Stability of Transonic Shock Solutions for Euler-Poisson and Euler Equations

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Euler-Poisson Equations

One dimensional Euler-Poisson equations:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (p(\rho) + \rho u^2)_x = \rho E, \\ E_x = \rho - b(x). \end{cases} \tag{1}$$

Background: the propagation of electrons in submicron semiconductor devices and plasmas, and the biological transport of ions for channel proteins.

In the hydrodynamical model of semiconductor devices or plasma, u, ρ and p represent the average particle velocity, electron density and pressure, respectively, E is the electric filed, which is generated by the Coulomb force of particles. $b(x) > 0$ stands for the density of fixed, positively charged background ions.

Assumption on p:

$$p(0) = p'(0) = 0,\ \ p'(\rho) > 0,\ \ p''(\rho) \geq 0,\ \ \text{for}\, \rho > 0,\ \ p(+\infty) = +\infty.$$

Steady Equations and Boundary Conditions

Steady Euler-Poisson equations:

$$\begin{cases} (\rho u)_x = 0, \\ (p(\rho) + \rho u^2)_x = \rho E, \\ E_x = \rho - b(x). \end{cases} \tag{2}$$

Boundary conditions:

$$(\rho, u, E)(0) = (\rho_l, u_l, E_l), \quad (\rho, u)(L) = (\rho_r, u_r). \tag{3}$$

We assume $u_l > 0$ and $u_r > 0$. By the first equation in (2), we know that $\rho u(x) = constant (0 \le x \le L)$, so the boundary data should satisfy

$$\rho_l u_l = \rho_r u_r.$$

Alternative Equations and Boundary Conditions

If one denotes

$$\rho_l u_l = \rho_r u_r = J > 0,$$

then $\rho u(x) = J (0 \leq x \leq L)$ and the velocity is given by

$$u = J/\rho.$$

Thus the boundary value problem for system (2) reduces to

$$\begin{cases} (p(\rho) + \frac{J^2}{\rho})_x = \rho E, \\ E_x = \rho - b(x), \end{cases} \tag{4}$$

with the boundary conditions:

$$(\rho, E)(0) = (\rho_l, E_l), \quad \rho(L) = \rho_r. \tag{5}$$

Transonic Shock Solutions

We use the terminology from gas dynamics to call $c = \sqrt{p'(\rho)}$ the sound speed. There is a unique solution $\rho = \rho_s$ satisfying $p'(\rho) = J^2/\rho^2$, which is the sonic state (recall that $J = \rho u$). Later on, the flow is called supersonic (subsonic) if

$$p'(\rho) < (>) J^2/\rho^2, \ i.e. \ \rho < (>) \rho_s.$$

Transonic shock solutions:

$$(\rho, E) = \begin{cases} (\rho_{sup}, E_{sup})(x), & 0 < x < x_0, \\ (\rho_{sub}, E_{sub})(x), & x_0 < x < L, \end{cases}$$

satisfying the Rankine-Hugoniot conditions

$$\left(p(\rho) + \frac{J^2}{\rho}\right)(x_0-) = \left(p(\rho) + \frac{J^2}{\rho}\right)(x_0+), \ E(x_0-) = E(x_0+),$$

and is supersonic behind the shock and subsonic ahead of the shock, i.e.,

$$\rho_{sup}(x_0-) < \rho_s < \rho_{sub}(x_0+).$$

Known Results

- A boundary value problem for (4) was discussed for a linear pressure function of the form $p(\rho) = k\rho$ with the boundary condition $\rho(0) = \rho(L) = \bar{\rho}$ where $\bar{\rho}$ being a subsonic state and the density of the background charge satisfied $0 < b < \rho_s$ (Ascher et al).
- A phase plane analysis was given for system (4) without the construction of the transonic shock solution (Rosini).
- The vanishing viscosity method was used to study (4). The structure of the solutions is not clear(Gamba).
- Existence of transonic shock solution with constant background charge (Luo and Xin).
- Asymptotic behavior of solutions for Euler-Poisson equations with relaxations (Huang, Pan and Yu, etc)
- Formation of singularity of Euler-Poisson equations (Chen and Wang)

Structural Stability

Theorem 1 Let $J > 0$ be a constant, and let b_0 be a constant satisfying $0 < b_0 < \rho_s$ and (ρ_l, E_l) be a supersonic state $(0 < \rho_l < \rho_s)$, ρ_r be a subsonic state $(\rho_r > \rho_s)$. If the boundary value problem (4) and (5) admits a unique transonic shock solution $(\rho^{(0)}, E^{(0)})$ for the case when $b(x) = b_0$ $(x \in [0, L])$ with a single transonic shock locating at $x = x_0 \in (0, L)$ satisfying

$$E^{(0)}(x_0+) = E^{(0)}(x_0-) > 0,$$

then there exists $\epsilon_0 > 0$ such that if

$$\|b - b_0\|_{C^0[0,L]} = \epsilon \leq \epsilon_0,$$

then the boundary problem (4) and (5) admits a unique transonic shock solution $(\tilde{\rho}, \tilde{E})$ with a single transonic shock locating at some $\tilde{x}_0 \in [x_0 - C\epsilon, x_0 + C\epsilon]$ for some constant $C > 0$.

Dynamical Stability

Theorem 2 Let $(\bar{\rho}, \bar{u}, \bar{E})$ be a steady transonic shock solution. Moreover, we assume that

$$\bar{E}_{-}(x_0) = \bar{E}_{+}(x_0) > 0.$$

If the initial data (ρ_0, u_0, E_0) satisfy and the $k+2$-th $(k \geq 15)$ order compatibility conditions at $x = 0$, $x = x_0$ and $x = L$, then the initial boundary value problem (1) and (3) admits a unique piecewise smooth solution $(\rho, u, E)(x, t)$ for $(x, t) \in [0, L] \times [0, \infty)$, which contains a single transonic shock $x = s(t)$ $(0 < s(t) < L)$ satisfying the Rankine-Hugoniot condition and the Lax geometric shock condition for $t \geq 0$ provided that

$$\|(\rho_0, u_0, E_0) - (\bar{\rho}, \bar{u}, \bar{E})\|_{H^{k+2}} = \varepsilon$$

is suitably small.

Decay of the Solutions

Let

$$(\rho, u, E) = \begin{cases} (\rho_-, u_-, E_-), & \text{if} \quad 0 < x < s(t), \\ (\rho_+, u_+, E_+), & \text{if} \quad s(t) < x < L. \end{cases}$$

Then there exists $T_0 > 0$ and $\alpha > 0$ such that

$$(\rho_-, u_-, E_-)(t, x) = (\bar{\rho}_-, \bar{u}_-, \bar{E}_-)(x), \quad \text{for } 0 \le x < s(t),$$

for $t > T_0$ and

$$\|(\rho_+, u_+, E_+)(\cdot, t) - (\bar{\rho}_+, \bar{u}_+, \bar{E}_+)(\cdot)\|_{W^{k-6,\infty}(s(t),L)} \le C\varepsilon e^{-\alpha t},$$

$$\sum_{m=0}^{k-6} |\partial_t^m (s(t) - x_0)| \le C\varepsilon e^{-\alpha t},$$

for $t \ge 0$, where we have extended $(\bar{\rho}_\pm, \bar{u}_\pm, \bar{E}_\pm)$ to be the solutions of the Euler-Poisson equations in the associated regions.

Instability and Some Remarks

- There exist $L > 0$ and a linearly unstable transonic shock solution $(\bar{\rho}, \bar{u}, \bar{E})$ satisfying

$$\bar{E}_-(x_0) = \bar{E}_+(x_0) < 0.$$

- In Theorem 2, the results are also true if we impose small perturbations for the boundary conditions (5).
- It follows from the results by Luo and Xin and Theorem 1, the background transonic shock solution does exist. Moreover, we do not assume that $b(x)$ is a small perturbation of a constant in Theorem 2, which may have large variation.
- In Theorem 2, the regularity assumption is not optimal. By adapting the methods by Metivier, less regularity assumptions will be enough . However, our proof only involves the elementary weighted energy estimates rather than paradifferential calculus.

Monotone Relation

Lemma 3 Let $(\rho^{(1)}, E^{(1)})$ and $(\rho^{(2)}, E^{(2)})$ be two transonic shock solutions of (4), and $(\rho^{(i)}, E^{(i)})(i = 1, 2)$ are defined as follows

$$(\rho^{(i)}, E^{(i)}) = \begin{cases} (\rho_{sup}^{(i)}, E_{sup}^{(i)}), & \text{for} \quad 0 < x < x_i, \\ (\rho_{sub}^{(i)}, E_{sub}^{(i)}), & \text{for} \quad x_i < x < L, \end{cases}$$

where

$$\rho_{sup}^{(i)} < \rho_s < \rho_{sub}^{(i)} \quad \text{for} \quad i = 1, 2.$$

Moreover, they satisfy the same upstream boundary conditions,

$$\rho^{(1)}(0) = \rho^{(2)}(0) = \rho_l, \ E^{(1)}(0) = E^{(2)}(0) = E_l.$$

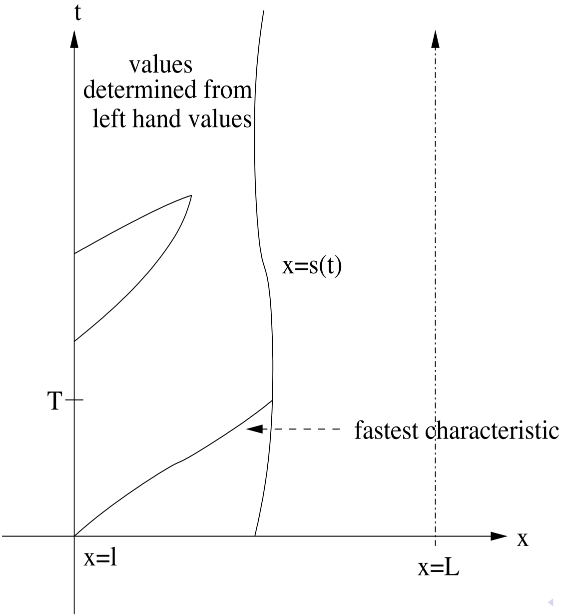
If $b < \rho_s$, $x_1 < x_2$ and $E_{sup}^{(2)}(x_1) > 0$, then

$$\rho^{(1)}(L) > \rho^{(2)}(L).$$

Proof of Structural Stability

- A priori estimates for subsonic and supersonic flows via multiplier method
- Monotone relation implies uniqueness of shock position
- Continuous dependence on shock positions for the exit pressures

Local Solutions



RH Conditions Revisited

$$
\begin{aligned}
(J_+ - \bar{J})(t, s(t)) = &- \frac{(p'(\bar{\rho}_+) - \frac{\bar{J}^2}{\bar{\rho}_+^2})(x_0)}{2\bar{J}/\bar{\rho}_+}(\rho_+ - \bar{\rho}_+)(t, s(t)) \\
&- \frac{(\bar{\rho}_+ - \bar{\rho}_-)\bar{E}_+(x_0)}{2\bar{J}/\bar{\rho}_+}(s(t) - x_0) + \text{quadratic terms}
\end{aligned}
$$

$$
\begin{aligned}
s'(t) = &- \frac{p'(\bar{\rho}_+) - \bar{J}^2/\rho_+^2}{2\bar{u}_+(\bar{\rho}_+ - \bar{\rho}_-)}(x_0)(\rho_+ - \bar{\rho}_+) \\
&- \frac{\bar{E}_+(x_0)}{2\bar{u}_+(x_0)}(s(t) - x_0) + \text{quadratic terms.}
\end{aligned}
$$

$$
s(t) - x_0 = \frac{1}{\bar{\rho}_-(x_0) - \bar{\rho}_+(x_0)}(E - \bar{E}_+) + \text{quadratic terms}
$$

The Second Order Equation

Set

$$Y = E_+(x,t) - \bar{E}_+(x).$$

Then

$$Y_t = \bar{J} - J_+, \quad Y_x = \rho_+ - \bar{\rho}_+.$$

Therefore, it follows from the second equation in the Euler-Poisson system (1) that

$$\partial_{tt} Y + \partial_x \left(p(\bar{\rho}_+) + \frac{\bar{J}^2}{\bar{\rho}_+} - p(\bar{\rho}_+ + Y_x) - \frac{(\bar{J} - Y_t)^2}{\bar{\rho}_+ + Y_x} \right)$$
$$+ \bar{E}_+ \partial_x Y + \bar{\rho}_+ Y + Y Y_x = 0.$$

The Linearized Problem

Introducing the transformation

$$\tilde{t} = t, \; \tilde{x} = (L - x_0)\frac{x - s(t)}{L - s(t)} + x_0, \; \sigma(\tilde{t}) = s(t) - x_0,$$

to transform the problem in the fixed domain $[x_0, L]$. After removing all $\tilde{}$ away, the linearized equation is

$$\begin{aligned} &\partial_{tt} Y - \partial_x((p'(\bar{\rho}_+) - \frac{\bar{J}^2}{\bar{\rho}_+^2})\partial_x Y) + \partial_x(\frac{2\bar{J}}{\bar{\rho}_+}\partial_t Y) \\ &+ \bar{E}_+\partial_x Y + \bar{\rho}_+ Y = 0. \end{aligned} \tag{6}$$

The associated boundary conditions are

$$\partial_x Y = \frac{2\bar{u}_+(x_0)}{c^2(\bar{\rho}_+)(x_0) - \bar{u}_+^2(x_0)}\partial_t Y + \frac{\bar{E}_+(x_0)}{c^2(\bar{\rho}_+)(x_0) - \bar{u}_+^2(x_0)} Y \tag{7}$$

at $x = x_0$ and

$$\partial_x Y = 0 \text{ at } x = L. \tag{8}$$

Theorem 4 Assume that $\bar{E}_+$ satisfies

$$\bar{E}_+(x_0) > 0.$$

Let Y be a smooth solution of the linearized problem (6)-(8). Then there exist $\alpha_0 \in (0, 1)$ and $T > 0$ such that

$$\varphi(Y, t + T) < \alpha_0 \varphi(Y, t),$$

where φ is defined as follows

$$\begin{aligned}\varphi(Y, t) =& \frac{\bar{E}_+}{\bar{\rho}_+}(x_0) Y^2(t, x_0) + \int_{x_0}^{L} \frac{1}{\bar{\rho}_+} \Big\{ (\partial_t Y)^2 \\ &+ \left(p'(\bar{\rho}_+) - \frac{\bar{J}_+^2}{\bar{\rho}_+^2} \right) (\partial_x Y)^2 + \bar{\rho}_+ Y^2 \Big\} (t, x) dx.\end{aligned}$$

Proof-Energy Estimate

Multiplying the equation (6) with $\frac{1}{\bar{\rho}_+(x)}\partial_t Y$ on both sides, integrating by parts, and applying the boundary conditions to get

$$\varphi(Y,t) + D(Y,t) = \varphi(Y,0), \tag{9}$$

where

$$D(Y,t) = 2\left(\int_0^t \frac{\bar{J}}{\bar{\rho}_+^2}(\partial_t Y)^2(s,L)ds + \int_0^t \frac{\bar{J}}{\bar{\rho}_+^2}(\partial_t Y)^2(s,x_0)ds\right).$$

Proof-Rauch-Taylor type estimates

Following from an argument by Rauch and Taylor, there exists a $T > 0$ such that

$$\int_0^T (Y_t^2 + Y_x^2)(x_0, t)dt \geq \int_{\frac{T}{2}-\delta}^{\frac{T}{2}+\delta} \varphi(Y, s)ds - C\int_0^T Y^2(x_0, t)dt$$
$$\geq C\delta\varphi(Y, T) - C\int_0^T Y^2(x_0, t)dt$$

Using the boundary conditions and the fact that $\frac{c^2(\bar{\rho}_+)-\bar{u}_+}{2u_+}(x_0) \geq C$ for some constant $C > 0$, one has

$$\varphi(Y, t) + C_1\int_0^t (Y_t^2 + Y_x^2)(s, x_0)ds \leq \varphi(Y, 0) + C_2\int_0^t Y^2(s, x_0)ds.$$

Thus

$$(1 + C_3)\varphi(Y, T) \leq \varphi(Y, 0) + C_4\int_0^T Y^2 dt,$$

for some positive constants C_3 and C_4, independent of t.

Proof-Spectrum Estimates I

Define a new norm $\|\cdot\|_{\mathbf{X}}$ for the function $h = (h_1, h_2) \in H^1 \times L^2([x_0, L])$,

$$\|h\|_{\mathbf{X}}^2 = \frac{\bar{E}_+}{\bar{\rho}_+}(x_0)|h_1|^2(x_0) + \int_{x_0}^{L} \frac{1}{\bar{\rho}_+}\left\{|h_2|^2 + \left(p'(\bar{\rho}_+) - \frac{\bar{J}_+^2}{\bar{\rho}_+^2}\right)|h_1'|^2 + \bar{\rho}_+|h_1|^2\right\}(x)dx.$$

The associated complex Hilbert space will be denoted by $(\mathbf{X}, \|\cdot\|_{\mathbf{X}})$. Define the solution operator $S_t : \mathbf{X} \mapsto \mathbf{X}$ as

$$S_t(h) = (Y(t,\cdot), Y_t(t,\cdot))$$

where Y is the solution of the problem (6)-(8) with initial data $h = (h_1, h_2)$. By (9), we can see that S_t is bounded and satisfies

$$\|S_t\| \leq 1.$$

It follows from the spectrum radius theorem that $|\sigma(S_t)| \leq 1$.

Proof-Spectrum Estimates II

Furthermore, we can define a map $K : \mathbf{X} \mapsto L^2([0, T])$ as

$$K(h) = Y(t, x_0).$$

Thus the Rauch-Taylor type estimate can be written as

$$(1 + C_5)\|S_T(h)\|_{\mathbf{X}} \leq \|h\|_{\mathbf{X}} + C_2\|Kh\|_{L^2([0,T])},$$

for some positive constant C_5. Note that for the initial data $f \in \mathbf{X}$, there exists a solution $Y \in H^1([0, T] \times [x_0, L])$, Note that K is compact. One has the following proposition.

Proposition Outside the disk $\{|z| \leq \frac{1}{1+C_5}\}$, there are only finite generalized eigenvalues for the operator S_T in the annulus $\{\frac{1}{1+C_5} < |z| \leq 1\}$ on the complex plane, each of these eigenvalues has the finite multiplicity.

Proposition There is no generalized eigenvalues of S_T on the circle $|z| = 1$.

A Priori Estimates for the Nonlinear Problem

- The energy estimates similar to those for the linearized problem yield the boundedness of higher order energy.
- Contraction of lower order energy of $\hat{Y}$ which is a solution of linearized problem implies contraction of lower order energy of Y, because $Y - \hat{Y}$ (quadratic term) is much smaller than $\hat{Y}$.

Suppose that $\bar{E}_+(x_0) < 0$. We look for the solutions for the linearized problem of the form $Y = e^{\lambda t} Z$. Then

$$\begin{cases} (p'(\bar\rho_+) - \bar u_+^2)\partial_x^2 Z + (\partial_x(p'(\bar\rho_+) - \bar u_+^2) - 2\bar u_+\lambda - \bar E_+)Z_x \\ \quad - (\lambda^2 + 2\lambda\partial_x \bar u_+ + \rho)Z = 0, \text{ for } x_0 < x < L, \\ \partial_x Z = \frac{2\bar u_+}{p'(\bar\rho_+) - \bar u_+^2}(x_0)(\frac{\bar E_+(x_0)}{2\bar u_+} + \lambda)Z, \text{at } x = x_0, \\ \partial_x Z = 0, \text{ at } x = L. \end{cases} \tag{10}$$

If $Z(x_0) = \alpha > 0$ and $\lambda = 0$, then $Z_x(x_0) < 0$. Therefore, there exists $L_1 > x_0$ such that $Z_x(x) < 0$ for $x_0 \le x \le L_1$.
If $\lambda = -2\frac{E}{2\bar u_+}(x_0)$, then $Z_x(x_0) > 0$. then there exists $L_2 > x_0$ such that $Z_x(x) > 0$ for $x_0 \le x \le L_2$.
By the continuous dependence of ODE with respect to the initial data and the parameters, there exist a $0 < \lambda < -2\frac{E_+}{2\bar u_+}(x_0)$ and an $L > 0$ such that the problem (10) admits a solution Z.

Flows in quasi-one-dimensional nozzles

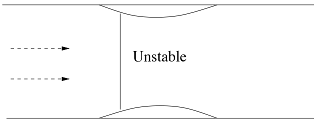
The governing equations are

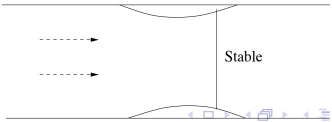
$$\begin{cases} \rho_t + (\rho u)_x = -\dfrac{a'(x)}{a(x)}\rho u, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = -\dfrac{a'(x)}{a(x)}\rho u^2. \end{cases} \tag{11}$$

The particular cases for $a(x)$ are 1, x, and x^2, which correspond to one dimensional, two dimensional radially symmetric, three dimensional spherical symmetric Euler equations.

Known results on stability and instability of transonic shock

- Glimm scheme (Liu),
- the characteristic method (Xin and Yin).





Theorem 5 Let $(\bar{\rho}, \bar{u})$ be a steady transonic shock solution. Assume that

$$a'(x_0) > 0. \tag{12}$$

If the initial data (ρ_0, u_0) satisfy and the $k+2$-th $(k \geq 15)$ order compatibility conditions at $x = 0$, $x = x_0$ and $x = L$, then the initial boundary value problem (11) with boundary conditions

$$\rho(0,t) = \bar{\rho}(0), \quad u(0,t) = \bar{u}(0), \quad \rho(L,t) = \bar{\rho}(L)$$

admits a unique piecewise smooth solution $(\rho, u)(x,t)$ for $(x,t) \in [0,L] \times [0,\infty)$, which contains a single transonic shock $x = s(t)$ $(0 < s(t) < L)$ satisfying the Rankine-Hugoniot condition and the Lax geometric shock condition for $t \geq 0$, and tend to the steady solutions exponentially fast, if

$$\|(\rho_0, u_0) - (\bar{\rho}, \bar{u})\|_{H^{k+2}} = \varepsilon$$

is suitably small.

Highlight of Stability and its Proof

Highlights:

- No assumptions on the smallness of $|a'|$ and shock strength.
- Exponential decay of the shock fronts and the deviation of the solutions.

Key ingredients of the proof

- Exponential decay of linearized problem via energy estimate, Rauch-Taylor type estimate, and spectral analysis.
- A priori estimates for the nonlinear problem

Summary and Prospects

Summary

- Structural stability of steady transonic shock solutions with respect to the perturbations of the background charge for Euler-Poisson system.
- Dynamical stability and instability of transonic shock solutions for Euler-Poisson system.
- Dynamical stability of the transonic shock solutions for the nozzle flows.

Prospects

- Non-isentropic flows
- Multidimensional wave patterns.

Thanks!