Optimal Transport, Ricci Curvature and Gradient Flows

We present the concept of generalized lower Ricci curvature bounds for metric measure spaces (M, d, m), introduced by Lott & Villani and the author. These curvature bounds are defined in terms of optimal transports, more precisely, in terms of convexity properties of the relative entropy regarded as function on the Wasserstein on the given space M. For Riemannian manifolds, this turns out to characterize lower Ricci bounds. We focus on new classes of examples, including Alexandrov spaces, Euclidean cones and spherical suspensions, and on new stability results (tensorization, local-to-global).

The heat flow on a metric measure space (M, d, m) can be introduced

- either as gradient flow on $L^2(M, m)$ for the *energy* $\frac{1}{2}\int_M \|\nabla u\|^2 \, dm$;
- or as gradient flow on the L^2-Wasserstein space $\mathcal{P}_2(M)$ for the *relative entropy* $\int_M u \log u \, dm$.

We will discuss various examples – including Finsler spaces, Alexandrov spaces, Wiener space and Heisenberg group – where both approaches lead to the same evolution semigroup.

This nonlinear heat semigroup will then be discussed in detail for Finsler spaces. A Finsler manifold is a smooth manifold M equipped with a norm $F(x, \cdot) : T_xM \to \mathbb{R}_+$ on each tangent space. The particular case of a Hilbert norm leads to the important subclasses of Riemannian manifolds. Bochner's inequality, Bakry-Emery estimate and Li-Yau type Harnack inequality will be verified for the nonlinear heat equation. On the other hand, exponential expansion bounds in Wasserstein distance hold true if and only if we are in a Riemannian setting.