1}^{2}v_{2}^{2} - v^{2}v_{12}^{2} - \pi_{12}$$

$$= (v_{2}^{1} - v_{1}^{2})_{t} + v_{1}^{1}(v_{2}^{1} - v_{1}^{2}) + v^{1}(v_{12}^{1} - v_{11}^{2}) + v_{2}^{2}(v_{2}^{1} - v_{1}^{2}) + v^{2}(v_{22}^{1} - v_{12}^{2})$$

$$= (v_{2}^{1} - v_{1}^{2})_{t} + (v_{1}^{1} + v_{2}^{2})(v_{2}^{1} - v_{1}^{2}) + v^{1}(v_{2}^{1} - v_{1}^{2})_{1} + v^{2}(v_{2}^{1} - v_{12}^{2})$$

$$= \omega_{t} + v^{1}\omega_{1} + v^{2}\omega_{2} = \omega_{t} + v \cdot \nabla\omega$$

 ω constant along streamlines $((t, \vec{x})$ integral curves of $(1, \vec{v}))$.

 $\omega = 0$ at $t = 0 \Rightarrow 0 = \omega = \nabla \times \vec{v}$ for all $t \ge 0$

Then $\vec{v} = \nabla \phi$ for scalar ϕ , and

$$\Delta \phi = \nabla \cdot \vec{v} = 0.$$

Complex variables

Assume $\omega = 0$ in some region.

v 2-d harmonic \leadsto more conveniently discussed with complex analysis

$$0 = \omega = \frac{\partial}{\partial x} v^y - \frac{\partial}{\partial y} v^x \tag{1}$$

$$0 = \nabla \cdot \vec{v} = \frac{\partial}{\partial x} v^x + \frac{\partial}{\partial y} v^y \tag{2}$$

With

Z = x + iy,

Cauchy-Riemann equations for complex velocity

$$W = v^x - iv^y.$$

Seek holomorphic $Z \mapsto W(Z)$.

Given W holomorphic in Ω , v = (ReW, -ImW) is an irrotational incompressible Euler solution in Ω (with appropriate π).

Force on aircraft wing: Laurent series



$$(F^x, F^y) = \int_{\text{body surface}} \pi \ \vec{n}_{\text{inner}} \ ds, \qquad F^x - iF^y = -\frac{i\varrho}{2} \Gamma W_{\infty}.$$

Kutta-Joukowski: choose unique Γ so that W = 0 at trailing edge. Result: approximation for correct lift (force perpendicular to \vec{v}_{∞}).

d'Alembert paradox: no drag (force in \vec{v}_{∞} direction). Drag generation very complex, requires some viscosity, boundary layer separation, vortex shedding, carrying energy away in wake.

d'Alembert paradox

Zero ω , zero Γ (symmetry): no force on body





 \Rightarrow vorticity ω crucial for most basic questions, such as lift and drag

Point vortices

$$W(Z) = \frac{\Gamma}{2\pi i (Z - Z')}$$

 $\sim \frac{1}{r}$ velocity field induced by vortex of circulation Γ located in Z'.

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$$\Gamma = \oint_C v \cdot ds$$

where C is any simple contour passing around Z' counterclock-wise.

For k = 1, ..., N:

$$\frac{\partial}{\partial t}Z_n = \partial_t (x_n + iy_n) = (v_n^x - iv_n^y)^* = \left(\sum_{k \neq n} \frac{\Gamma_k}{2\pi i(Z_n - Z_k)}\right)^*$$

Like gravitational N body problem, but "forces" angular, not radial.

Vortex sheets: ω concentrated on smooth curve.

Consider composed of small vortices $d\Gamma'$.

$$\frac{\partial}{\partial t}Z_n(t) = \left(\sum_{k \neq n} \frac{\Gamma_k}{2\pi i (Z_n(t) - Z_k(t))}\right)^* \qquad (k, n = 1, ..., N)$$

→ Birkhoff-Rott equation:

$$\frac{\partial}{\partial t}Z(\Gamma,t) = W^* := \left(p.v.\int_{-\infty}^{\infty} \frac{d\Gamma'}{2\pi i(Z(\Gamma,t) - Z(\Gamma',t))}\right)$$
where p.v. $\int := \lim_{\epsilon \downarrow 0} \int_{\mathbb{R} \setminus [\Gamma - \epsilon, \Gamma + \epsilon]}$.
$$Z(\Gamma_1,t)$$
Curve
$$\Gamma \mapsto Z(\Gamma,t)$$
Closed simple
contour C (c.c.w.)

 $\int_C v \cdot ds = 2\pi(\Gamma_2 - \Gamma_1)$

 $t \mapsto Z(t, \Gamma)$ for fixed Γ is supposed to follow particle paths (arithmetic average of v on each side).

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Birkhoff-Rott \leftrightarrow incompressible Euler

For sufficiently smooth Z (neglect question of $\Gamma \rightarrow \pm \infty$ here): define v off-sheet by

$$W(x+iy) = v^{x} - iv^{y} = \int_{G} \frac{d\Gamma'}{x+iy-Z(t,\Gamma')} \quad (x+iy \notin Z(t,G)),$$

then v is weak solution of incompressible Euler equations

$$v_t + \nabla \cdot (v \otimes v) + \nabla \pi = 0$$
, $\nabla \cdot v = 0$

I.e. for test functions ϕ ,

$$\int \int \vec{v} \,\partial_t \phi + (\vec{v} \cdot \nabla \phi)\vec{v} + \pi \nabla \cdot \vec{v} \,dx \,dt = 0 \quad , \quad \int \nabla \phi \cdot v \,dx = 0$$

Proof:

1. off-sheet: v, π smooth irrotational incompressible Euler solution.

- 2. on sheet: verify Rankine-Hugoniot conditions.
- 3. in singular points (spiral centers): v does not blow up too fast.

Jump conditions for incompressible Euler discontinuities Generally, in any point on discontinuity with speed σ and normal n:

 $0 = U_t + \nabla \cdot f(U) \quad \rightsquigarrow \quad \sigma[U] = [f(U)] \cdot n, \quad [A] = \text{jump of } A.$

(Integrate over volume and consider boundary integrals.)

For incompressible Euler: $\nabla \cdot v = 0 \rightsquigarrow$

$$\mathbf{0} = [v] \cdot n = [v \cdot n]$$

$$v_t + \nabla \cdot (v \otimes v) + \nabla \pi = 0 \rightsquigarrow$$

 $\sigma[v] = [(v \cdot n)v] + [\pi]n$ (several components)



$$n \rightarrow \sigma[v \cdot n] = [(v \cdot n)^2] + [\pi] \Rightarrow [\pi] = 0$$

When solving $\Delta \pi = -\operatorname{tr}((\nabla v)^2)$ on each side, impose " π continuous" as coupling condition on sheet.

$$t \cdot \Rightarrow \sigma[v \cdot t] = [(v \cdot n)(v \cdot t)] = [v \cdot t](v \cdot n) \Rightarrow \sigma = v_{+} \cdot n = v_{-} \cdot n$$

(unless $[v \cdot t] = 0$ which means no jumps in any variable).

Principal-value integrals: If R(s) may have singularity at s = 0, a < 0 < b:

p.v.
$$\int_{a}^{b} R(s) ds := \lim_{\epsilon \downarrow 0} \left(\int_{a}^{-\epsilon} + \int_{\epsilon}^{b} \right) R(s) ds$$

 $\mathbb{R} \supset G \ni \Gamma \mapsto Z(\Gamma) \in \mathbb{C}$ injective $(Z_{\Gamma} \neq 0)$, real-analytic \rightsquigarrow holomorphic extension to neighbourhood U of \mathbb{R} in \mathbb{C} .



 \overline{C}_{-} radius ϵ upper half-circle around Γ , C_{+} lower half-circle. No \int change when shifting contours without crossing Γ !

$$\frac{1}{Z(\Gamma) - Z(\Gamma')} = \frac{1}{Z_{\Gamma}(\Gamma)(\Gamma - \Gamma') - \frac{1}{2}Z_{\Gamma\Gamma}(\Gamma)(\Gamma - \Gamma')^{2} + \dots}$$
$$= \frac{1}{Z_{\Gamma}(\Gamma)} \cdot \frac{1}{\Gamma - \Gamma'} \cdot \frac{1}{1 + O(\Gamma - \Gamma')}$$
$$|\Gamma - \frac{\Gamma'}{Z_{\Gamma}(\Gamma)} \cdot \frac{1}{\Gamma - \Gamma'} \cdot \left(1 + O(\Gamma - \Gamma')\right) = \frac{1}{Z_{\Gamma}(\Gamma)} \cdot \frac{1}{\Gamma - \Gamma'} + O(1)$$

Simple pole in principal-value integral

$$\int_{\Gamma+\epsilon}^{\Gamma+1} \frac{d\Gamma'}{\Gamma-\Gamma'} = \left[\log(\Gamma'-\Gamma)\right]_{\Gamma'=\Gamma+\epsilon}^{\Gamma+1} = \log 1 - \log \epsilon$$
$$\int_{\Gamma-1}^{\Gamma-\epsilon} \frac{d\Gamma'}{\Gamma-\Gamma'} = \left[\log(\Gamma-\Gamma')\right]_{\Gamma'=\Gamma-1}^{\Gamma-\epsilon} = \log \epsilon - \log 1$$
$$\lim_{\epsilon \downarrow 0} \int_{[\Gamma-1,\Gamma+1] \setminus [\Gamma-\epsilon,\Gamma+\epsilon]} \frac{1}{Z_{\Gamma}(\Gamma)} \cdot \frac{1}{\Gamma-\Gamma'} d\Gamma' = 0$$
$$([\Gamma-a,\Gamma+b] \text{ yields } \neq 0 \text{ value depending on } a, b \in (0,\infty), \text{ but still convergent.})$$



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Contour integrals

 C_- upper half-circle (clockwise) from $\Gamma + \epsilon$ to $\Gamma - \epsilon$

$$\int_{C_{-}} \frac{1}{\Gamma - \Gamma'} d\Gamma' = \int_{\pi}^{0} \frac{1}{\Gamma - (\Gamma + \epsilon e^{i\theta})} \frac{d(\Gamma + \epsilon e^{i\theta})}{d\theta} d\theta = -\int_{\pi}^{0} \frac{1}{\epsilon e^{i\theta}} i\epsilon e^{i\theta} d\theta = \pi i$$
$$\lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{C_{-}} \frac{d\Gamma'}{Z(\Gamma) - Z(\Gamma')} = \frac{1}{2} \frac{1}{Z_{\Gamma}(\Gamma)}$$
$$\lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{C_{+}} \frac{d\Gamma'}{Z(\Gamma) - Z(\Gamma')} = -\frac{1}{2} \frac{1}{Z_{\Gamma}(\Gamma)}$$
$$\lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} p.v. \int_{[-\epsilon,\epsilon]} \frac{d\Gamma'}{Z(\Gamma) - Z(\Gamma')} d\Gamma' = 0.$$

Hence: complex velocity \boldsymbol{W} has tangential jump from lower to upper side by

$$\frac{1}{Z_{\Gamma}(\Gamma)}$$

Kelvin-Helmholtz instability:



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Pullin (1989) separated sheet



Self-similar:

$$v(t,x) = v(\frac{x}{t}) \quad , \quad \pi(t,x) = \pi(\frac{x}{t})$$

Self-similar vortex sheets

Let $\frac{1}{2} < \mu < \infty$ ($x \sim t^{\mu}$; similarity exponent; $\mu = 1$ most important). $Z(\Gamma, t) = t^{\mu} z(\gamma), \qquad \gamma = t^{-\alpha} \Gamma.$

 $\gamma \in \mathbb{R}$ is Γ at t = 1, and then $z(\gamma)$ is $Z(\Gamma, t)$ with t = 1.

$$\frac{\partial}{\partial t}Z = t^{\mu}(-\alpha)t^{-1}t^{-\alpha}\Gamma\frac{\partial}{\partial\gamma}z + \mu t^{\mu-1}z = t^{\mu-1}(-\alpha\gamma\frac{\partial}{\partial\gamma}z + \mu z),$$

p.v.
$$\int \frac{d\Gamma'}{Z(\Gamma,t) - Z(\Gamma',t)} = p.v. \int \frac{t^{\alpha} d\gamma'}{t^{\mu} [z(\gamma) - z(\gamma')]}$$

Comparison: need

$$\mu - 1 = \alpha - \mu \qquad \Rightarrow \qquad \alpha = 2\mu - 1.$$

Self-similar Birkhoff-Rott equation:

$$(1-2\mu)\gamma\frac{\partial}{\partial\gamma}z+\mu z=W^*=\left(\frac{1}{2\pi i}\mathsf{p.v.}\int_{\mathbb{R}}\frac{d\gamma'}{z(\gamma)-z(\gamma')}\right)^*$$

ODE, integral equation \Rightarrow like a boundary-value problem

Multi-branched rollup Example: N = 4.



Result discussed here: existence for N sufficiently large (but finite).

[E., submitted to Arch. Rat. Mech. Anal.]

N large: (1) N^{-1} yields a form of "smallness", (2) symmetry eliminates unpleasant terms

Key difficulty: spiral center very dense, corrections must have good decay, must not self-intersect [> m1N4ell.vs]

Birkhoff-Rott for multiple branches

Sheets p = 0, ..., N - 1 at equal angles $\frac{2\pi}{N}$, $N \in \mathbb{N}$.

 $z(\gamma)$ contour for sheet 0 \rightsquigarrow sheet p is $u^p z(\gamma)$ with $u = \exp \frac{2\pi i}{N}$.

Redefine Γ as circulation for all branches combined \Rightarrow each individual branch generates $\frac{\Gamma}{N}$

$$(1-2\mu)\gamma\frac{\partial}{\partial\gamma}z(\gamma) + \mu z(\gamma) = W^* = \left(\frac{1}{2\pi i} \text{ p.v.} \int_{\mathbb{R}} \underbrace{\frac{1}{N} \sum_{p=0}^{N-1} \frac{d\gamma'}{z(\gamma) - u^p z(\gamma')}}_{=:A_p}\right)^*.$$

Approach: express problem as

F(z) = 0 (F nonlinear C^1 operator; $z, F(z) \in Banach$ spaces) Solve by quasi-Newton iteration

$$z \leftarrow K(z) := z - A^{-1}F(z)$$

 A^{-1} "approximate inverse" for F'(z).

$$|K(z) - K(w)| = |z - w - A^{-1}(F(z) - F(w))| = |A^{-1}(A(z - w) - (F(z) - F(w))|$$

$$\leq |A^{-1}| |A(z - w) - F'(z)(z - w) + F'(z)(z - w) - (F(z) - F(w))|$$

$$\leq |A^{-1}| (|A - F'(z)||z - w| + |F(w) - F(z) - F'(z)(w - z)|)$$

$$= |A - F'(z)|O(z - w) + o(z - w)$$

If $A \approx F'(z)$ and |z - w| small, then for some $0 \leq L < 1$

$$\leq L|z-w|,$$

Iteration map K is uniform contraction, Banach fixed point theorem: converges to fixed point z = K(z) so that F(z) = 0. Simpler example: steady sheet

$$0 \stackrel{!}{=} \partial Z_t = W^* - \text{given } W_e^*$$
$$W = \frac{1}{2\pi i} \text{p.v.} \int_{\mathbb{R}} \frac{d\Gamma'}{Z(\Gamma) - Z(\Gamma')} \stackrel{!}{=} W_e.$$

Linearize:

$$-\frac{1}{2\pi i}\mathsf{p.v.}\int_{\mathbb{R}}\frac{\tilde{Z}(\Gamma)-\tilde{Z}(\Gamma')}{(Z(\Gamma)-Z(\Gamma'))^2}d\Gamma'.$$

For straight uniform sheet $Z(\Gamma) = \Gamma$:

$$-\frac{1}{2\pi i} \mathrm{p.v.} \int_{\mathbb{R}} \frac{\tilde{Z}(\Gamma) - \tilde{Z}(\Gamma')}{(\Gamma - \Gamma')^2} d\Gamma' =: L\tilde{Z}(\Gamma)$$

Fourier transform $\tilde{Z} = \exp(i\xi\Gamma)$:

$$(L\tilde{Z})^{\wedge}(\xi) = C|\xi|\tilde{Z}^{\wedge}(\xi).$$

Nice multiplier operator $L : \dot{H}^s \to \dot{H}^{s-1}$: Hilbert transform $\circ \partial$. Can invert (Fourier multiplier $|\xi|^{-1}$) modulo some polynomials. No dynamic instabilities in steady/self-similar problem. Self-similarity strong constraint; eliminates unstable modes.
Why are self-similar vortex sheets better behaved? Kelvin-Helmholtz instability: analogous to inverse heat equation

 $u_t = -\Delta u$

or Cauchy problem for Laplace equation

 $u_{tt} = -\Delta u.$

Many Fourier modes yield exponential growth in t:

$$u_{tt}^{\wedge}(t,\xi) = |\xi|^2 u^{\wedge}(t,\xi)$$

$$\rightarrow \quad u^{\wedge}(t,\xi) = C_{+} \exp(|\xi|t) + C_{-} \exp(-|\xi|t)$$

Consider steady solutions: $\Delta u = 0$. No problem!

Self-similar: change coordinates from (t, x) to $(t, \frac{x}{t})$, then ansatz u(t, x/t) = u(x/t): "quasi-steady".