Heat Flow on Non-Riemannian Spaces

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L^2-Wasserstein Space

Let (M,d) complete separable metric space, define

$$\mathcal{P}_2(M) \quad = \quad \left\{ \text{prob. meas. } \mu \text{ on } M \text{ with } \int_M d^2(x,x_0)\,\mu(dx) < \infty \right\}$$

and

$$W_2(\mu_0,\mu_1) = \inf_q \left[\int_{M\times M} d^2(x,y)\, d\,q(x,y) \right]^{1/2}.$$

Then

- $(\mathcal{P}_2(M), W_2)$ is a complete separable metric space.
- $(\mathcal{P}_2(M), W_2)$ is a compact space or a length space or an Alexandrov space[1)] with curvature ≥ 0 if and only if (M,d) is so.

1) Pythagorean inequality $a^2+b^2 \geq c^2$

L^2-Wasserstein Space for Riemannian M

Given compl. Riem. manifold M and $\mu_0, \mu_1 \in \mathcal{P}_2(M)$ with $d\mu_0 \ll d\mathrm{vol}$.

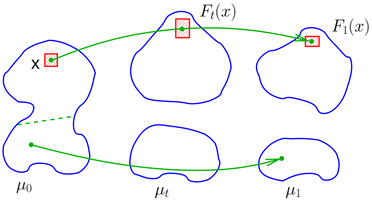
There exists a **unique geodesic** $(\mu_t)_{0 \le t \le 1}$ connecting μ_0, μ_1, given as

$$\mu_t := (F_t)_* \mu_0,$$

where

$$F_t(x) = \exp_x(t \nabla \varphi(x))$$

with suitable $d^2/2$-convex $\varphi : M \to \mathbb{R}$.



In the case $M = \mathbb{R}^n$ this states that there exists a **convex function** φ_1 such that

$$F_t(x) = (1-t)x + t\nabla\varphi_1(x).$$

In particular,

$$F_1(x) = \nabla\varphi_1(x).$$

The φ from above is $\varphi(x) = \varphi_1(x) - |x|^2/2$.

Riemannian Structure of $\mathcal{P}_2(M)$

Tangent space:

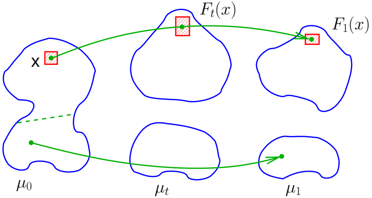
$$T_{\mu_0}\mathcal{P}_2 \quad = \text{closure of} \quad \{\Phi = \nabla\varphi : \ M \to TM, \quad \int_M |\nabla\varphi|^2 d\mu_0 < \infty\}$$

Riemannian tensor:

$$\langle \nabla\varphi, \nabla\psi \rangle_{T_{\mu_0}\mathcal{P}_2} \quad = \quad \int_M \langle \nabla\varphi, \nabla\psi \rangle_{T_x} \, d\mu_0(x)$$

Exponential map:

$$\exp_{\mu_0}(t\,\nabla\varphi) \quad = \quad [\exp(t\nabla\varphi)]_* \mu_0$$



Gradient Flows on $\mathcal{P}_2(M)$

The gradient $\nabla S(\nu) \in T_\nu \mathcal{P}_2(M)$ of the relative entropy

$$S(\nu) = \begin{cases} \int \rho \log \rho \, dm, & \text{if } d\nu = \rho \, dm \\ +\infty, & \text{if } d\nu \not\ll dm \end{cases}$$

as a function on $\mathcal{P}_2(M)$ is given by $\nabla S(\nu) = \nabla \log \rho$.

The gradient flow

$$\frac{\partial \nu}{\partial t} = -\nabla S(\nu) \qquad \text{on } \mathcal{P}_2(M)$$

for the relative entropy S is given by $\nu_t(dx) = \rho_t(x) m(dx)$ where ρ solves the heat equation

$$\frac{\partial}{\partial t} \rho = \triangle \rho \qquad \text{on } M.$$

$\mathbb{R}^n$: Otto '01,
Finsler (M, F, m): Ohta/Sturm '09,
Heisenberg group: Juillet '09,

Riemann (M, g): Ohta '09, Savare '09, Villani '09, Erbar '09,
Alexandrov space: Gigli/Kuwada/Ohta '10,
Wiener space: Fang/Shao/Sturm '09

Gradient Flows on $\mathcal{P}_2(M)$

$M = \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$, $m =$ Wiener measure, $d =$ Cameron-Martin distance

$$d(x,y) = \left(\int_0^\infty |\dot{x}(t) - \dot{y}(t)|^2 \, dt \right)^{1/2}$$

Transport cost / concentration inequalities

Talagrand, Ledoux, Wang, Fang, Shao, ... (1996, ...)

Existence & uniqueness of optimal transport map between m and $\rho\, m$

Feyel/Ustunel (2004)

Gradient flow for the relative entropy $\mathrm{Ent}(.|m)$ on $\mathcal{P}_2(M,d)$

$=$ Ornstein-Uhlenbeck semigroup on M.

Fang/Shao/St.: PTRF (2009)

Gradient Flows on $\mathcal{P}_2(M)$

Consider

$$S(\nu) = \frac{1}{s-1}\int \rho^s \, dx + \int V d\nu + \int\int W d\nu \, d\nu$$

for $d\nu = \rho \, dx + d\nu^{sing}$.
Here $s > 0$ real, $V: \mathbb{R}^n \to \mathbb{R}$ some external potential and $W: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ some interaction potential.

Theorem. (Jordan/Kinderlehrer/Otto '98, Otto '01, Villani '03, Ambrosio/Gigli/Savare '05, ...)

The gradient flow $\frac{\partial \nu}{\partial t} = -\nabla S(\nu)$ on $\mathcal{P}_2(\mathbb{R}^n)$ is given by $\nu_t(dx) = \rho_t(x)dx$ where ρ solves the nonlinear PDE

$$\frac{\partial}{\partial t}\rho = \triangle(\rho^s) + \nabla(\rho \cdot \nabla V) - \nabla(\rho \cdot \int (\nabla W \rho))$$

This includes porous medium equation, fast diffusion, Fokker-Planck, McKean-Vlasov.

Other examples:

quantum-drift diffusion (Fisher information), Ginzburg-Landau dynamics (squared H^{-1}-norm), p-Laplacian.

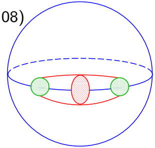
Optimal Transport, Heat Flow and Ricci Curvature

M complete Riemannian manifold, m Riemannian volume measure,
$S(\rho\, dm) = \int \rho \log \rho \; dm$.
Recall that the gradient flow of S satisfies $\frac{\partial}{\partial t}\rho = \triangle \rho$.

Theorem. (Otto '01, Otto/Villani '00, Cordero/McCann/Schmuckenschläger '01, vRenesse/Sturm '05)

$$\mathrm{Ric}_M \geq K \quad \Leftrightarrow \quad \mathrm{Hess}\, S \;\geq K$$
$$\Leftrightarrow \quad W_2(p_t\mu, p_t\nu) \leq e^{-Kt}\, W_2(\mu, \nu)$$

$\rightsquigarrow$ Ricci bounds for Markov chains (Ollivier '08)



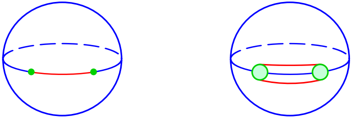
Theorem. (Bakry/Émery '84, Kendall, Cranston, Wang '97)

$$\mathrm{Ric}_M \geq K \quad \Leftrightarrow \quad \nabla |p_t u|^2(x) \leq e^{-Kt} \cdot p_t\left(|\nabla u|^2\right)(x)$$
$$\Leftrightarrow \quad \forall x, y : \; \exists \text{BMs } X_t, Y_t \text{ s.t. } \; d(X_t, Y_t) \leq e^{-Kt} d(x, y)$$

Optimal Transport, Heat Flow and Ricci Curvature

Let $(M, g(t))$ evolve under backward Ricci flow

$$\frac{\partial}{\partial t} g(t) = 2\mathrm{Ric}_{g(t)}.$$



Theorem (McCann/Topping '08).

$$W_2^{(t)}(p_{t_0,t}\mu, p_{t_0,t}\nu) \leq W_2^{(t_0)}(\mu, \nu)$$

with $W_2^{(t)}$ = Wasserstein distance for $d_{g(t)}$ and $p_{t_0,t}\mu$ = solution to forward heat flow $\frac{\partial}{\partial t}\eta = \triangle_{g(t)}\eta$ with $\eta(t_0) = \mu$.

Extension to $\mathcal{L}$-distance:
Monotonicity formula for Perelman's $\mathcal{L}$-functional (Topping '09, Lott '09)

Probabilistic/robust def. of Ricci flow

Optimal Transport, Heat Flow and Ricci Curvature

Let $(M, g(t))$ evolve under backward Ricci flow $\frac{\partial}{\partial t} g(t) = 2\mathrm{Ric}_{g(t)}$.

Detailed, pathwise version:

Theorem (Arnaudon/Coulibaly/Thalmaier '09).

For each pair $x, y \in M$: $\exists$ coupling of Brownian motions $(X_t), (Y_t)$ with $X_{t_0} = x$, $Y_{t_0} = y$ s.t.

$$d^{(t)}(X_t, Y_t) \leq d^{(t_0)}(x, y) \qquad \mathbb{P}\text{-a.s. for all } t \geq t_0.$$

Extension: Coupling of BMs s.t. $\mathcal{L}$-distance becomes supermartingale (Kuwada/Philipowski '10)

Ricci Bounds for Metric Measure Spaces

M complete Riemannian manifold, m Riem. volume measure, $\dim_M = n$

Let $S(\rho\, dm) = \int \rho \log \rho \, dm$. Then

$$\mathrm{Hess}\, S \;\geq K \qquad \Leftrightarrow \qquad \mathrm{Ric}_M \geq K$$

$\rightsquigarrow$ Ricci bound for metric measure spaces

$\rightsquigarrow$ logarithmic Sobolev inequality, concentration of measures

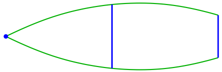
Ricci Bounds for Metric Measure Spaces

M complete Riemannian manifold, m Riem. volume measure, $\dim_M = n$

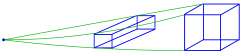
Let $S(\rho\, dm) = \frac{1}{s-1}\int_M \rho^s\, dm$. Then

$$\mathrm{Hess}\, S \geq 0 \quad \Leftrightarrow \quad \begin{cases} s & \geq \ 1 - \frac{1}{n} \\ \mathrm{Ric}_M & \geq \ 0 \end{cases}$$

$\rightsquigarrow$ Curvature-Dimension condition CD(K,N) for mms
$\rightsquigarrow$ Sobolev inequality, Bishop-Gromov volume growth estimate



sec ≥ 0 $\iff$ dist concave



ric ≥ 0 $\iff$ $\mathrm{vol}^{1/n}$ concave

Ricci Bounds for Metric Measure Spaces

(M, d) complete separable metric space, m locally finite measure on M

Definition. $\mathrm{Ric}(M, \mathsf{d}, m) \geq K$ or "$CD(K, \infty)$"

$\Longleftrightarrow$ $\forall \mu_0, \mu_1 \in \mathcal{P}_2(M)$: $\exists$ geodesic μ_t s.t. $\forall t \in [0,1]$:

$$\mathrm{Ent}(\mu_t|m) \leq (1-t)\mathrm{Ent}(\mu_0|m) + t\,\mathrm{Ent}(\mu_1|m) - \frac{K}{2}\,t(1-t)\,W_2^2(\mu_0, \mu_1)$$

Recall relative entropy

$$\mathrm{Ent}(\nu|m) = \begin{cases} \int_M \rho \log \rho \, dm, & \text{if } \nu = \rho \cdot m \\ +\infty, & \text{if } \nu \not\ll m \end{cases}$$

The Condition $CD(K,N)$

Definition. A metric measure space (M,d,m) satisfies the **Curvature-Dimension Condition** $CD(K,N)$ for $K,N\in\mathbb{R},\ N\geq 1$, iff

$\forall\ \rho_0 m,\ \rho_1 m:\ \exists$ geodesic $\rho_t m$ and optimal coupling q satisfying

$$\int \rho_t^{1-1/N}(z)\,dm(z) \quad\geq\quad \int\Big[\tau_{K,N}^{(1-t)}(\gamma_0,\gamma_1)\cdot\rho_0^{-1/N}(\gamma_0) + \tau_{K,N}^{(t)}(\gamma_0,\gamma_1)\cdot\rho_1^{-1/N}(\gamma_1)\Big]\,dq(\gamma_0,\gamma_1).$$

Here $\quad \tau_{K,N}^{(t)}(x,y) = t^{1/N}\left(\frac{\sin\left(\sqrt{\frac{K}{N-1}}\,t\,\mathrm{d}(x,y)\right)}{\sin\left(\sqrt{\frac{K}{N-1}}\,\mathrm{d}(x,y)\right)}\right)^{\frac{N-1}{N}}$,

e.g. $\tau_{0,N}^{(t)}(x,y) = t$

The Condition $CD(K, N)$

Theorem. For Riemannian manifolds:

$$CD(K, N) \iff \mathrm{Ric}_M \geq K \quad \text{and} \quad \dim_M \leq N$$

Further examples:
Alexandrov spaces, Finsler manifolds (e.g. Banach spaces), Wiener space ($K = 1, N = \infty$), quotients, products, cones, suspensions.

Theorem. The curvature-dimension condition is **stable** under convergence.

Theorem. For all $K, N, L \in \mathbb{R}$ the space of all (M, d, m) with $CD(K, N)$ and with diameter $\leq L$ is **compact**.

- **St.: Acta Math. 196 (2006)**
- **Lott, Villani: Annals of Math. 169 (2009)**

The Condition $CD(K, N)$

Assume $m(M) = 1$.

Theorem. $CD(K, N)$ with $K > 0$ and $N \leq \infty$ implies

- **Logarithmic Sobolev Inequality**
- **Talagrand Inequality**
- **Concentration of Measure**
- **Poincaré / Lichnerowicz Inequality**: for all functions f with $\int_M f\, dm = 0$

$$K \frac{N}{N-1} \cdot \int_M f^2 dm \leq \int_M |\nabla f|^2 dm.$$

The Condition $CD(K,N)$

Theorem. $CD(K,N)$ with $N<\infty$ implies Bishop-Gromov Volume Growth Estimate

$$\frac{s(r)}{s(R)} \geq \frac{\sin\left(\sqrt{\frac{K}{N-1}}r\right)^{N-1}}{\sin\left(\sqrt{\frac{K}{N-1}}R\right)^{N-1}} \qquad \text{for } s(r)=\frac{\partial}{\partial r}m(B_r(x_0))$$

Moreover: Brunn-Minkowski, Prekopa-Leindler, Borell-Brascamp-Lieb Inequalities.

Corollary. $CD(K,N)$ with $K>0$ and $N<\infty$ implies Bonnet-Myers Diameter Bound

$$\mathrm{diam}(M) \leq \sqrt{\frac{N-1}{K}}\cdot\pi$$

Heat Flow on Metric Measure Spaces (M, d, m)

Heat equation on M

- either as gradient flow on $L^2(M, m)$ for the **energy**

$$\mathcal{E}(u) = \frac{1}{2}\int_M |\nabla u|^2 \, dm$$

(with "$|\nabla u|$" local Lipschitz constant or minimal upper gradient or Finsler norm or ...)

- or as gradient flow on $\mathcal{P}_2(M)$ for the **relative entropy**

$$\text{Ent}(u) = \int_M u \log u \, dm.$$

Heat Flow on Metric Measure Spaces (M, d, m)

Theorem (Cheeger '99).

- On each (M, d, m) which satisfies Poincaré & doubling there exists a unique gradient flow for the energy.

Theorem (Ambrosio/Savare/Zambotti '07).

- For each log-concave probability measure m on a Hilbert space M the above energy defines a closable Dirichlet form with m being the unique invariant measure of the associated Markov process $(X_t, \mathbb{P}_x)$.
- If $(m_n)_{n \in \mathbb{N}}$ is a family of log-concave measures on M with $m_n \longrightarrow m$ then $(X_t^n, \mathbb{P}_x^n) \longrightarrow (X_t, \mathbb{P}_x)$ in f.d.d. sense.

Theorem (Gigli '09).

- On each (M, d, m) which satisfies $CD(K, \infty)$ there exists a unique gradient flow for the relative entropy.
- If $(M_n, d_n, m_n) \rightarrow (M, d, m)$ and $\nu_n(0) \rightarrow \nu(0)$ then $\nu_n(t) \rightarrow \nu(t)$ for all $t > 0$.

Heat Flow on Metric Measure Spaces (M, d, m)

Theorem (Gigli/Kuwada/Ohta '10).

For **Alexandrov spaces**[1)] (M, d, m) both approaches coincide.

- L^2-Wasserstein contraction

$$W_2(p_t\mu, p_t\nu) \leq e^{-Kt}\, W_2(\mu, \nu)$$

- Bakry-Émery gradient estimate

$$\nabla|p_t u|^2(x) \leq e^{-2Kt} \cdot p_t\left(|\nabla u|^2\right)(x)$$

- Coupling of Brownian motions

1) Geodesic space with Pythagorean <u>in</u>equality $a^2 + b^2 \geq c^2$

Heat Flow on Finsler Spaces

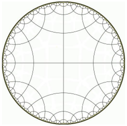
- M smooth n-dimensional manifold

- $F: TM \to \mathbb{R}$ smooth on $TM \setminus \{0\}$ with

$$F(x,.): \; T_xM \to \mathbb{R} \text{ norm} \qquad (\text{for each } x \in M).$$

- m measure on M with smooth density in local coordinates

E.g. Banach spaces, scaling limits of periodic graphs



Ohta/St.: Comm. Pure Appl. Math. (2009)

Heat Flow on Finsler Spaces

- Either as gradient flow on $L^2(M, m)$ for the energy

$$\mathcal{E}(u) = \frac{1}{2}\int_M F^2(\nabla u)\, dm = \frac{1}{2}\int_M F^{*2}(Du)\, dm$$

- Or as gradient flow on the L^2-Wasserstein space $\mathcal{P}_2(M)$ of probability measures on M for the relative entropy

$$\mathrm{Ent}(u) = \int_M u \log u\, dm.$$

Theorem (Ohta/St. '09):

For compact **Finsler spaces** (M, F, m) both approaches coincide.

$$\forall u_0 \in L^2 : \quad \exists! \text{ weak solution to } \Delta u = \frac{\partial}{\partial t} u.$$

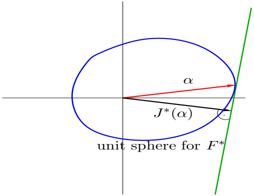
Example. $M = \mathbb{R}^n$, $F(x, .) = \|.\|$, $u(t, x) = t^{-n/2} \exp(-\|x\|^2/4t)$.

Heat Flow on Finsler Spaces

Basic notions:

- Dual norm $F^*(x,.) : T_x^*M \to \mathbb{R}$

- Legendre transform $J^*(x,.) : T_x^*M \to T_xM$
  In local coordinates: $J^*(x,\alpha)_i = \frac{1}{2}\frac{\partial}{\partial\alpha^i}F^{*2}(x,\alpha)$ for $i = 1,\dots,n$



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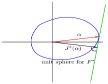


- Differential $Du(x) \in T_x^*M$ of smooth function $u : M \to \mathbf{R}$

- Gradient $\nabla u(x) = J^*(x, Du(x)) \in T_xM$ nonlinear in u !

- Divergence of vector field Φ defined via
  $\int_M u\,\mathrm{div}\Phi\,dm = -\int_M \Phi\,u\,dm$

1

- Laplacian $\Delta u = \mathrm{div}(J^*(Du))$ cf. Chern, Shen

Finsler Laplacian: Nonlinear, Uniformly Elliptic

Attention.

- Nonlinear heat equation
- Never $\mathcal{C}^2$, but $\mathcal{C}^{1,\alpha}$.

L^2-Contraction.

$\forall u_0, v_0 \in L^2(M)$:

$$\|u_t - v_t\|_{L^2} \leq e^{-\lambda\cdot\kappa\cdot t} \cdot \|u_0 - v_0\|_{L^2}$$

where $1/\lambda =$ Poincaré const. for $\mathcal{E}$, $\kappa =$ unif. convexity bound for F^{*2}.

Integrated Gaussian Estimates á la Davies.

$\forall u, v \in L^2(M)$

$$\int_M u P_t v \, dm \leq \exp\left(- \frac{d^2(v,u)}{4t} \right) \|u\|_{L^2} \|v\|_{L^2}$$

with $d(v,u) = \operatorname{ess\,inf}\{d(y,x) : x \in \operatorname{supp}[u],\ y \in \operatorname{supp}[v]\}$.

Heat Flow and Curvature on Finsler Spaces

- Curvature-Dimension Condition $CD(K,\infty)$

  K-convexity of relative entropy

- L^2-Wasserstein contraction

  $$W_2(p_t\mu, p_t\nu) \le e^{-Kt}\, W_2(\mu,\nu)$$

- Bakry-Émery gradient estimate

  $$\nabla|p_t u|^2(x) \le e^{-2Kt} \cdot p_t\left(|\nabla u|^2\right)(x)$$

Heat Flow and Curvature on Finsler Spaces

- Curvature-Dimension Condition $CD(K, \infty)$ **YES**

  K-convexity of relative entropy

- L^2-Wasserstein contraction **NO**

  $$W_2(p_t\mu, p_t\nu) \leq e^{-Kt}\, W_2(\mu, \nu)$$

- Bakry-Émery gradient estimate **YES**

  $$\nabla|p_t u|^2(x) \leq e^{-2Kt} \cdot p_t\left(|\nabla u|^2\right)(x)$$

The Finsler Structure on the L^2-Wasserstein Space

Distance function d (non-symmetric) on M

$$d(x,y) = \inf_{\gamma} \left(\int_0^1 F^2\big(\gamma(t), \dot{\gamma}(t)\big)\, dt \right)^{1/2}$$

where the infimum is taken over all differentiable curves $\gamma : [0,1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$.

L^2-Wasserstein distance W_2 on $\mathcal{P}_2(M)$

$$\begin{aligned} W_2(\mu,\nu) \quad &:= \quad \inf_{q \in \Pi(\mu,\nu)} \left(\int_{M \times M} d^2(x,y)\, dq(x,y) \right)^{1/2} \\ &= \quad \inf_{(\mu_t)_{t\in[0,1]}} \left(\int_0^1 F_W^2(\mu_t, \dot{\mu}_t)\, dt \right)^{1/2} \end{aligned}$$

where the infimum is taken over all locally Lipschitz continuous curves $(\mu_t)_{t\in[0,1]} \subset \mathcal{P}_2(M)$ with $\mu_0 = \mu$ and $\mu_1 = \nu$.

The Finsler Structure on the L^2-Wasserstein Space

For each $\mu \in \mathcal{P}_2(M)$, we define

$$T_\mu \mathcal{P} \quad = \quad \overline{\{\Phi = \nabla \varphi \,:\, \varphi \in \mathcal{C}^\infty(M)\}}^{F_W(\mu,\cdot)}$$

with closure taken with respect to the **Finsler structure**

$$F_W(\mu, \Phi) \quad = \quad \left(\int_M F^2(x, \Phi(x))\, \mu(dx) \right)^{1/2}.$$

Analogously, $F_W^*(\mu, \alpha) = \left(\int_M F^{*2}(x, \alpha(x))\, \mu(dx) \right)^{1/2}$ and $T_\mu^* \mathcal{P} = \overline{\{\alpha = D\varphi\}}^{F_W^*(\mu,\cdot)}$.
F_W and F_W^* are dual to each other w.r.t. the pairing between $T_\mu^*\mathcal{P}$ and $T_\mu \mathcal{P}$ given by

$$\langle \alpha, \Phi \rangle_\mu := \int_M \langle \alpha(x), \Phi(x) \rangle_x \, \mu(dx)$$

where $\langle \cdot, \cdot \rangle_x$ denotes the natural pairing between $T_x^* M$ and $T_x M$.

Excursion: Convex Functions on Finsler Spaces

$S : M \to \overline{\mathbb{R}}$ is called *K-convex* (or geodesically K-convex) if

$$S(\gamma_t) \leq (1-t)\, S(\gamma_0) + t\, S(\gamma_1) - \frac{K}{2} t(1-t)\, d^2(\gamma_0, \gamma_1)$$

for each geodesic γ in M.
If S is smooth this is equivalent to

$$\begin{array}{ccc} \partial_t^2\; S(\gamma_t) & \geq & K \cdot F^2(\dot\gamma_t) \\ \shortparallel & & \\ \partial_t \left[DS(\gamma_t)\, \dot\gamma_t \right] & & \end{array}$$

For each K-convex S there exists a unique gradient flow:
$\forall x \in S : \exists!$ Lipschitz curve $(\xi_t)_{t \geq 0}$ with $\xi_0 = x$ and

$$\dot\xi_t = \nabla(-S)(\xi_t).$$

Excursion: Convex Functions on Finsler Spaces

A smooth function $S : M \to \mathbb{R}$ is *K*-convex if

$$\partial_t \Big[\underbrace{DS(\gamma_t)}_{\substack{\pitchfork \\ T^*M}} \quad \underbrace{\dot\gamma_t}_{\substack{\pitchfork \\ TM}} \Big] \geq K \cdot F^2(\dot\gamma_t)$$

Theorem

The gradient flow for S is *L*-contractive

$$d(\xi_t, \eta_t) \leq e^{-Lt} d(\xi_0, \eta_0)$$

if and only if S is *L*-skew convex

$$\partial_t \Big[\underbrace{-\nabla(-S)(\gamma_t)}_{\substack{\pitchfork \\ TM}} \quad \underbrace{J(\dot\gamma_t)}_{\substack{\pitchfork \\ T^*M}} \Big] \geq L \cdot F^2(\dot\gamma_t)$$

Excursion: Convex Functions on Finsler Spaces

On Riemannian spaces M, a function S is K-convex if and only if it is K-skew convex.

For each compact Finsler space (M, F, m) the relative entropy

$$S(\mu) = \int_M \log\left(\frac{d\mu}{dm}\right)\, d\mu$$

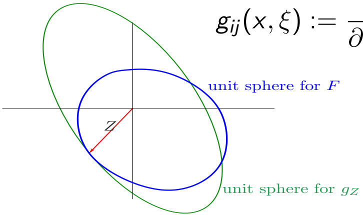
is K-convex on $\mathcal{P}_2(M)$ for some K.

Now consider the Finsler space $M = \mathbb{R}^n$, $dm(x) = dx$ and $F(x, v) = \|v\|$ for some norm $\|.\|$ on $\mathbb{R}^n$.

The relative entropy S is K-skew convex on $\mathcal{P}_2(\mathbb{R}^n)$ for some K if and only if $\|.\|$ is a Hilbert norm (i.e. inner product).

$CD(K,N)$ and Induced Riemannian Structure

Given a non-vanishing vector field $Z: M \to TM$ we define a Riemannian structure g_Z on M by $g_Z(x) := g(x, Z(x))$ where (in local coordinates):



$$g_{ij}(x,\xi) := \frac{\partial^2}{\partial \xi^i \, \partial \xi^j}(\frac{1}{2}F^2(x,\xi)).$$

Theorem.
(M, F, m) satisfies the curvature-dimension condition $\mathrm{CD}(K,N)$ if and only if $\mathrm{Ric}_{N,g_Z,m}(Z,Z) \geq K \cdot |Z|^2$ for all Jacobi fields Z.

Here for a Riemannian metric $g = g_Z$ and for any number $N \geq n$

$$\mathrm{Ric}_{N,g,m} = \mathrm{Ric}_g + \mathrm{Hess}\, V - \frac{1}{N-n}(DV \otimes DV)$$

where $V = V_g$ is chosen s.t. $dm = e^{-V} d\mathrm{vol}_g$.

$CD(K,N)$ and Induced Riemannian Structure

Theorem ("Cheeger-Yau Estimate")
Assume $CD(K,N)$ and let u be a solution to the heat equation on $[0,\infty)\times M$ with $u(0,\cdot)\ge h_0(d(\cdot,z))$ for some $z\in M$ and some smooth decreasing function h_0 on $[0,L)$. Then

$$u(t,x)\ge h^{K,N}(t,d(x,z))$$

for all $t>0$ and $x\in M$ where $h^{K,N}$ denotes the solution to the PDE

$$\partial_t h=\partial_r^2 h+\partial_r h\sqrt{(N-1)K}\cdot\cot\left(\sqrt{\frac{K}{N-1}}r\right)$$

on $(0,\infty)\times(0,L)$ with initial condition $h^{K,N}(0,\cdot)=h_0$.

Here $L=\pi\sqrt{(N-1)/K}$ if $K>0$ and $L=\infty$ else.

In particular, under $CD(0,n)$

$$p(t,x,z):=P_t\delta_z(x)\ge\frac{1}{\rho(z)}\,t^{-n/2}\cdot\exp\big(-\frac{d^2(x,z)}{4t}\big).$$

Recent progress

- Bochner inequality, Bakry-Emery estimates
- Li-Yau type Harnack inequality