Sobolev spaces over metric measure spaces

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From a joint work with L.Ambrosio and G.Savaré

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- Introduction
- The vertical approach
- The horizontal approach
- Identification of the two
- Who is $|\nabla f|$?
- Applications: the heat flow
- Open problems

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The point

Let (X, d) be a Polish space equipped with a reference measure $\boldsymbol{m} \in \mathscr{P}(X)$.

Our goal is to define the Sobolev space $W^{1,2}(X, d, \boldsymbol{m})$.

Few things which is better to realize soon

- The space $W^{1,2}(X, d, \boldsymbol{m})$ in general will *not* be an Hilbert space,
- There is not really hope to define the distributional gradient ∇f of a Sobolev function.
- Still, we may hope to give a good definition of $|\nabla f|$.
- Up to technicalities, we need to produce a notion of 'norm of weak gradient' $|\nabla f|$ such that

$$f \qquad \mapsto \qquad \int |\nabla f|^2 d\boldsymbol{m},$$

  is lower semicontinuous in $L^2(X, \boldsymbol{m})$

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An incorrect way of defining the Cheeger energy

Given $f : X \to \mathbb{R}$ the local Lipschitz constant $\mathrm{lip}_f : X \to [0, \infty]$ is defined as

$$\mathrm{lip}_f(x) := \varlimsup_{y \to x} \frac{|f(x) - f(y)|}{d(x, y)}.$$

An incorrect way of defining the Cheeger energy

Given $f : X \to \mathbb{R}$ the local Lipschitz constant $\mathrm{lip}_f : X \to [0, \infty]$ is defined as

$$\mathrm{lip}_f(x) := \overline{\lim_{y \to x}} \frac{|f(x) - f(y)|}{d(x, y)}.$$

Wrong definition of Cheeger energy:

$$f \quad \mapsto \quad \frac{1}{2} \int \mathrm{lip}_f^2 \, d\boldsymbol{m}.$$

The correct approach: relaxation

$$\mathrm{Ch}(f) := \inf_{(f_n)} \varliminf_{n} \frac{1}{2} \int \mathrm{lip}_{f_n}^2 \, d\boldsymbol{m},$$

where the infimum is taken among all sequences of Lipschitz functions (f_n) converging to f in L^2.

The correct approach: relaxation

$$\mathrm{Ch}(f) := \inf_{(f_n)} \varliminf_{n} \frac{1}{2}\int \mathrm{lip}^2_{f_n}\, d\boldsymbol{m},$$

where the infimum is taken among all sequences of Lipschitz functions (f_n) converging to f in L^2.

The map $\mathrm{Ch} : L^2 \to [0,\infty]$ is:

- Convex
- Lower semicontinuos
- 2-homogeneous
- Typically not quadratic

Vertical weak gradients

$G : X \to [0, \infty]$ is a vertical weak gradient of $f : X \to \mathbb{R}$ if there is a sequence (f_n) of Lipschitz functions, $f_n \to f$ in $L^2(X, \boldsymbol{m})$ and $\mathrm{lip}(f_n)$ weakly converges in $L^2(X, \boldsymbol{m})$ to $\tilde{G}$, with $\tilde{G} \le G$ $\boldsymbol{m}$-a.e..

Note: the set of all possible vertical weak gradients of f is:

- closed in $L^2(X, \boldsymbol{m})$ (possibly empty),
- convex.

Hence, when non empty, there exists a unique element of minimal L^2 norm. Call it $|\nabla f|_v$

Basic properties of $|\nabla f|_v$

1)

$$\mathrm{Ch}(f) = \frac{1}{2}\int |\nabla f|_v^2 \, d\boldsymbol{m}$$

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2) If (f_n) is a sequence of Lipschitz functions converging to f in $L^2(X, \boldsymbol{m})$ such that

$$\frac{1}{2}\int \mathrm{lip}_{f_n}^2 \, d\boldsymbol{m} \qquad \to \qquad \mathrm{Ch}(f).$$

Then $\mathrm{lip}_{f_n} \to |\nabla f|$ in $L^2(X, \boldsymbol{m})$.

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3) If G is a vertical weak gradient of f then

$$|\nabla f|_v \leq G \qquad \boldsymbol{m} - a.e..$$

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4) if $u : \mathbb{R} \to \mathbb{R}$ is C^1 it holds

$$|\nabla(u \circ f)|_v \leq |u' \circ f||\nabla f|_v.$$

If $u' > 0$ then we have equality

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The idea

For a C^1 function $f : \mathbb{R}^d \to \mathbb{R}$ we have the inequality

$$|f(\gamma_0) - f(\gamma_1)| \leq \int_0^1 |\nabla f|(\gamma_t)|\gamma_t'|dt,$$

for any smooth curve $\gamma : [0, 1] \to \mathbb{R}^d$.

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for any smooth curve $\gamma : [0, 1] \to \mathbb{R}^d$.

The point is that if f is Sobolev, then the same is true along 'almost any curve'.

Absolutely continuous curves

A curve $\gamma : [0,1] \to X$ is absolutely continuous if

$$d(\gamma_t, \gamma_s) \leq \int_t^s g(r)dr, \qquad \forall t \leq s,$$

for some $g \in L^1(0,1)$.
In this case the limit

$$\lim_{h \to 0} \frac{d(\gamma_{t+h}, \gamma_t)}{h},$$

exists for a.e. t, and defines an L^1 function denoted by $|\dot{\gamma}_t|$.
We will denote by $AC^2([0,1], X)$ the set of abs.cont. curves γ such that $|\dot{\gamma}_t| \in L^2(0,1)$.

Negligible set of curves

$\boldsymbol{\pi} \in \mathscr{P}(AC^2([0,1], X))$ is a test plan provided

$$(\mathrm{e}_t)_\sharp \boldsymbol{\pi} \leq C\boldsymbol{m}, \qquad \forall t \in [0,1],$$

for some $C > 0$.

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for some $C > 0$.

A set of curves $A \subset AC^2([0,1],X)$ is negligible provided

$$\boldsymbol{\pi}(A) = 0,$$

for any test plan $\boldsymbol{\pi}$.

Functions which are Sobolev along a.e. curve

Let $f : X \to \mathbb{R}$. We say that f is Sobolev along a.e. curve provided for a.e. γ the function $f \circ \gamma$ coincides in $\{0, 1\}$ and a.e. in $[0, 1]$ with an absolutely continuous map f_γ.

Horizontal weak gradients

Let $f : X \to \mathbb{R}$ be Sobolev along a.e. curve.
We say that $G : X \to [0, \infty]$ is an horizontal weak gradient provided

$$|f(\gamma_1) - f(\gamma_0)| \leq \int_0^1 G(\gamma_t)|\dot{\gamma}_t|dt, \qquad \text{for a.e. } \gamma.$$

Invariance

Let $f = \tilde{f}$, $\boldsymbol{m}$-a.e. and $G = \tilde{G}$, $\boldsymbol{m}$-a.e..
Assume that f is Sobolev along a.e. curve and that G is an horizontal weak gradient.
Then $\tilde{f}$ is Sobolev along a.e. curve and $\tilde{G}$ is an horizontal weak gradient.

Invariance

Let $f = \tilde{f}$, $\boldsymbol{m}$-a.e. and $G = \tilde{G}$, $\boldsymbol{m}$-a.e..
Assume that f is Sobolev along a.e. curve and that G is an horizontal weak gradient.
Then $\tilde{f}$ is Sobolev along a.e. curve and $\tilde{G}$ is an horizontal weak gradient.

Indeed for any test plan $\boldsymbol{\pi}$ it holds

$$\boldsymbol{\pi}\{\gamma : f(\gamma_i) \neq \tilde{f}(\gamma_i)\} = (\mathrm{e}_i)_\sharp \boldsymbol{\pi}(f \neq \tilde{f}) = 0, \qquad i = 0, 1,$$

$$(\boldsymbol{\pi} \times \mathcal{L}^1)\big\{(\gamma, t) : G(\gamma_t) \neq \tilde{G}(\gamma_t)\big\} = 0 \;\;\Rightarrow\;\; \boldsymbol{\pi}\big\{(\gamma : \int_\gamma G \neq \int_\gamma \tilde{G})\big\} = 0$$

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Lower semicontinuity

Along the same lines, one can show that if $f_n \to f$ $\boldsymbol{m}$-a.e., f_n is Sobolev along a.e. curve, G_n is an horizontal weak gradient for f_n and (G_n) weakly converges in $L^2(X, \boldsymbol{m})$ to G, then:

f is Sobolev along a.e. curve and G is an horizontal weak gradient.

Locality

If f is Sobolev along a.e. curve and G_1, G_2 are horizontal weak gradients, then so is $\min\{G_1, G_2\}$.

It follows that there exists a minimal function $|\nabla f|_h$ such that

$$|\nabla f|_h \leq G, \qquad \boldsymbol{m} - a.e.,$$

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Rmk. If f is Lipschitz, then it is Sobolev along a.e. curve and lip_f is an horizontal weak gradient. In particular

$$|\nabla f|_h \le \mathrm{lip}_f, \qquad \boldsymbol{m} - a.e..$$

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The result

Theorem Let $(X, d, \boldsymbol{m})$ be a Polish space with $\boldsymbol{m}$ locally finite.

Then for any $f \in L^2(X, \boldsymbol{m})$ it holds

$$|\nabla f|_v = |\nabla f|_h, \qquad \boldsymbol{m} - a.e.$$

The easy inequality

Let $f \in D(\mathrm{Ch})$. Then it is Sobolev along a.e. curve and

$$|\nabla f|_h \leq |\nabla f|_v$$

Indeed, pick a sequence (f_n) of Lipschitz functions converging in L^2 to f such that (lip_{f_n}) converges to $|\nabla f|_v$ in $L^2(X, \boldsymbol{m})$. Then we conclude by the lower semicontinuity of $|\nabla f|_h$.

What we need to conclude

To conclude it is sufficient to prove that

$$\int \frac{|\nabla f|_v^2}{f} d\boldsymbol{m} \leq \int \frac{|\nabla f|_h^2}{f} d\boldsymbol{m},$$

for $f \geq 0$ far from 0 and ∞ with $\int f d\boldsymbol{m} = 1$.

Laplacian and gradient flow of Ch

We say that $f \in D(\Delta)$ if $\partial^- \mathrm{Ch}(f) \neq 0$. In this case we define $\Delta f := -v$, where v is the element of minimal norm in $\partial^- \mathrm{Ch}(f)$.
For any $f_0 \in L^2(X, \boldsymbol{m})$ there exists a unique map $t \mapsto f_t \in L^2(X, \boldsymbol{m})$ such that

$$\frac{d^+}{dt} f_t = \Delta f_t, \qquad \forall t \geq 0.$$

Maximum/minimum principle

We claim that for any $f \in L^2$ such that $f \geq c$ and any $\tau > 0$ the minimum of

$$g \mapsto \frac{1}{2}\int |\nabla g|_v^2 d\boldsymbol{m} + \frac{\|f-g\|_{L^2}^2}{2\tau},$$

satisfies $g \geq c$.

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A similar argument shows that the mass is preserved in time

Integration by parts

For $f \in D(\Delta)$ and $g \in D(\mathrm{Ch})$ it holds

$$\left| \int g \Delta f \, d\boldsymbol{m} \right| \leq \int |\nabla g|_v |\nabla f|_v d\boldsymbol{m}.$$

For a C^1 map $u : f(X) \to \mathbb{R}$ we have

$$\int u(f) \Delta f \, d\boldsymbol{m} = -\int u'(f) |\nabla f|_v^2 \, d\boldsymbol{m}.$$

Entropy dissipation

Let $f_0 \in D(\mathrm{Ch})$ be such that $0 < c \leq f_0 \leq C < \infty$ and (f_t) the gradient flow of Ch starting from f.
Then

$$-\frac{d}{dt}\int f_t \log(f_t)\, d\boldsymbol{m} = \int \frac{|\nabla f_t|_v^2}{f_t}\, d\boldsymbol{m}, \qquad \forall t \geq 0.$$

Hamilton-Jacobi semigroup alias Hopf-Lax formula alias Moreau-Yosida approximation alias inf-convolution

Let (X, d) be a metric space.
For $\psi : X \to \mathbb{R}$ Lipschitz and bounded $t > 0$ we define $Q_t\psi : X \to \mathbb{R}$ by

$$Q_t\psi(x) := \inf_{y \in X} \psi(y) + \frac{d^2(x, y)}{2t}$$

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Then:

- $Q_t\psi$ is Lipschitz for every $t > 0$,
- $t \mapsto Q_t\psi$ is a Lipschitz curve w.r.t. the sup distance ($Q_0\psi := \psi$),

Subsolutions of Hamilton-Jacobi equation

For any $x \in X$ the map $t \mapsto Q_t\varphi(x)$ is locally Lipschitz and it holds

$$\frac{d}{dt} Q_t\varphi(x) + \frac{\mathrm{lip}^2_{Q_t\varphi}(x)}{2} \leq 0,$$

for every $t \geq 0$ with the possible exception of a countable set.

Estimating $W_2(f_t\boldsymbol{m}, f_{t+s}\boldsymbol{m})$ by duality

$$\frac{1}{2}W_2^2(f_t\boldsymbol{m}, f_{t+s}\boldsymbol{m}) = \sup_{\varphi\in C^{Lip}} \int \varphi f_t d\boldsymbol{m} + \int \varphi^c f_{t+s} d\boldsymbol{m}$$

$$= \sup_{\psi\in C^{Lip}} \int Q_1\psi f_{t+s} d\boldsymbol{m} - \int \psi f_t d\boldsymbol{m}$$

Key computation

$r \mapsto Q_r\psi$ and $r \mapsto f_{t+rs}$ are Lipschitz curves with values in $L^2(X, \boldsymbol{m})$.

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$r \mapsto Q_r\psi$ and $r \mapsto f_{t+rs}$ are Lipschitz curves with values in $L^2(X, \boldsymbol{m})$. Thus

$$\int Q_1\psi f_{t+s} - \psi f_t \, d\boldsymbol{m}$$

$$= \iint_0^1 \frac{d}{dr}\Big(Q_r\psi f_{t+rs}\Big) dr d\boldsymbol{m}$$

$$= \iint_0^1 -\frac{\mathrm{lip}^2_{Q_r\psi}}{2} f_{t+rs} + sQ_r\psi\Delta f_{t+rs} dr d\boldsymbol{m}$$

$$\leq \iint_0^1 -\frac{\mathrm{lip}^2_{Q_r\psi}}{2} f_{t+rs} + |\nabla Q_r\psi|_v \frac{s|\nabla f_{t+rs}|^2_v}{f_{t+rs}} f_{t+rs} dr d\boldsymbol{m}$$

$$\leq \iint_0^1 -\frac{\mathrm{lip}^2_{Q_r\psi}}{2} f_{t+rs} + \frac{|\nabla Q_r\psi|^2_v}{2} f_{t+rs} + \frac{s^2}{2}\frac{|\nabla f_{t+rs}|^2_v}{f_{t+rs}} dr d\boldsymbol{m}$$

$$\leq \frac{s^2}{2}\int\int_0^1 \frac{|\nabla f_{t+rs}|^2_v}{f_{t+rs}} dr d\boldsymbol{m}.$$

Conclusion of the argument

From

$$
\begin{aligned}
W_2^2(f_t\boldsymbol{m}, f_{t+s}\boldsymbol{m}) &\le s^2 \int_0^1 \int \frac{|\nabla f_{t+rs}|_v^2}{f_{t+rs}} d\boldsymbol{m} dr \\
&\le \frac{s^2}{c} \int_0^1 \int |\nabla f_{t+rs}|_v^2 d\boldsymbol{m} dr \\
&\le \frac{s^2}{c} \int |\nabla f_t|_v^2 d\boldsymbol{m},
\end{aligned}
$$

we get that $(f_t\boldsymbol{m})$ is locally Lipschitz.
We also have the estimate

$$
\overline{\lim_{s\downarrow 0}} \frac{1}{s^2} \int_0^s |\dot{f_r \boldsymbol{m}}|^2 dr \le \int \frac{|\nabla f_0|_v^2}{f_0} d\boldsymbol{m}
$$

An existence result

Theorem (Lisini) Let $(\mu_t) \subset \mathscr{P}_2(X)$ be a curve in $AC^2([0,1], (\mathscr{P}_2(X), W_2))$. Then there exists $\boldsymbol{\pi} \in \mathscr{P}(AC^2([0,1], X))$ such that

$$(\mathrm{e}_t)_\sharp \boldsymbol{\pi} = \mu_t,$$
$$|\dot{\mu}_t|^2 = \int |\dot{\gamma}_t|^2 d\boldsymbol{\pi}(\gamma), \qquad a.e.\ t.$$

Entropy dissipation and horizontal weak gradients

Let (f_t) be the gradient flow of Ch starting from $f_0 \in D(\mathrm{Ch})$, bounded away from 0 and infinity with mass 1. Also, let π be a plan associated to $(f_t \boldsymbol{m})$.

Entropy dissipation and horizontal weak gradients

Let (f_t) be the gradient flow of Ch starting from $f_0 \in D(\mathrm{Ch})$, bounded away from 0 and infinity with mass 1. Also, let $\boldsymbol{\pi}$ be a plan associated to $(f_t\boldsymbol{m})$. Then

$$
\begin{aligned}
&\int f_0\log(f_0) - f_t\log(f_t)\,d\boldsymbol{m}\\
&\le \int \log(f_0)(f_0 - f_t)\,d\boldsymbol{m}\\
&= \int \log(f_0)(\gamma_0) - \log(f_t)(\gamma_t)\,d\boldsymbol{\pi}(\gamma)\\
&\le \int\int_0^t \frac{|\nabla f_0|_h}{f_0}(\gamma_s)|\dot\gamma_s|\,ds\,d\boldsymbol{\pi}(\gamma)\\
&\le \frac{1}{2}\int\int_0^t \frac{|\nabla f_0|_h^2}{f_0^2}(\gamma_s)\,ds\,d\boldsymbol{\pi}(\gamma) + \frac{1}{2}\int\int_0^t |\dot\gamma_s|^2\,ds\,d\boldsymbol{\pi}(\gamma)\\
&= \frac{1}{2}\int\int_0^t \frac{|\nabla f_0|_h^2}{f_0^2} f_s\,ds\,d\boldsymbol{m} + \frac{1}{2}\int_0^t |\dot\mu_s|^2\,ds
\end{aligned}
$$

Conclusion of the identification

Dividing by t and letting $t \downarrow 0$ we get

$$-\frac{d}{dt}|_{t=0} \int f_t \log(f_t) d\boldsymbol{m} \leq \frac{1}{2} \int \frac{|\nabla f_0|_h^2}{f_0} \, d\boldsymbol{m} + \frac{1}{2} \int \frac{|\nabla f_0|_v^2}{f_0} \, d\boldsymbol{m}.$$

Since we knew that

$$-\frac{d}{dt}|_{t=0} \int f_t \log(f_t) d\boldsymbol{m} = \int \frac{|\nabla f_0|_v^2}{f_0} \, d\boldsymbol{m},$$

we conclude.

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Everything could be trivial

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Then $|\nabla f| = 0$ for any f.

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Let $(X, d, \boldsymbol{m})$ be $\mathbb{R}$ with the Euclidean distance and $\boldsymbol{m}$ a measure concentrated on $\mathbb{Q}$.
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Then $|\nabla f| = 0$ for any f.

In general we only know that for f Lipschitz it holds

$$|\nabla f| \leq \mathrm{lip}_f, \qquad \boldsymbol{m} - a.e.$$

Cheeger's result

Theorem Let $(X, d, \boldsymbol{m})$ be a locally compact metric measure space and assume that $\boldsymbol{m}$ is doubling and the space supports a local Poincaré inequality.
Then for any $f : X \to \mathbb{R}$ Lipschitz it holds

$$\mathrm{lip}_f(x) = |\nabla f|(x), \qquad \boldsymbol{m} - a.e.x$$

Our result: a metric Brenier theorem

Theorem Let $(X, d, \boldsymbol{m})$ be a Polish geodesic space, $\mu, \nu \in \mathscr{P}_2(X)$, (μ_t) a geodesic connecting them and φ a Kantorovich potential. Assume that

$$\left\| \frac{d\mu_t}{d\boldsymbol{m}} \right\|_{L^\infty} \leq C, \qquad \text{for } t \in [0, \varepsilon].$$

Then

$$\mathrm{lip}_\varphi^+ = |\nabla\varphi|, \qquad \boldsymbol{m} - a.e..$$

Furthermore, if $\gamma \in \mathrm{Opt}(\mu, \nu)$, then

$$d(x, y) = |\nabla\varphi|(x), \qquad \gamma - a.e.(x, y).$$

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Heat flow

Theorem Assume that $Ric((X, d, \boldsymbol{m})) \geq K$. Then the gradient flow of Ch is L^2 coincides with the gradient flow of $\mathrm{Ent}_{\boldsymbol{m}}$ in $(\mathscr{P}_2(X), W_2)$

A particular case

Theorem The following are equivalent:

1. $Ric((X, d, \boldsymbol{m})) \geq K$ in the strong sense and the Heat Flow is linear.
2. $Ric((X, d, \boldsymbol{m})) \geq K$ in the strong sense and Ch is a quadratic functional.
3. The functional $\mathrm{Ent}_{\boldsymbol{m}}$ has Gradient Flows in the EVI sense relative to K.

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Theorem The following are equivalent:

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2. $Ric((X, d, \boldsymbol{m})) \geq K$ in the strong sense and Ch is a quadratic functional.
3. The functional $\mathrm{Ent}_{\boldsymbol{m}}$ has Gradient Flows in the EVI sense relative to K.

Definition We say that $(X, d, \boldsymbol{m})$ has *Riemannian Ricci* curvature bounded from below by K if (one of) the three conditions above is fulfilled.

Properties of Riemannian Ricci bounds

Compatibility with the Riemannian case

Stability w.r.t. measured-GH convergence

Properties of Riemannian Ricci bounds

Compatibility with the Riemannian case

Stability w.r.t. measured-GH convergence

Locality

Tensorizability

} for non branching spaces

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Some open questions

- What about $W^{1,p}$?
- What about fractals?

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