The Curvature-Dimension Condition for Metric Measure Spaces

Karl-Theodor Sturm

Universität Bonn

Metric Measure Spaces

Metric measure space (M, d, m): (M, d) compl. sep. metric space, m locally finite measure on M.

Generalizations of Riemannian manifolds, including singularities, rich geometric structure

Alexandrov '51: generalized sectional curvature bounds for metric spaces

Gromov '81: GH-metric on the space of compact metric spaces

Key Results.

- $\{(M, d)$ with sect. curv. $\geq K\}$ is closed under GH-convergence;
- $\{(M, d)$ with sect. curv. $\geq K$, dim $\leq N$, diam $\leq L\}$ is compact.

Generalized Curvature Bounds

Crucial for analysis and stochastics: Ricci curv. $\geq K$

S.T.Yau, Cheeger, Colding, Kendall, Malliavin, Perelman et al.

Aim:

Generalized Ricci Curvature Bound $\mathrm{Ric}(M, \mathrm{d}, m) \geq K$

- equivalent to $\mathrm{Ric}_p(\xi, \xi) \geq K \|\xi\|^2$ if M is Riemannian manifold
- stable under convergence
- intrinsic, synthetic

Two independent approaches:

- St.: Acta Math. **196** (2006)
- Lott, Villani: Annals of Math. **169** (2009)

based on fundamental contributions of Brenier '91, McCann '01, Otto '01, Otto/Villani '00,

Cordero-Erausquin/McCann/Schmuckenschläger '01, v.Renesse/St. '05

L^2-Wasserstein Space

Let (M, d) complete separable metric space, define

$$\mathcal{P}_2(M) \quad = \quad \left\{ \text{prob. meas. } \mu \text{ on } M \text{ with } \int_M d^2(x, x_0)\, \mu(dx) < \infty \right\}$$

and

$$d_W(\mu_0, \mu_1) = \inf_q \left[\int_{M \times M} d^2(x, y)\, d\, q(x, y) \right]^{1/2}.$$

Then

- $(\mathcal{P}_2(M), d_W)$ is a complete separable metric space.
- $(\mathcal{P}_2(M), d_W)$ is a compact space or a length space or an Alexandrov space with curvature ≥ 0 if and only if (M, d) is so.

Ricci Bounds for Metric Measure Spaces

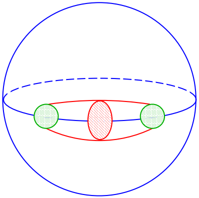
Relative entropy $\quad \mathrm{Ent}(\cdot, m): \quad \mathcal{P}_2(M) \to [-\infty, \infty]$

$$\mathrm{Ent}(\nu, m) = \begin{cases} \int_M \rho \log \rho \, dm & , \text{if } \nu = \rho \cdot m \\ +\infty & , \text{if } \nu \not\ll m \end{cases}$$

Definition. $\mathrm{Ric}(M, \mathsf{d}, m) \geq 0$

$\iff \quad \forall \mu_0, \mu_1 \in \mathcal{P}_2(M): \ \exists$ geodesic μ_t s.t. $\forall t \in [0, 1]$:

$$\mathrm{Ent}(\mu_t | m) \ \leq \ (1 - t)\mathrm{Ent}(\mu_0 | m) + t\, \mathrm{Ent}(\mu_1 | m)$$



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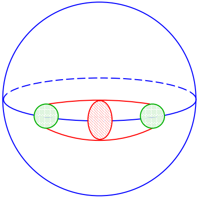
Definition. $\quad \mathrm{Ric}(M, \mathsf{d}, m) \geq K$

$\Longleftrightarrow \quad \forall \mu_0, \mu_1 \in \mathcal{P}_2(M): \ \exists$ geodesic μ_t s.t. $\forall t \in [0, 1]$:

$$\mathrm{Ent}(\mu_t | m) \ \leq \ (1-t)\mathrm{Ent}(\mu_0 | m) + t\,\mathrm{Ent}(\mu_1 | m) - \frac{K}{2}\, t(1-t)\, \mathrm{d}_{\mathrm{w}}^2(\mu_0, \mu_1)$$



Basic Examples

Riemannian manifolds (Otto/Villani '00, Cordero-Er./McCann/Schmuckenschl. '01, vRenesse/Sturm '05):

$$\mathrm{Ric}(M,\ \mathsf{d}, m) \geq K \qquad \Longleftrightarrow \qquad \mathrm{Ric}(\xi, \xi) \geq K \quad \text{for all} \quad \xi \in TM,\ |\xi| = 1$$

Weighted spaces:

$$\mathrm{Ric}(M,\ \mathsf{d}, m) \geq K \quad \text{and} \quad \mathrm{Hess}\, \mathrm{V} \geq \mathrm{L} \qquad \Longrightarrow \qquad \mathrm{Ric}(\mathrm{M},\ \mathsf{d}, \mathrm{e}^{-\mathrm{V}}\mathrm{m}) \geq \mathrm{K} + \mathrm{L}$$

Finsler spaces (Ohta '09): Flag Ricci curvature plus weights

$$\mathrm{Ric}_\xi(\xi, \xi) + \ \mathrm{Hess}\ \mathrm{V}_\xi(\xi, \xi) \geq K|\xi|^2 \quad \text{where} \quad dm = \exp(-V_\xi) d\mathrm{vol}_\xi$$

Alexandrov spaces (Petrunin '09): $\mathrm{Sec}\,(M, d) \geq 0 \qquad \Longrightarrow \qquad \mathrm{Ric}(M,\ \mathsf{d}, m) \geq 0$

Wiener space (Fang/Shao/Sturm '09): $\mathrm{Ric}(M,\ \mathsf{d}, m) \geq 1$

Heisenberg group (Juillet '09): $\mathrm{Ric}(M,\ \mathsf{d}, m) \not\geq K$, but MCP(0,5)

Discrete spaces (Bonciocat/Sturm '09, Ollivier '08): h-rough curvature, h-midpoints

Tensorization, Globalization, Stability

Assume M is non-branching.

Theorem ('Tensorization'). If $M = \bigotimes_{i \in I} M_i$ then

$$\mathrm{Ric}(M_i,\, \mathsf{d}, m) \geq K \quad (\forall i) \quad \Longrightarrow \quad \mathrm{Ric}(M,\, \mathsf{d}, m) \geq K.$$

Theorem ('Globalization'). If $M = \bigcup_{i \in I} M_i^0$ then

$$\mathrm{Ric}(M_i,\, \mathsf{d}, m) \geq K \quad (\forall i) \quad \Longrightarrow \quad \mathrm{Ric}(M,\, \mathsf{d}, m) \geq K.$$

Theorem ('Stability'). If $(M_n,\, \mathsf{d}_n, m_n) \xrightarrow{\mathbb{D}} (M,\, \mathsf{d}, m)$ then

$$\mathrm{Ric}(M_n,\, \mathsf{d}_n, m_n) \geq K \quad (\forall n) \quad \Longrightarrow \quad \mathrm{Ric}(M,\, \mathsf{d}, m) \geq K.$$

The L^2-Transportation Metric $\mathbb{D}$

$$\mathbb{D}((M,\, \mathrm{d},m),(M',\, \mathrm{d}',m')) \quad = \quad \inf\left(\int_{M\times M'} \hat{\mathrm{d}}^2(x,y)d\,\hat{m}(x,y)\right)^{1/2}$$

where the inf is taken over all couplings $\hat{\mathrm{d}}$ of d and d' and over all couplings $\hat{m}$ of m and m'.

A measure $\hat{m}$ on the product space $M\times M'$ is a coupling of m and m' iff

$$\hat{m}(A\times M')=m(A), \quad \hat{m}(M\times A')=m'(A')$$

for all measurable sets $A\subset M, A'\subset M'$.

A pseudo metric $\hat{\mathrm{d}}$ on the disjoint union $M\sqcup M'$ is a coupling of d and d' iff

$$\hat{\mathrm{d}}(x,y)=\mathrm{d}(x,y), \quad \hat{\mathrm{d}}(x',y')=\mathrm{d}'(x',y')$$

for all $x,y\in \mathrm{supp}[m]\subset M$ and all $x',y'\in \mathrm{supp}[m']\subset M'$.

For each pair of spaces $(M,\, \mathrm{d},m)$ and $(M',\, \mathrm{d}',m')$ there exists an optimal pair of couplings $\hat{\mathrm{d}}$ and $\hat{m}$.

$\mathbb{D}$ is a complete separable length metric on the space of isomorphism classes of normalized metric measure spaces.

Measured GH-convergence $\Longrightarrow$ Gromov's $\square$-convergence $\Longleftrightarrow$ $\mathbb{D}$-convergence

Ricci Curvature and Dimension

M complete Riemannian manifold, m Riemannian volume measure, $\dim_M = n$

Let $S(\rho\, dm) = \int \rho \cdot \log \rho \, dm$. Then gradient flow satisfies $\frac{\partial}{\partial t}\rho = \triangle \rho$ and

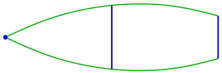
$$\mathrm{Hess}\, S \ \geq K \qquad \Leftrightarrow \qquad \mathrm{Ric}_M \geq K$$

$\rightsquigarrow$ Ricci bound for metric measure spaces
$\rightsquigarrow$ logarithmic Sobolev inequality, concentration of measures

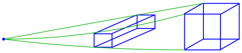
Let $S(\rho\, dm) = \frac{1}{s-1}\int_M \rho^s \, dm$. Then gradient flow satisfies $\frac{\partial}{\partial t}\rho = \triangle(\rho^s)$ and

$$\mathrm{Hess}\, S \ \geq 0 \quad \Leftrightarrow \quad \begin{cases} s & \geq \quad 1 - \frac{1}{n} \\ \mathrm{Ric}_M & \geq \quad 0 \end{cases}$$

$\rightsquigarrow$ Curvature-Dimension condition CD(K,N) for mms
$\rightsquigarrow$ Sobolev inequality, Bishop-Gromov volume growth estimate



sec ≥ 0 $\iff$ dist concave



ric ≥ 0 $\iff$ $\mathrm{vol}^{1/n}$ concave

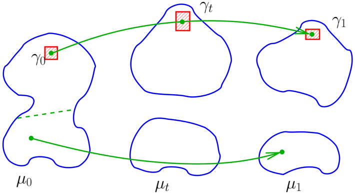
The Condition $CD(K, N)$

Definition. A metric measure space (M, d, m) satisfies the Curvature-Dimension Condition $CD(K, N)$ for $K, N \in \mathbb{R}$, $N \geq 1$, iff

$\forall\ \rho_0 m,\ \rho_1 m$ with bounded supports: $\exists$ geodesic $\rho_t m$ and optimal coupling q satisfying

$$\int_M \rho_t^{1-1/N}(z)\, dm(z) \ \geq\ \int_{M\times M} \left[\tau_{K,N}^{(1-t)}(\gamma_0,\gamma_1)\cdot \rho_0^{-1/N}(\gamma_0) + \tau_{K,N}^{(t)}(\gamma_0,\gamma_1)\cdot \rho_1^{-1/N}(\gamma_1)\right]\, dq(\gamma_0,\gamma_1).$$

Here $\quad \tau_{K,N}^{(t)}(x,y) = t^{1/N}\left(\dfrac{\sin\left(\sqrt{\frac{K}{N-1}}\, t\, \mathrm{d}(x,y)\right)}{\sin\left(\sqrt{\frac{K}{N-1}}\, \mathrm{d}(x,y)\right)}\right)^{\frac{N-1}{N}}, \qquad$ e.g. $\tau_{0,N}^{(t)}(x,y) = t$



The Condition $CD(K, N)$

Riemannian manifolds

$$CD(K, N) \iff \mathrm{Ric}_M \geq K \quad \text{and} \quad \dim_M \leq N$$

Weighted Riemannian spaces $dm = e^{-V} d\,vol$

$$CD(K, N) \iff \mathrm{Ric}_M + \mathrm{Hess}\,V - \frac{1}{N-n} DV \otimes DV \geq K \quad \text{and} \quad n = \dim_M \leq N$$

Finsler spaces

Alexandrov spaces

$$\mathrm{Sec}(M, d) \geq 0 \implies CD(0, N)$$

Consequences of $CD(K,N)$ with finite N

Theorem ("Bishop-Gromov Volume Growth Estimate"). $CD(K,N)$ with $N<\infty$ implies

$$\frac{s(r)}{s(R)} \geq \frac{\sin\left(\sqrt{\frac{K}{N-1}}r\right)^{N-1}}{\sin\left(\sqrt{\frac{K}{N-1}}R\right)^{N-1}} \qquad \text{for } s(r)=\frac{\partial}{\partial r}m(B_r(x_0))$$

Corollary ("Bonnet-Myers Diameter Bound"). $CD(K,N)$ with $K>0$ and $N<\infty$ implies

$$\mathrm{diam}(M) \leq \sqrt{\frac{N-1}{K}}\cdot\pi$$

Theorem ("Poincaré / Lichnerowicz Inequality"). $CD(K,N)$ with $K>0$ and $N\leq\infty$ implies

$$K\frac{N}{N-1}\cdot\int_M f^2\,dm \leq \int_M |\nabla^- f|^2\,dm$$

for f with $\int_M f\,dm=0$ where $|\nabla^- f|(x)=\limsup_{y\to x}\frac{[f(y)-f(x)]_-}{d(y,x)}$.

Consequences of $CD(K, N)$ with finite N

Open question ("Splitting theorem for the measure")

(M, d, m) satisfies $CD(0, N)$ for some $N < \infty$ and $\exists$ line in M

$$?\Downarrow?$$

$$M = M_0 \times \mathbb{R}, \qquad dm(\xi, r) = dm_0(\xi) \otimes dr$$

No splitting theorem for the metric! Example: Banach spaces.

A result in this spirit:

Theorem (Ohta)

(M, d, m) satisfies $CD(K, N)$ for some $N < \infty$ and $K > 0$ and $\exists$ geodesic in M of length $L = \sqrt{\frac{N-1}{K}} \cdot \pi$

$$\Downarrow$$

$$M = M_0 \times [0, L], \qquad dm(\xi, r) = dm_0(\xi) \otimes \sin^{N-1}\left(r\sqrt{\frac{K}{N-1}}\right) dr$$

Metric Cones

Given mms (M, d, m), define n-Euclidean metric cone $(\hat{M}, \hat{d}, \hat{m})$ by

- $\hat{M} := M \times [0, \infty)$ with $M \times \{0\}$ contracted to one point
- $\hat{d}((x,r),(y,s)) := \sqrt{r^2 + s^2 - 2rs\cos(d(x,y))}$
- $d\hat{m}(x,r) := dm(x) \otimes r^n dr$.

If M is complete n-dim Riemannian manifold then $\hat{M}_0 := M \times (0, \infty)$ is non-complete $n+1$-dimensional Riemannian manifold.

Theorem (Bacher/Sturm JFA'10)

Let M be complete n-dim Riemannian manifold, $n \geq 2$. Then

$$\mathrm{Ric}_M \geq n-1 \quad \Longleftrightarrow \quad (\hat{M}, \hat{d}, \hat{m}) \text{ satisfies } CD(0, n+1)$$

Two remarks, indicating that the result is neither surprising nor trivial:

(i) Cheeger/Taylor'82: $\mathrm{Ric}_M \geq n-1 \iff \mathrm{Ric}_{\hat{M}_0} \geq 0$

(ii) If $n = 1$, $M = [0, L]$ then $\mathrm{Ric}_{\hat{M}_0} \geq 0$. But

$$\hat{M} \text{ satisfies } CD(0,2) \quad \Longleftrightarrow \quad L \leq 2\pi$$

Otherwise, singularity at 0 (curvature $-\infty$). E.g. if $L = k2\pi$ then $\hat{M} = k-$fold cover of $\mathbb{R}^2 \setminus \{0\}$.

Open question

$$(M, d, m) \text{ satisfies } CD(n-1, n) \quad \overset{?}{\Longleftrightarrow} \quad (\hat{M}, \hat{d}, \hat{m}) \text{ satisfies } CD(0, n+1)$$

Spherical Cones

Given mms (M, d, m) with diameter $\leq \pi$, define *n-spherical cone* (or suspension) $(\hat{M}, \hat{d}, \hat{m})$ by

- $\hat{M} := M \times [0, \pi]$
- $\cos\left(\hat{d}(x, r), (y, s))\right) := \cos r \cos s + \sin r \sin s \cos(d(x, y))$
- $d\hat{m}(x, r) := dm(x) \otimes \sin^n(r) dr.$

If M is complete n-dim Riemannian manifold then $\hat{M}_0 := M \times (0, \infty)$ is non-complete $n+1$-dimensional Riemannian manifold.

Theorem (Bacher/Sturm JFA'10)

Let M be complete n-dim Riemannian manifold, $n \geq 2$. Then

$$\mathrm{Ric}_M \geq n - 1 \quad \Longleftrightarrow \quad (\hat{M}, \hat{d}, \hat{m}) \text{ satisfies } CD(n, n+1)$$

The Condition $CD(K, N)$

Theorem. The curvature-dimension condition is stable under convergence.

Theorem. For all $K, N, L \in \mathbb{R}$ the space of all (M, d, m) with $CD(K, N)$ and with diameter $\leq L$ is compact.

Open questions:

- Local-to-global property of $CD(K, N)$
- Tensorization property of $CD(K, N)$

The Reduced Curvature-Dimension Condition $CD^*(K,N)$

Replace

$$\tau_{K,N}^{(t)}(x,y) = t^{1/N}\left(\frac{\sin\left(\sqrt{\frac{K}{N-1}}\,t\,\mathrm{d}(x,y)\right)}{\sin\left(\sqrt{\frac{K}{N-1}}\,\mathrm{d}(x,y)\right)}\right)^{\frac{N-1}{N}}$$

in definition of $CD(K,N)$ by

$$\sigma_{K,N}^{(t)}(x,y) = \frac{\sin\left(\sqrt{\frac{K}{N}}\,t\,\mathrm{d}(x,y)\right)}{\sin\left(\sqrt{\frac{K}{N}}\,\mathrm{d}(x,y)\right)}.$$

That is,

(M,d,m) satisfies $CD^*(K,N)$ iff $\forall\ \rho_0 m,\ \rho_1 m$ with bdd supp: $\exists$ geodesic $\rho_t m$ and optimal coupling q s.t.

$$\int_M \rho_t^{1-1/N}(z)\,dm(z) \ \geq\ \int_{M\times M}\left[\sigma_{K,N}^{(1-t)}(\gamma_0,\gamma_1)\cdot\rho_0^{-1/N}(\gamma_0) + \sigma_{K,N}^{(t)}(\gamma_0,\gamma_1)\cdot\rho_1^{-1/N}(\gamma_1)\right]\,dq(\gamma_0,\gamma_1).$$

Lemma.

- $CD(K,N) \implies CD^*(K,N)$ (since $\sigma_{K,N} \leq \tau_{K,N}$)
- $CD(0,N) \iff CD^*(0,N)$ (since $\sigma_{0,N} = \tau_{0,N}$)
- $CD^*(K,N) \implies CD(K^*,N)$ with $K^* := K\frac{N-1}{N}$ whenever $K \geq 0$.

Theorem ("Tensorization"). If M_i is non-branching and satisfies $CD^*(K,N_i)$ then $\bigotimes M_i$ satisfies $CD^*(K,\sum N_i)$.

Crucial: If $\sum N_i = N, \sum \vartheta_i^2 = \vartheta$ then $\prod \sigma_{K,N_i}^{(t)}(\vartheta_i)^{N_i} \geq \sigma_{K,N}^{(t)}(\vartheta)^N$.

The Reduced Curvature-Dimension Condition $CD^*(K,N)$

Definition.

- $CD_{loc}(K,N) \iff \forall x \in M: \quad \exists$ neighborhood M_x s.t. $\forall$ prob. meas. μ_0, μ_1 supported in M_x: $\exists$ geodesic μ_t (not necessarily supported in M_x) s.t. ... holds.
- $CD_{loc}(K-,N) \iff CD_{loc}(K',N)$ for all $K' < K$.

Proposition.

$$CD_{loc}(K-,N) \iff CD^*_{loc}(K-,N).$$

("Proof": For small distances $\sigma \approx \tau$ up to fourth order terms; these can be dominated by changing K' into K''.)

Theorem. If $M = \mathrm{supp}[m]$ is geodesic and non-branching then

$$CD^*_{loc}(K,N) \iff CD^*(K,N).$$

Corollary.

$$CD(K,N) \implies CD_{loc}(K,N) \implies CD^*(K,N).$$

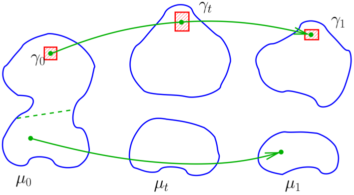
Application. Assume $M = \mathrm{supp}[m]$ non-branching and semi-locally simply connected. Let $(\hat{M}, \hat{d}, \hat{m})$ be universal cover of (M,d,m).
If (M,d,m) satisfies $CD(K,N)$ for some finite N and $K > 0$ then $\hat{M}$ has finite fundamental group.

The Reduced Curvature-Dimension Condition $CD^*(K, N)$

Ingredients of the proof:

- For non-branching spaces $CD^*(K, N)$ is equivalent to a pathwise version:
  $\forall\ \rho_0 m,\ \rho_1 m:\ \exists$ geodesic $\rho_t m$: for a.e. geodesic γ and $\forall\ t \in [0,1]$:

$$\rho_t^{-1/N}(\gamma_t) \quad \geq \quad \sigma_{K,N}^{(1-t)}(\gamma_0, \gamma_1) \cdot \rho_0^{-1/N}(\gamma_0) + \sigma_{K,N}^{(t)}(\gamma_0, \gamma_1) \cdot \rho_1^{-1/N}(\gamma_1)$$



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- The above inequality for $t = \frac{1}{2}$ implies the inequality for all $t \in [0,1]$.
  (Not true with τ in the place of σ.)

The Reduced Curvature-Dimension Condition $CD^*(K, N)$

Recall Jacobi field calculus in the Riemannian case:

$F_t = \exp(t\nabla\varphi): \; M \to M,$
$\mu_t = (F_t)_* \; \mu_0$ Geodesic in $\mathcal{P}_2(M)$
$J_t := \det dF_t \,, \quad y_t := \log J_t$

$$\begin{aligned} -\ddot{y}_t \quad &\geq \quad \frac{1}{n}\dot{y}_t^2 \; + \; \mathrm{Ric}(\dot{F}_t, \dot{F}_t) \\ &\geq \quad \frac{1}{N}\dot{y}_t^2 \; + \; K \cdot |\dot{F}_t|^2 \qquad (**) \end{aligned}$$

if $\dim \leq N, \quad \mathrm{Ric} \geq K.$

Two easy cases

1. Ignore n (i.e. $N = \infty$). Then

$$\begin{aligned} (**) \quad &\Leftrightarrow \quad -y \;\text{ is }\; K \cdot |\dot{F}_t|^2 \text{ - convex} \\ &\Leftrightarrow \quad \mathrm{Ent}(\cdot \mid m) \;\text{ is }\; K \text{ - convex} \end{aligned}$$

2. Assume $\mathrm{Ric} \geq 0$ (i.e. $K = 0$). Then

$$\begin{aligned} (**) \quad &\Leftrightarrow \quad J^{1/N} \;\text{ is concave} \\ &\Leftrightarrow \quad S_N(\cdot \mid m) \;\text{ is convex} \end{aligned}$$

Rényi entropy functional for $\mu = \rho \cdot m + \mu^{\text{sing}}$

$$S_N(\mu \mid m) := -\int \rho^{1-\,1/N}\, dm$$

The Reduced Curvature-Dimension Condition $CD^*(K, N)$

General case is much more sophisticated

$$(**) \quad \Leftrightarrow \quad \partial_t^2 J_t^{1/N}(x) \leq -\vartheta^2 \cdot J_t^{1/N}(x)$$

where $\vartheta = \mathrm{d}(x, F_1(x)) \cdot \sqrt{\frac{K}{N}}$.

Stable integrated version:

$$J_t^{1/N}(x) \geq \sigma^{(1-t)}(\vartheta) \cdot J_0^{1/N} + \sigma^{(t)}(\vartheta) \cdot J_1^{1/N}$$

with

$$\sigma^{(t)}(\vartheta) = \frac{\sin(t\vartheta)}{\sin(\vartheta)}.$$

Basic observation: Curvature has no effect in direction of transport!

$\Longrightarrow$ Get rid of 1 dimension! Improved Estimate:

$$J_t^{1/N} \geq \tau^{(1-t)}(\vartheta) \cdot J_0^{1/N} + \tau^{(t)}(\vartheta) \cdot J_1^{1/N}$$

Open questions for general mms (M, d, m):

- $CD^*(K, N)$ and separation of transport direction $\Longrightarrow CD(K, N)$?
- Self-improving property of $CD^*(K, N)$, similar to self-improving property of Bakry-Emery condition?