Existence of algebraic vortex spirals and ill-posedness of inviscid flow (Part III)

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Birkhoff-Rott for multiple branches m4N1ell.vs

Sheets $p = 0, ..., N-1$ at equal angles $\frac{2\pi}{N}$, $N \in \mathbb{N}$.

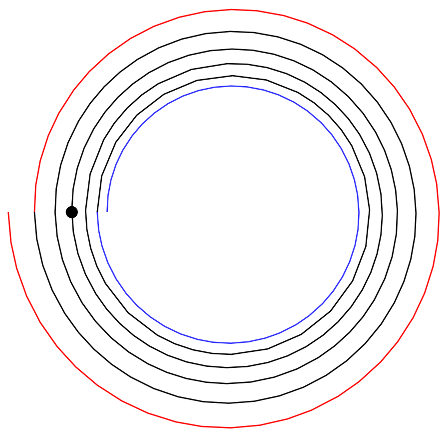
$z(\gamma)$ contour for sheet 0 $\rightsquigarrow$ sheet p is $u^p z(\gamma)$ with $u = \exp \frac{2\pi i}{N}$.

Redefine Γ as circulation for all branches combined
$\Rightarrow$ each individual branch generates $\frac{\Gamma}{N}$

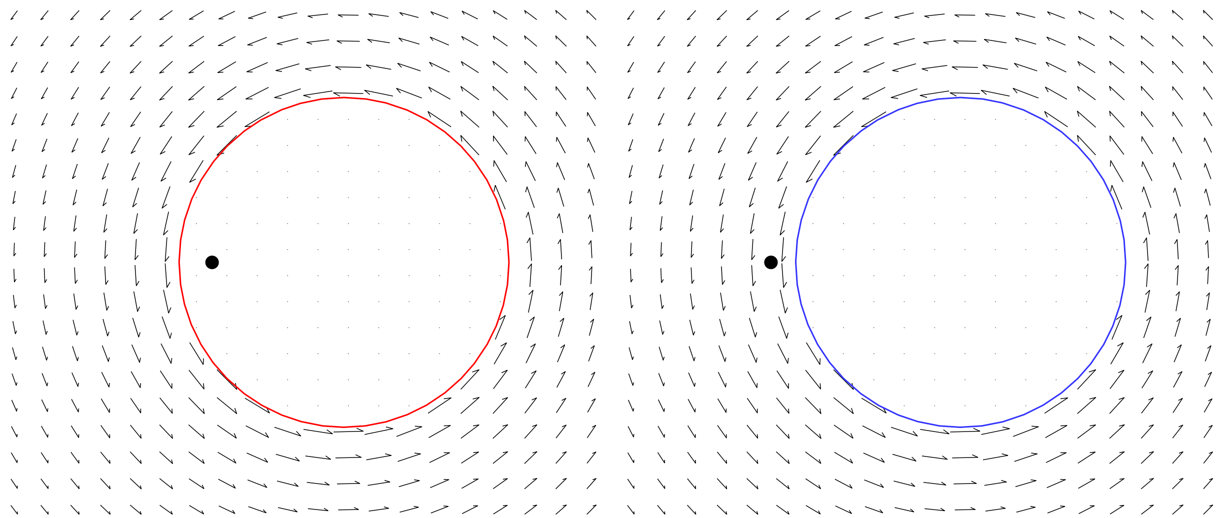
$$(1-2\mu)\gamma\frac{\partial}{\partial\gamma}z(\gamma) + \mu z(\gamma) = W^* = \left(\frac{1}{2\pi i}\,\text{p.v.}\int_{\mathbb{R}} \underbrace{\frac{1}{N}\sum_{p=0}^{N-1}}_{=:A_p} \frac{d\gamma'}{z(\gamma) - u^p z(\gamma')}\right)^*.$$

*

Kaden approximation (Kaden (1931), Rott (1956)):



$\gamma = 0$ in center;
increasing to outside



Spiral turns almost circular, with almost uniform $d\Gamma/ds$.
Inside circle: $W = 0$. Outside circle: $W = \frac{\Gamma}{2\pi i Z}$ (= point vortex).
Approximation of W integral:

$$(1-2\mu)\gamma\frac{\partial}{\partial\gamma}z(\gamma) + \mu z(\gamma) = \left(\frac{1}{2\pi i}\cdot\frac{\gamma}{z(\gamma)}\right)^*.$$

ODE for $z(\gamma)$. Ansatz: ϱ distance from center, θ angle travelled (= 0 at infinity, $\to \infty$ towards spiral center):

$$z = z_k = \varrho(\gamma)\exp(i\theta(\gamma)).$$

$$(1-2\mu)\gamma\frac{\partial}{\partial\gamma}z+\mu\gamma=\left(\frac{1}{2\pi i}\frac{\gamma}{z}\right)^*, \qquad z=\varrho(\gamma)\exp(i\theta(\gamma)).$$

$$(1-2\mu)\gamma\left(\frac{\partial}{\partial\gamma}\varrho+i\varrho\frac{\partial}{\partial\gamma}\theta\right)e^{i\theta}+\mu\varrho e^{i\theta}=\frac{i}{2\pi}\frac{\gamma}{\varrho}e^{i\theta}.$$

$$(1-2\mu)\frac{\partial}{\partial\gamma}\varrho+\mu\varrho=0 \qquad \Rightarrow \qquad \gamma=\varrho^{2-1/\mu}$$

$$(1-2\mu)\varrho\frac{\partial}{\partial\gamma}\theta=\frac{\gamma}{2\pi\varrho} \qquad \Rightarrow \qquad \boxed{\gamma=(2\pi\theta)^{1-2\mu}}$$

$$\boxed{\varrho=(2\pi\theta)^{-\mu}}, \qquad \boxed{z_k=(2\pi\theta)^{-\mu}e^{i\theta}}.$$

* Algebraic spirals ($\mu=1$: hyperbolic).

Prandtl (1920s): logarithmic spirals. Exact solutions, easier to construct [Saffman, Vortex Dynamics] but seem uncommon.

Pseudo-angle

Kaden approximation:

$$\gamma = (2\pi\theta)^{1-2\mu} \quad \Leftrightarrow \quad \theta = (2\pi)^{-1}\gamma^{1/(1-2\mu)}$$

More convenient: define pseudo-angle t as *

$$t = t(\gamma) = (2\pi)^{-1}\gamma^{1/(1-2\mu)} \qquad \Leftrightarrow \qquad \gamma = (2\pi t)^{1-2\mu}$$

If Kaden approximation good, can expect $t \approx \theta$ for $\theta \approx \infty$.
$\gamma, t \in (0, \infty)$; $\gamma \downarrow 0$ means $t \uparrow \infty$ and vice versa.

Advantage: instead of $\varrho(\theta), \gamma(\theta)$ can use single ($\mathbb{C}$-valued) $z(t)$.

Left-hand side:

$$(1-2\mu)\gamma z_\gamma + \mu z = (1-2\mu)\frac{\gamma}{\partial_t \gamma} z_t + \mu z = t z_t + \mu z$$

New velocity integral:

$$\frac{1}{2\pi i}\text{p.v.}\int_0^\infty A_p \frac{d\gamma'}{z(\gamma) - u^p z(\gamma')} = \frac{1}{2\pi i}\text{p.v.}\int_0^\infty A_p \frac{1}{z(t) - u^p z(s)} \left|\frac{d\gamma}{dt}(s)\right| ds$$

$$= -i(2\mu - 1)(2\pi)^{-2\mu}\text{p.v.}\int_0^\infty A_p \frac{1}{z(t) - u^p z(s)} s^{-2\mu} ds$$

Asymptotic series ansatz

$$z(t) = z_k(t)g(t) \quad , \quad g(t) = 1 + \tilde{g}(t) \quad , \quad \tilde{z}(t) = \tilde{g}(t)z_k(t).$$

$\tilde{g} = 0$ yields $z = z_k$ (Kaden approximation).

Idea: try (for example)

$$\tilde{g}(t) = \sum_{j=1}^{\infty} a_j t^{-j} \quad , \quad a_j \in \mathbb{C}$$

$$0 \stackrel{!}{=} F(z) = F_1(a_1)t^{-\beta} + F_2(a_1, a_2)t^{-\beta-1} + F_3(a_1, a_2, a_3)t^{-\beta-2} + ...$$

Determine a_1, then a_2, then a_3, ...

Problem: does not seem to converge [no surprise], not even for t large.

Why asymptotic series rarely work:

$$\sum_{j=1}^{\infty} a_j t^{-j}$$

1. Series does not converge (coefficients a_j grow too fast);
2. If it does, may converge only locally ($t \approx \infty$);
3. If it converges, need not converge to a solution;
4. Even if true, any of these is usually very hard to prove.

Examples:

$$u(1) = 1, \qquad \frac{du}{d\theta} + \theta^{-1} u = 0 \qquad \text{works (Frobenius)}$$

$$u(1) = 1, \qquad \frac{du}{d\theta} + u = 0 \qquad \text{fails! (reason 3.)}$$

$$\int_{\infty}^{t} s^{\beta} e^{is} ds = e^{it}\Big(\frac{t^{\beta}}{i} + \frac{\beta t^{\beta-1}}{i^2} + \frac{\beta(\beta-1)t^{\beta-2}}{i^3} + ...\Big) \qquad \text{fails! (reason 1.)}$$

Function space

$$z(t) = z_k(t)g(t) \quad , \quad g(t) = 1 + \tilde{g}(t) \quad , \quad \tilde{z}(t) = \tilde{g}(t)z_k(t).$$

* Idea: use truncated asymptotic series ansatz:

$$\tilde{g}(t) = \chi(t) \sum_{j=1}^{M} a_m t^{-m} + (1+t)^{-M} r(t)$$

$$= \sum_{j=1}^{M} \tilde{g}_m(t) + \tilde{g}_{M+1}(t) \qquad *$$

$$a_j \in \mathbb{C}, \quad r \in H^S \quad (S \text{ large})$$

$\chi(t)$ smooth function, $= 1$ near $t = \infty$, $= 0$ near $t \in [-\infty, 0]$.
Determine a_1, then a_2, then a_3, ... as before,
but r determined by iteration.

For N large, will have a_m, r small.

Approach: express problem as

$F(z) = 0 \qquad (F$ nonlinear C^1 operator; $z, F(z) \in$ Banach spaces)

Solve by quasi-Newton iteration

$$z \leftarrow K(z) := z - A^{-1}F(z)$$

A^{-1} "approximate inverse" for $F'(z)$.

$$|K(z)-K(w)| = |z-w-A^{-1}(F(z)-F(w))| = \left|A^{-1}\Big(A(z-w)-(F(z)-F(w)\Big)\right|$$

$$\leq |A^{-1}|\Big|A(z-w) - F'(z)(z-w) + F'(z)(z-w) - (F(z) - F(w))\Big|$$

$$\leq |A^{-1}|\Big(|A - F'(z)||z-w| + |F(w) - F(z) - F'(z)(w-z)|\Big)$$

$$= |A - F'(z)|O(z-w) + o(z-w)$$

If $A \approx F'(z)$ and $|z-w|$ small, then for some $0 \leq L < 1$

$$\leq L|z-w|,$$

Iteration map K is uniform contraction, Banach fixed point theorem: converges to fixed point $z = K(z)$ so that $F(z) = 0$.

Motivation for truncated series ansatz:

$$z(t) = z_k(t)g(t), \qquad \mathcal{D} \ni g = 1 + \sum_{m=1}^{M} a_m t^{-m} + t^{-M} r(t) \quad \text{(at } t \approx \infty)$$

To find "rest" $r \in H^S$ have to solve

$$G(t^{-M} r) = 0$$

for some nonlinear operator (self-similar Birkhoff-Rott)

$$G : \mathcal{D} \to \mathcal{R}.$$

Need $G'(0)$ isomorphism: $G'(0)^{-1} : \mathcal{R} \to \mathcal{D}$
$\Rightarrow \mathcal{R}$ must enforce sufficient decay in t, say $O(t^{-M+O(1)})$.

$$0 = G(t^{-M} r) = G(0) + G'(0)[t^{-M} r] + \frac{G''(0)}{2}[t^{-M} r, t^{-M} r] + \ldots$$

For large N, $G''(0)$ integral operator with near-diagonal kernel:

$$G''(0)[t^{-M} r, t^{-M} r] = O(t^{-2M+O(1)}).$$

If M large, $2M \gg M + O(1)$: nonlinear terms allow wasteful analysis.

$\Rightarrow$ essential difficulty is only linear !

Integral expansion

$$\text{p.v.} \int_0^\infty A_p \frac{s^{-2\mu} ds}{z(t) - u^p z(s)} = \text{p.v.} \int_0^\infty A_p \frac{s^{-2\mu} ds}{\Delta z} \overset{z = z_k + \tilde{z}}{=} \text{p.v.} \int_0^\infty A_p \frac{s^{-2\mu} ds}{\Delta z_k + \Delta \tilde{z}}$$

For N large, due to smallness $|\Delta \tilde{z}| < |\Delta z_k|$.

$$= \sum_{j=0}^{\infty} (-1)^j \text{p.v.} \int_0^\infty A_p \frac{(\Delta \tilde{z})^j}{(\Delta z_k)^{j+1}} s^{-2\mu} ds$$

$$(\tilde{z} = \sum_{m=1}^{M+1} \tilde{z}_m \quad , \quad \tilde{z}_k = \tilde{g}_m z_k)$$

$$\text{p.v.} \int_0^\infty A_p \frac{(\Delta \tilde{z})^j}{(\Delta z_k)^{j+1}} s^{-2\mu} ds = \text{p.v.} \int_0^\infty A_p \frac{(\sum_{m=1}^{M+1} \Delta \tilde{z}_m)^j}{(\Delta z_k)^{j+1}} s^{-2\mu} ds$$

$$= \text{p.v.} \int_0^\infty A_p \sum_{|\alpha|=j} \binom{j}{\alpha} \frac{\prod_{m=1}^{M+1} (\Delta \tilde{z}_m)^{\alpha(m)}}{(\Delta z_k)^{j+1}} s^{-2\mu} ds$$

Integral expansion (2): $\tilde{z}_m = \tilde{g}_m z_k$, $|\alpha| = \sum_{m=1}^{M+1} \alpha(m)$,

$$\Delta \tilde{z}_m = \tilde{z}_m(t) - u^p \tilde{z}_m(s) = \tilde{g}_m(t) z_k(t) - u^p \tilde{g}_m(s) z_k(s)$$

$$= \tilde{g}_m(t)\Big(z_k(t) - u^p z(s)\Big) + \Big(\tilde{g}_m(t) - \tilde{g}_m(s)\Big) u^p z(s) = \tilde{g}_m(t)\Delta z_k + u^p z(s)\Delta g_m$$

$$\text{p.v.} \int_0^\infty A_p \frac{\prod_{m=1}^{M+1} (\Delta \tilde{z}_m)^{\alpha(j)}}{(\Delta z_k)^{j+1}} s^{-2\mu} ds$$

$$= \text{p.v.} \int_0^\infty A_p \frac{\prod_{m=1}^{M+1} \Big(\tilde{g}_m(t)\Delta z_k + u^p z_k(s)\Delta \tilde{g}_m(s)\Big)^{\alpha(j)}}{(\Delta z_k)^{j+1}} s^{-2\mu} ds$$

$$= \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \text{p.v.} \int_0^\infty A_p \frac{\prod_{m=1}^{M+1} (u^p z_k(s)\Delta \tilde{g}_m)^{\beta(m)} (\tilde{g}_m(t)\Delta z_k)^{\alpha(m)-\beta(m)}}{(\Delta z_k)^{j+1}} s^{-2\mu} ds$$

$$\overset{|\alpha|=j}{\rightsquigarrow} \prod_{m=1}^{M+1} \tilde{g}_m(t)^{\alpha(m)-\beta(m)} \text{p.v.} \int_0^\infty A_p (u^p z_k(s))^{|\beta|} \frac{\prod_{m=1}^{M+1} (\Delta \tilde{g}_m)^{\beta(m)}}{(\Delta z_k)^{1+|\beta|}} s^{-2\mu} ds$$

Integral expansion (3):

have expanded velocity integral into series of integrals like

$$\text{p.v.} \int_0^\infty A_p(u^p z_k(s))^{|\beta|} \frac{\prod_{m=1}^{M+1} (\Delta \tilde{g}_m)^{\beta(m)}}{(\Delta z_k)^{1+|\beta|}} s^{-2\mu} ds$$

$|\beta| = 0$: constant part

$$\text{p.v.} \int_0^\infty A_p \frac{1}{\Delta z_k} s^{-2\mu} ds$$

$|\beta| = 1$: linear parts

$$\text{p.v.} \int_0^\infty A_p u^p z_k(s) \frac{\Delta \tilde{g}_m}{(\Delta z_k)^2} s^{-2\mu} ds$$

$|\beta| = 2, 3, \ldots$: quadratic and higher parts like

$$\text{p.v.} \int_0^\infty A_p (u^p z_k(s))^2 \frac{\Delta \tilde{g}_m \Delta \tilde{g}_\ell}{(\Delta z_k)^3} s^{-2\mu} ds$$

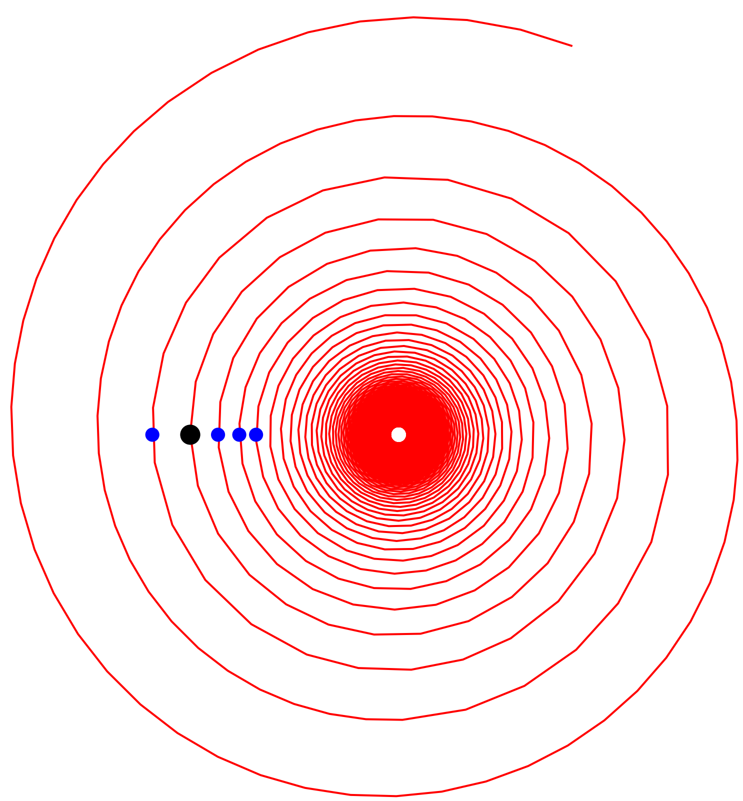
Key difficulty: spiral rollup.

With Kaden approximation $z_k(t) = (2\pi t)^{-\mu} e^{i\theta}$,
typical linearized operator like

$$\text{p.v.} \int_0^\infty A_p \frac{\tilde{z}(t) - \tilde{z}(s)}{[t^{-\mu}\exp(it) - u^p s^{-\mu}\exp(is)]^2} s^{-2\mu} ds$$

Key problem: near-singularity whenever (e.g.) $p = 0$,

$$s \approx t + 2\pi k \qquad (k \in \mathbb{Z}).$$



1. Spiral must not self-intersect
2. delicate cancellations (left, right side of each turn) must not be upset.

Analogies in elliptic PDE

$(a_{ij}) = A = A(x)$ continuous

Variable coefficients = constant coefficients + small

$$f = u - \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} u = \underbrace{u - \sum_{i,j} a_{ij}(0) \frac{\partial^2}{\partial x_i \partial x_j} u}_{L_0 u:\ \text{constant coeff.}} + \sum_{i,j} \underbrace{\left(a_{ij}(x) - a_{ij}(0)\right)}_{\text{small for } x \approx 0} \frac{\partial^2}{\partial x_i \partial x_j} u$$

$$\left(u - \sum_{i,j} a_{ij}(0) \frac{\partial^2}{\partial x_i \partial x_j} u\right)^{\wedge}(\xi) = \underbrace{\left(1 + \sum_{i,j} a_{ij}(0)\xi_i \xi_j\right)}_{\sigma(\xi)} u^{\wedge}(\xi)$$

Elliptic means $\sigma(\xi)$ invertible $\Rightarrow$ Fourier transform to invert operator:

$$L_0 u = g \quad \Rightarrow \quad u^{\wedge}(\xi) = \sigma(\xi)^{-1} g^{\wedge}(\xi) \quad , \quad L_0^{-1} : H^s \to H^{s+2}$$

$$(L_0 + E)u = f, \quad (L_0 + E)^{-1} = L_0^{-1} \sum_{j=0}^{\infty} (E L_0^{-1})^j$$

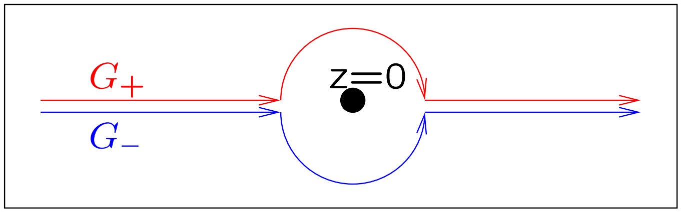
converges if $\|E\| < \frac{1}{\|L_0^{-1}\|}$ so that $\|E L_0^{-1}\| \leq \|E\| \cdot \|L_0^{-1}\| < 1$.

Singular integrals by Fourier transform

$$Lw(t) := \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{w(t)-w(s)}{(t-s)^2}ds$$

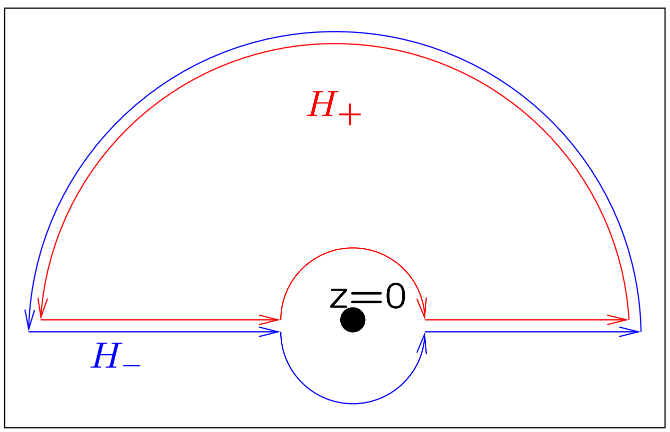
$$\frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{\exp(i\xi t)-\exp(i\xi s)}{(t-s)^2}ds = \exp(i\xi t)\frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{1-\exp(i\xi(s-t))}{(t-s)^2}ds$$

(remember: may also average contour above and below:)



$$\frac{1}{2\pi}\Big(\int_{G_-} + \int_{G_+}\Big)\frac{1-\exp(i\xi z)}{z^2}dz$$

$\xi > 0 \Rightarrow \exp(i\xi z)$ decays as $\Im z \uparrow +\infty$: close contours near ∞ above:



$\int_{G_+} \leftarrow \int_{H_+}$, no pole enclosed, $= 0$

$\int_{G_-} \leftarrow \int_{H_-}$ encloses pole $z = 0$ c.c.w.

$$\frac{1-\exp i\xi z}{z^2} = \frac{1-(1+i\xi z+O(z^2))}{z^2} = \frac{-i\xi}{z}+O(1)$$

Residue: $\frac{1}{2\pi}2\pi i(-i\xi) = \xi \overset{\xi>0}{=} |\xi|$.

For $\xi < 0$ close contours below (clockwise), get $-\xi \overset{\xi<0}{=} |\xi|$.

L is Fourier multiplier operator: $w^\wedge(\xi) \mapsto |\xi| w^\wedge(\xi)$.

Brute-force singular integrals: Know: Fourier transform can do

$$Hw(t) := \frac{1}{\pi}\text{p.v.}\int \frac{w(s)}{t-s}ds = \mathcal{F}^{-1}\Big(\operatorname{sgn}\xi \cdot \mathcal{F}w(\xi)\Big)$$

Idea: if f smooth, $f'(t) \neq 0$:

$$\text{p.v.}\int \frac{w(s)}{f(s)-f(t)}ds = \text{p.v.}\int \frac{w(s)}{f'(t)(s-t)+\frac{1}{2}f''(t)(s-t)^2+\ldots}ds$$

$$= \text{p.v.}\int \frac{w(s)}{f'(t)(s-t)(1+\frac{f''(t)}{f'(t)}(s-t)+O(s-t)^2}ds$$

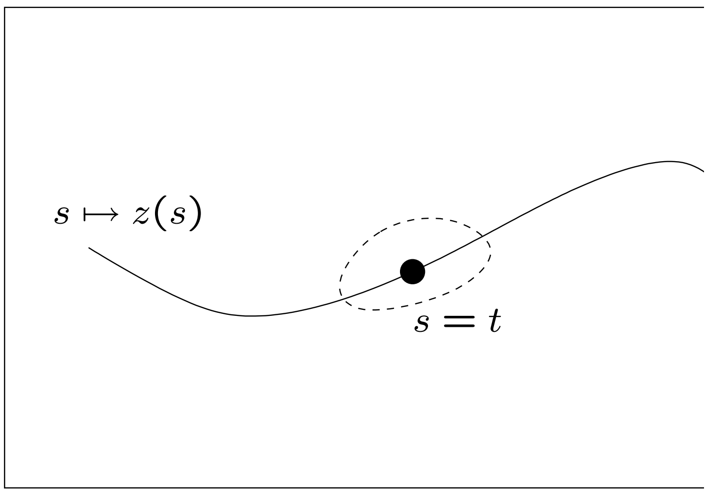
$$= \text{p.v.}\int \frac{w(s)}{f'(t)(s-t)}\Big(1-\frac{f''(t)}{f'(t)}(s-t)+O(s-t)^2\Big)ds$$

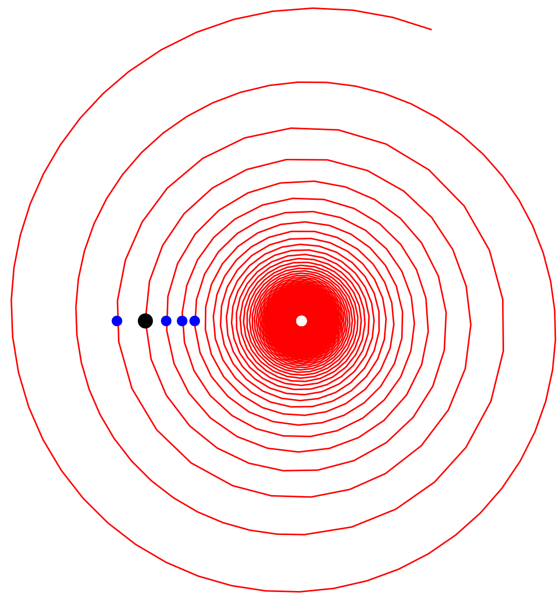
$$= \underbrace{\frac{1}{f'(t)}\text{p.v.}\int \frac{w(s)}{s-t}ds}_{\text{Hilbert}} - \underbrace{\frac{f''(t)}{f'(t)^2}\text{p.v.}\int w(s)ds + \ldots}_{\text{regular}}$$

Taylor expansion only $s \approx t$, so pick $\theta \in C^\infty$ with support near t and

$$\text{p.v.}\int \frac{w(s)}{f(s)-f(t)}ds = \underbrace{\text{p.v.}\int \frac{w(s)}{f(s)-f(t)}\theta(s)ds}_{\text{singular}} + \underbrace{\text{p.v.}\int \frac{w(s)}{f(s)-f(t)}(1-\theta(s))ds}_{\text{regular}}$$

Local expansion



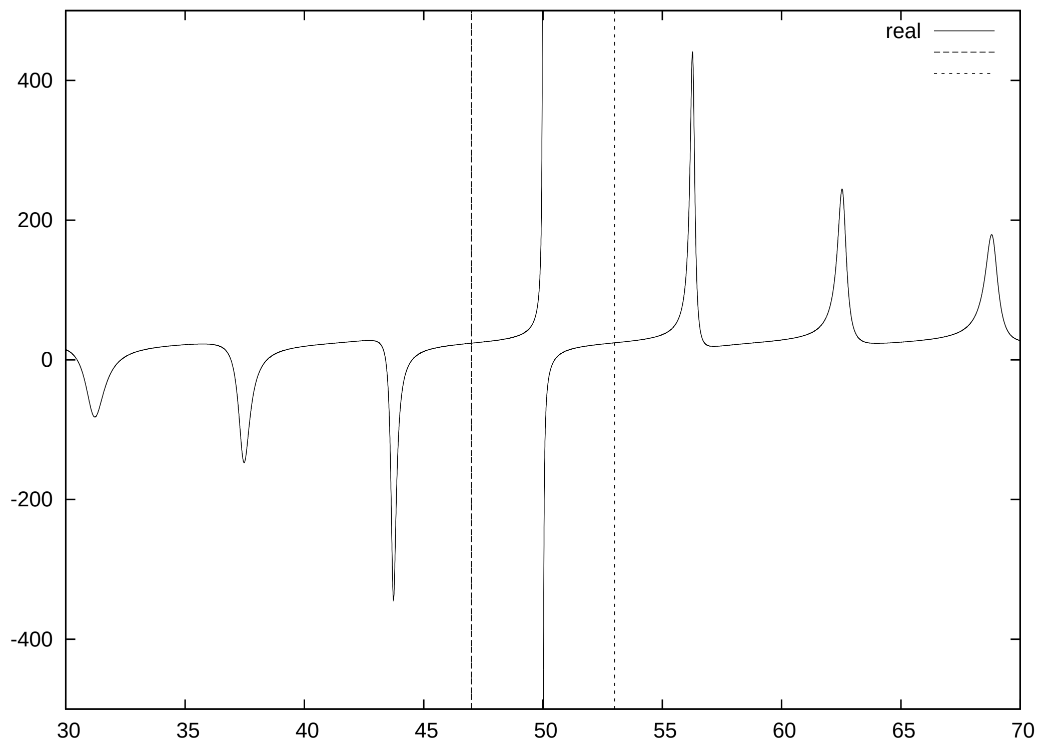


Truncation to near-singular and Taylor-expansion:
works well for left curve, but not for spirals:
regular part is not regular enough

Problem with "spiral transform": $z_k(t) = (2\pi t)^{-\mu} e^{it}$

$$\int \frac{h(s)}{z_k(t) - z_k(s)} (1 - \theta(s)) ds = C \int \frac{h(s)}{t^{-\mu} e^{it} - s^{-\mu} e^{is}} (1 - \theta(s)) ds$$

$\theta = 1$ near $t = s$: no singularity in $t = s$,
but still near-singular for $t \approx s + 2\pi k$, $k \in \mathbb{Z}$.



Singularities become worse as $t \to \infty$:

$$\frac{1}{t^{-\mu} e^{it} - (t + 2\pi)^{-\mu} e^{i(t+2\pi)}}$$

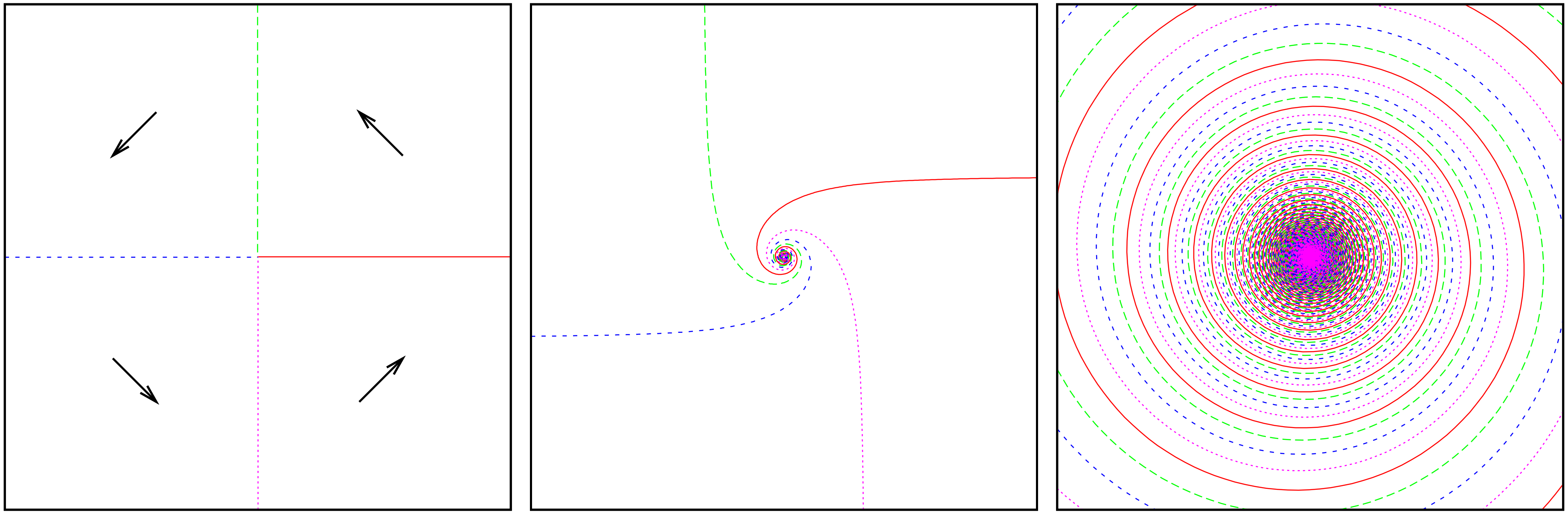
$$= e^{-it} t^{\mu} \frac{1}{1 - \underbrace{\left(1 + \frac{2\pi}{t}\right)^{-\mu}}_{\to 1} e^{2i\pi}}$$

$$= O(t^{\mu+1})$$

Brute-force analysis loses one t^{-1} for every $\frac{1}{\Delta z}$.
$\Rightarrow$ still good enough for quadratic and higher terms:
t^{-2M} integrand has to yield $t^{-M+O(1)}$ integral decay

Some slides omitted (unpublished results)

Multi-branched rollup Example: $N = 4$.

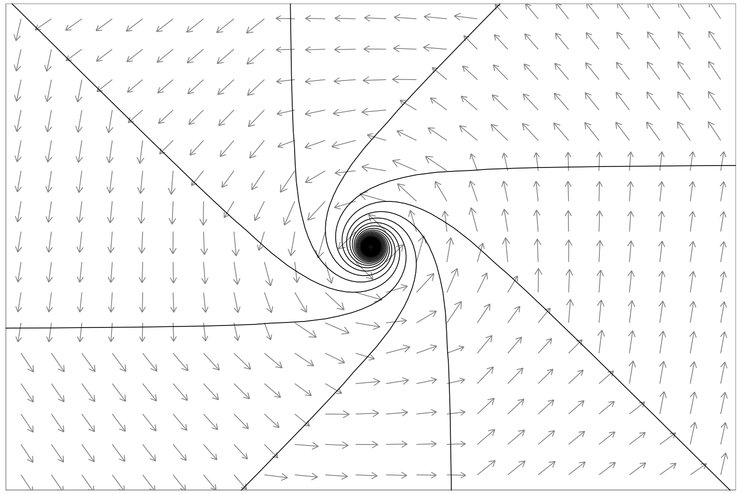


Result discussed here: existence for N sufficiently large (but finite).

[E., submitted to *Arch. Rat. Mech. Anal.*]

N large: (1) N^{-1} yields a form of "smallness", (2) symmetry eliminates unpleasant terms

Key difficulty: spiral center very dense, corrections must have good decay, must not self-intersect [▷ m1N4ell.vs]

Limit of $N \to \infty$ 

Self-similar solution of incompressible Euler: with $\xi = \frac{x}{t^\mu}$, for $\mu > \frac{1}{2}$,

$$v(\xi) = Ct^{\mu-1}|\xi|^{1-1/\mu}\frac{\xi^\perp}{|\xi|} = C|x|^{1-1/\mu}\frac{x^\perp}{|x|} \qquad \text{(purely angular)}$$

$$\omega(\xi) = C't^{-1}|\xi|^{-1/\mu} = C'|x|^{-1/\mu}$$

Self-similar vorticity equation:

$$(\mu\xi - v) \cdot \nabla\omega - \omega = 0$$

For v as above, integral curves of $\mu\xi - v$ are identical with the Kaden approximation spirals!

Single-branched rollup Self-similarity:

$$\boxed{x \sim t^{\mu}}$$

$\mu = 1$: "hyperbolic scaling" $x \sim t$, familiar from compressible flow/acoustics. Multiplying t with 2 is like dilation by 2^{μ}.

$t \uparrow \infty$: zooming in, $t \downarrow 0$: zooming out.

$$\vec{v}(t, \vec{x}) = t^{\mu-1}\vec{v}(\frac{\vec{x}}{t^{\mu}})$$

[E., in preparation]:

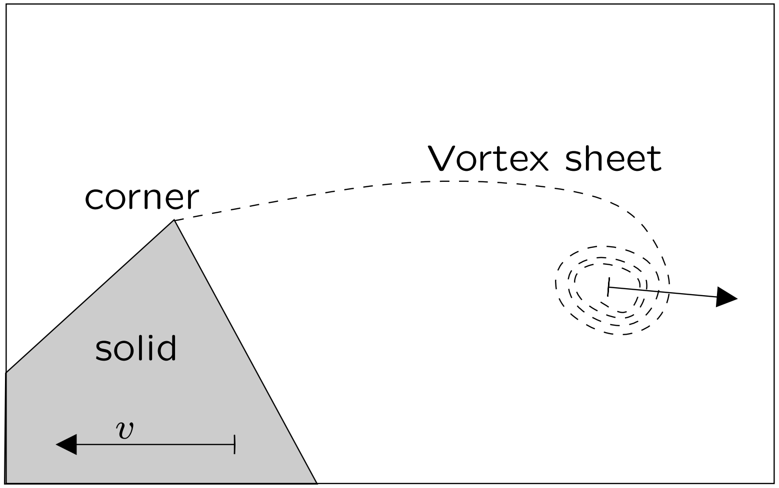
$$\boxed{\text{existence of single-branched spirals for } \mu = \frac{1}{2} + \epsilon,\ \epsilon \gtrapprox 0}$$

New difficulty: elliptic deformation

Close analogue of the Kaden spiral ($\mu = \frac{2}{3}$)

[▷ m0.53many.vs]

Vortex spirals from solid corners



$t < 0$: solid, fluid at rest. $t = 0$: solid is instantaneously accelerated to $v \neq 0$. $t > 0$: self-similar vortex sheet emanates, ends in spiral.

Project:

1. prove existence of such flows [doable soon]
2. Sheet C^1 at corner? (Likely; if so:) What angles can the sheet form to the two wall boundaries? Non-uniqueness?
3. What self-similar scalings $x \sim t^\mu$, i.e. which μ are possible?
4. Velocity of spiral center relative to corner?

(end)